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HANKEL DETERMINANT FOR CERTAIN SUBCLASS OF UNIVALENT FUNCTIONS DEFINED BY q-DIFFERENCE OPERATOR

OLATUNJI SUNDAY OLUWAFEMI AND FAGBEMIRO OLALEKAN*

Abstract. The most powerful tool that cannot be completely eroded in the history of Geometric Functions Theory (GFT) is determinant of any order. There is no gaining-saying that determinants have series of applications in Sciences, Engineering, Data analysis, Computing, and generally in other sectors of man's endeavor. In particular, the Hankel determinant has attracted attention of numerous researchers possibly because of its distinct geometric structural sequence, and despite gaining so much attention there still exist some perceived gaps in knowledge that are yet to be explored. It is on this positive direction that this present study derived its interest so that a new development in knowledge can be reached. The method used the q- Difference Operator with the second Hankel determinant as well as its inverse functions of order two along with the concept of subordination principle. With this approach in focus,this study examined some new subclasses of analytic functions. The sharp initial coefficient bounds obtained were used to derive some new subclasses of the Second Hankel along with its inverse functions.

1. Introduction and Preliminaries

Represented by $\mathcal{A}$ the class of analytic functions f defined inside the unit disk $\Gamma = \{z \in \mathbb{C} : i < |z| < 1\}$, normalized with the conditions

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

and hence the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots . \tag{1.1}$$

Also note that, $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent (i.e., one-to-one) with the same aforementioned conditions in the unit disk $\Gamma = \{z \in \mathbb{C} : i < |z| < 1\}$ is a well known class. The function in equation (1.1) has been used as the basic function to define several classes of analytic functions like: Starlike

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Olatunji Sunday Oluwafemi, *Fagbemiro Olalekan.

functions, Convex functions Close-to-convex functions and Spirallike functions which satisfy the geometric conditions,

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0,\quad \operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0,\quad \operatorname{Re}\left\{\frac{f(z)}{g(z)}\right\}>0, \operatorname{Re}\left\{\frac{e^{i\theta}zf'(z)}{f(z)}\right\}>0$$

and so on. See details in ([3],[5], [27], [30], [39], [40] and [43]).
The analytic function f in Γ is *subordinate* to an analytical function g in Γ, written as $f \prec g$, if there exists some function $w(z)$ satisfying $w(0)=0$ and $|w(z)|<1$ which are our well known Schwarz function then these can be obviously expressed as $f(z)=g(w(z))$. Specifically, if g is univalent, then $f \prec g$ holds if and only if $f(\Gamma)\subset g(\Gamma)$. The definitions are well known.

The idea of the qth Hankel determinant of the form

$$H_n(q)=\begin{vmatrix} a_q & a_{q+1} & a_{q+2} & \cdots & a_{q+n-1} \\ a_{q+1} & a_{q+2} & a_{q+3} & \cdots & a_{q+n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{q+n-1} & a_{q+n} & a_{q+n+1} & \cdots & a_{q+2n-2} \end{vmatrix}, \quad (n,q\in\mathbb{N}=\{1,2,3,\dots\}) \tag{1.2}$$

was initiated by Pommerenke [41] in 1963.
Studying Hankel determinants of inverse functions provides valuable insights into different aspects of analytic functions and their geometry, including univalence and distortion, boundary behavior, geometric properties under inversion, analytic continuation and coefficient relationships of the analytic functions and its inverses. Researchers in Geometric Function Theory explore these properties to understand the intricate relationship between analytic functions, their inverses and the geometric implications within the complex plane.
The determinant has applications in the study of power series having integer coefficients ([4], [8]) useful in the theory of singularities [6], for data analysis, engineering, and sciences in general. Noonan and Thomas [35] studied the second Hankel determinant of a real mean-p-valent function. Noor [36]investigated the second Hankel determinant for the class of univalent functions with bounded boundary rotation. Layman [32] worked on the Hankel transform and some of its properties, while Ethrenborg [9] investigated the Hankel determinant of exponential polynomials. Shi et al. [47] investigates sharp Hankel determinant $H_{2,1}\left(\frac{F_f}{2}\right)$ and $H_{2,2}\left(\frac{F_f}{2}\right)$ with a logarithmic coefficient for a special class of bounded turning functions connected with a three-leaf-shaped domain. The computations of the upper bounds of the absolute Hankel determinants for many subclasses of analytic and univalent functions for larger values of q and n are tedious to achieve in their general structure. Also, we have some other authors that have contributed in this direction who also obtained upper bounds and they are, Krishna and Ramreddy [31], Thomas et al. [49], Sim *et al.* [48] and Ramachandran *et al.* [43]. In particular, Obradovic and Tuneski [40] investigated the sharp upper bounds for the Hankel determinant of second order for the inverse functions of functions for some classes of univalent functions. They approached the Hankel determinants by considering both the second and third order for their investigations so

that they can examine their peculiar sequence for the development of some new special coefficient estimates. However, their work failed to give a distinct generalization. This present study draws inspiration from their work and carried out its investigation through the use of the q- difference operator that involved the use of subordination principle so that a new generalization of the Hankel determinant and their inverses can be achieved. The second-order Hankel determinant

$H_2, 1(f)$ and third-order Hankel determinant $H_3, 1(f)$ for f of the form in (1) are given as:

$$H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2, \tag{1.3}$$

and

$$H_{3,1}(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2), \tag{1.4}$$

while the famous Fekete-Szego [10] is the determinant of $H_{2,1}(f) = a_3 - a_2^2$, which is a special case of $a_3 - \mu a_2^2$ when $\mu = 1$ holds. See also [7], [11]-[17], [18], [19], [20], [21], [29], [33] and for recent articles on Hankel determinants, refer to [22], [23]-[25], [26], [27], [36] and [38]. In addition, seasonal researchers like [40], [47] have started contributing to the second Hankel determinant for f^{-1} since each $f \in \mathcal{S}$ is univalent.Thus, if $w = f(z)$, then there must be some disk $|w| < r_0(f)$ for which f^{-1} exists. This simply implies that there will be at least the disk $|r_0| < \frac{1}{4}$ assured by Koebe's Quarter Theorem. Denote

$$f(w) = w + \Delta_2 w^2 + \Delta_3 w^3 + \Delta_4 w^4 + \cdots . \tag{1.5}$$

where,

$$f(f^{-1}(\omega)) = \omega \quad \text{and} \quad f^{-1}(\omega) = \omega + \sum_{k=2}^{\infty} \Delta_k\, \omega^k. \tag{1.6}$$

Comparing coefficients in (1.1) and (1.6), we obtain:

$$\Delta_2 = -a_2$$

$$\Delta_3 = -a_3 + 2a_2^2$$

$$\Delta_4 = -a_4 + 5a_3a_2 - 5a_2^3$$

. The inverse functions of the form (1.6) for the second Hankel determinant is given by

$$H_{2,2}(f^{-1}) = \Delta_2\Delta_4 - \Delta_3 = a_2a_4 - a_3^2 - a_2^2(a_3 - a_2^2) \tag{1.7}$$

More information can be seen in Aouf and Seaoudy [1] as well as Nehad et al. [36].

Jackson ([28], [29]) introduced the q-difference operator of a functions of the form

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & \text{if } z \neq, 0 \\ 0, & \text{if } z = 0. \end{cases}$$

In particular, the q-derivative of a function f analytic in Γ is given by

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}, \tag{1.8}$$

where$[k]_q = \frac{1-q^k}{1-q}$, is the q-integer. and $[k]_q - > k$ contributors such as [2], [37], [42] and many others have applied equation (1.8) from different perspective, see details in [44].

Motivated by the study achieved in [40] ,the authors in this present study aimed to improve results on Hankel determinants and their inverses of order two associated with the q-difference operator for some subclasses of analytical functions in order to achieve robust knowledge for an age that is purely driven by critical intelligence.

To achieve this goal, the following lemma and definitions are highly important.

Lemma 1.1 [36] Let $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ be a Schwarz function, i.e., $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Gamma$. Then for any complex number ν, we have:

$$|c_2 - \nu c_1^2| \leq \max\{1, |\nu|\}.$$

Lemma 1.2 [34] Let $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$ be an analytic function in the open unit disk Γ, with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in \Gamma$ (i.e., w is a Schwarz function). Then the following coefficient inequalities hold:

$$|c_1| \leq 1, \quad |c_2| \leq 1 - |c_1|^2, \quad |c_3| \leq 1 - |c_1|^2 - \frac{|c_2|^2}{1 + |c_1|}.$$

Now, for the purpose of the present work, the definition of the following subclasses shall be necessary.

Definition 1.3 Let $0 \leq \beta < 1$. A function $f \in \mathcal{A}$ is said to belong to the class of q-starlike function of order β $\mathcal{CL}(\beta)$ if

$$CL(\beta) = \left\{ \frac{zD_q f(z)}{f(z)} \prec \beta + (1-\beta)\sqrt{1+z}, \quad z \in \mathbb{U} \right\}.$$

Definition 1.4 Let $0 < \beta < 1$. A function $f \in \mathcal{A}$ is said to belong to the class of q- convex function of order β $\mathcal{G}L(\beta)$ if

$$GL(\beta) = \left\{ \frac{(zD_q f(z))'}{D_q f(z)} \prec \beta + (1-\beta)\sqrt{1+z}, \quad z \in \mathbb{U} \right\}.$$

The generalized Hankel determinant defined by q-derivative would be established in the next section.

2. Results

In this section, the results of second and third Hankel determinants that involved the q- difference operator for the classes $\mathcal{C}L(q)$ and $\mathcal{C}L(q)$ would be examined. Also, the results for Hankel determinant of the inverse functions that involved q-difference operator would be established.

Theorem 2.1: Let $f \in CL(\beta)$, and suppose f is of the form (1.1), then

$$|H_{2,1}(f)| \leq \frac{1}{2([3]_q - 1)}(1-\beta)$$

and

$$|H_{2,2}(f)| \leq \frac{1}{4([3]_q - 1)^2}(1-\beta)^2.$$

These results are sharp.

Proof: If $f \in CL(\beta)$, then by definition:

$$\frac{zD_q f(z)}{f(z)} \prec \beta + (1-\beta)\sqrt{1+z}.$$

Using subordination principle, there exists a Schwarz function $w(z) = c_1 z + c_2 z^2 + c_3 z^3 + \cdots$, with $w(0) = 0$ and $|w(z)| < 1$, for all $z \in \Gamma$. Now, if equality holds, we have:

$$\frac{zD_q f(z)}{f(z)} = \beta + (1-\beta)\sqrt{1+w(z)}. \tag{2.1}$$

Expanding and comparing coefficients, we have:

$$a_2 = \frac{1}{2([2]_q - 1)}(1-\beta)c_1, \tag{2.2}$$

$$a_3 = \frac{1}{2([3]_q - 1)}(1-\beta)\left(c_2 - \frac{1}{4}\left(\frac{1-2(1-\beta)^2}{([2]_q-1)^2}\right)c_1^2\right), \tag{2.3}$$

$$a_4 = \frac{1}{2([4]_q - 1)}(1-\beta)(c_3 - \frac{1}{2}\left(1 - \frac{([2]_q + [3]_q - 2)(1-\beta)^2}{([2]_q - 1)([3]_q - 1)}\right)c_1c_2 + \frac{1}{8}\left(1 - \frac{([2]_q + [3]_q - 2)(1-\beta)^2(1-(1-\beta)^2)}{([2]_q - 1)^3([3]_q - 1)}\right)c_1^3). \tag{2.4}$$

The second-order Hankel determinant takes the form:

$$H_{2,1}(f) = a_3 - a_2^2.$$

Substituting for these expressions with (2.2) and (2.3) we have:

$$H_{2,1}(f) = \frac{1}{2([3]_q - 1)}(1-\beta)\left[c_2 - \frac{1}{4}\left(1 - \frac{2(1-\beta)([3]_q - 1)}{([2]_q - 1)^2}\right)c_1^2\right]. \tag{2.5}$$

Using Lemma 1,1, this gives:

$$|H_{2,1}(f)| \leq \frac{1}{2([3]_q - 1)}(1-\beta).$$

This bound is sharp for the choice $w(z) = z^2$, where $c_1 = 0, c_2 = 1$.

Next is to investigate the function of the form:

$$H_{2,2}(f) = a_2a_4 - a_3^2.$$

Substituting the expressions in (2.2), (2.3) and (2.4) this gives:

$$H_{2,2}(f) = \frac{1}{4XY^2Z}(1-\beta)^2 \quad [Y^2c_1c_3 - XZc_2^2 - \frac{1}{2}\left(Y^2 - \frac{Y(1-\beta)(X+Y)+2Z(1-\beta)^2}{X} - XZ\right)c_1^2c_2$$

$$-\frac{1}{16}\left(\frac{2Y(1-\beta)(X+Y)(1-(1-\beta)^2)+4Y^2(1-\beta)^2}{X^3} + XZ\left(1-\frac{2(1-\beta)^2}{X^2}\right)^2 - 2Y^2\right)c_1^4].$$

Taking absolute values and applying triangle inequality, this gives:

$$|H_{2,2}(f)| \leq$$

$$\frac{(1-\beta)^2}{4XY^2Z}\left[Y^2c_1||(1-|c_1|^2-\frac{|c_2|^2}{1+|c_1|})-\frac{1}{2}\left(Y^2 - \frac{Y(1-\beta)(X+Y)+2Z(1-\beta)^2}{X} - XZ\right)|c_1|^2(1-|c_1|^2)\right.$$

$$\left.-\frac{1}{16}\left(\frac{2Y(1-\beta)(X+Y)(1-(1-\beta)^2)+4Y^2(1-\beta)^2}{X^3} + XZ\left(1-\frac{2(1-\beta)^2}{X^2}\right)^2 - 2Y^2\right)|c_1|^4\right].$$

Using $|c_2| \leq 1-|c_1|^2$ with simple simplification, this gives:

$$|H_{2,2}(f)| \leq \frac{(1-\beta)^2}{4XY^2Z} \quad [-XZ - (\frac{2Z(1-\beta)^2 - Y(1-\beta)(X+Y)}{X} - 3XZ - Y^2)|c_1|^2 - Y^2|c_1|^3$$

$$-\frac{1}{16}\left(\frac{2Y(1-\beta)(X+Y)(1-(1-\beta)^2)+4Y^2(1-\beta)^2}{X^3} + XZ\left(1-\frac{2(1-\beta)^2}{X^2}\right)^2 - 2Y^2\right)|c_1|^4].$$

where,

$$X = ([2]_q - 1)$$

,

$$Y = ([3]_q - 1)$$

and

$$Z = ([4]_q - 1)$$

This expression attains its maximum when $|c_1| = 0$, thus:

$$|H_{2,2}(f)| \leq \frac{(1-\beta)^2}{4Y^2} = \frac{1}{4([3]_q - 1)^2}(1-\beta)^2.$$

Equality is attained for $w(z) = z^2$, i.e., $c_1 = 0, c_2 = 1$. This completes the proof.
Theorem 2.2: Let $f \in \mathcal{G}L(\beta)$, and be of the form (1.1), then

$$|H_{2,1}(f)| \leq \frac{1}{4[3]_q}(1-\beta),$$

and

$$|H_{2,2}(f)| \leq \frac{1}{16[3]_q}(1-\beta)^2.$$

These results are sharp.
Proof: If $f \in \mathcal{G}L(\beta)$, then by Definition 1,4, we have

$$\frac{(zD_qf(z))'}{D_qf(z)} \prec \beta + (1-\beta)\sqrt{1+z},$$

Applying subordination principle that ensures that there exists a Schwarz function $w(z) = c_1z + c_2z^2 + \ldots$ with $w(0) = 0$ and $|w(z)| < 1$, for $z \in \Gamma$. For the sake of equality, we have:

$$\frac{(zD_qf(z))'}{D_qf(z)} = \beta + (1-\beta)\sqrt{1+w(z)}. \tag{2.6}$$

where,

$$\frac{(zD_qf(z))'}{D_qf(z)} = 1+[2]_q a_2 z+(2[3]_q a_3-([2]_q)^2 a_2^2)z^2+(3[4]_q a_4-3[2]_q[3]_q a_2 a_3+[2]_q^3 a_2^3)z^3+\ldots$$

and

$$\beta+(1-\beta)\sqrt{1+w(z)} = 1+\frac{1}{2}(1-\beta)c_1z+\frac{1}{2}(1-\beta)\left(c_2-\frac{1}{4}c_1^2\right)z^2+\frac{1}{2}(1-\beta)\left(c_3-\frac{1}{2}c_1c_2+\frac{1}{8}c_1^3\right)z^3+\ldots.$$

Substituting (2.2) and (2.3) into (2.4) and also compare coefficients of z, z^2, z^3 with little simplification, this gives:

$$a_2 = \frac{1}{2[2]_q}(1-\beta)c_1, \tag{2.7}$$

$$a_3 = \frac{1}{4[3]_q}(1-\beta)\left[c_2+\frac{1}{4}(1-2\beta)c_1^2\right], \tag{2.8}$$

$$a_4 = \frac{1}{24[4]_q}(1-\beta)\left[4c_3-(2\beta-1)c_1c_2+\frac{1}{4}(1-2(1-\beta)^2)c_1^3\right]. \tag{2.9}$$

Using (2.7),(2.8) and (2.9) in the form given by : $H_{2,1}(f) = a_3 - a_2^2$ and simplifying, this gives

$$H_{2,1}(f) = \frac{1}{4[3]_q}(1-\beta)\left[c_2-\frac{1}{4}((2\beta-1)+\frac{4[3]_q(1-\beta)}{[2]_q^2})c_1^2\right]. \tag{2.10}$$

Taking the absolute and applying Lemma 1.1, this gives

$$|H_{2,1}(f)| \le \frac{1}{4[3]_q}(1-\beta)\max\left\{1, \left|\frac{1}{4}((2\beta-1)-\frac{4[3]_q(\beta-1)}{[2]_q^2}\right|\right\}.$$

Since $\left|\frac{1}{4}((2\beta-1)-\frac{4[3]_q(\beta-1)}{[2]_q^2}\right| \le 1$, we obtain

$$|H_{2,1}(f)| \le \frac{1}{4[3]_q}(1-\beta),$$

which is sharp for $w(z) = z^2$ in equation (2.3) as well as for $c_1 = 0$ and $c_2 = 1$ in equation (2.8).

Next, examine the case for $H_{2,2}(f) = a_2a_4 - a_3^2$. Using (2.2), (2.3) and (2.4) with simple simplification, this gives:

$$H_{2,2}(f) = \frac{(1-\beta)^2}{48[2]_q[3]_q^2[4]_q}\left[4[3]_q^2c_1c_3 - 3[2]_q[4]_qc_2^2 - (2\beta-1)([3]_q^2 - \frac{3[2]_q[4]_q}{2})c_1^2c_2 + J\right],$$

where $J = \frac{1}{16}(4[3]_q^2(1-2(1-\beta)^2) - 3[2]_q[4]_q(2\beta-1)^2)c_1^4$. Taking the absolute value and applying the triangle inequality, this gives:

$$|H_{2,2}(f)| \leq \frac{(1-\beta)^2}{48[2]_q[3]_q^2[4]_q} \quad [4[3]_q^2|c_1||c_3| - 3[2]_q[4]_q|c_2|^2 - (2-1)([3]_q^2 - \frac{3[2]_q[4]_q}{2})|c_1|^2|c_2|$$

$$+\frac{1}{16}(4[3]_q^2(1-2(1-\beta)^2) - 3[2]_q[4]_q(2\beta-1)^2)|c_1|^4].$$

Using Lemma 1.2, this gives:

$$|H_{2,2}(f)| \leq \frac{(1-\beta)^2}{48[2]_q[3]_q^2[4]_q}\left[4[3]_q^2|c_1|(1-|c_1|^2 - \frac{|c_2|^2}{1+|c_1|}) - 3[2]_q[4]_q|c_2|^2 - N\right.$$

$$\left.+\frac{1}{16}(4[3]_q^2(1-2(1-\beta)^2) - 3[2]_q[4]_q(2\beta-1)^2)|c_1|^4\right],$$

where $N = (2\beta-1)([3]_q^2 - \frac{3[2]_q[4]_q}{2})|c_1|^2(1-|c_1|^2)$. After simple computation of the right hand side (RHS), the observation is that the maximum of the right-hand side occurs at $|c_1| = 0$. Hence

$$|H_{2,2}(f)| \leq \frac{1}{16[3]_q}(1-\beta)^2.$$

This inequality is sharp for $w(z) = z^2$ in equation (2.3). This completes the proof.

The next Theorem shall examine the inverse functions of the Hankel determinant. **Theorem 2.3:** If $f \in \mathcal{G}L(\beta)$ and is of the form (1.1), then

$$H_{2,2}(f^{-1}) \leq \frac{1}{16[3]_q^2}(1-\beta)^2$$

The result is sharp.
Proof: If $f \in SL^*(\beta)$ then by using (2.8) - (2.10) in (2.7) for the expression in the form: $H_{2,2}(f^{-1}) = a_2a_4 - a_3^2 - a_2^2(a_3 - a_2^2)$.
Performing some simplifications, this gives :

$$H_{2,2}(f^{-1}) = \frac{(1-\beta)^2}{48[2]_q[3]_q^2[4]_q}(4[3]_q^2c_1c_3 - 3[2]_q[4]_qc_2^2 - \left[(2\beta-1)([3]_q^2 - \frac{3[2]_q[4]_q}{2}) - \frac{3[3]_q[4]_q}{[2]_q}\right]c_1^2c_2$$

$$+\frac{1}{16}(4[3]_q^2(1-2(1-\beta)^2) - 3[2]_q[4]_q(2\beta+1)^2 + \frac{12[[3]_q[4]_q}{[2]_q}\left((2\beta+1) + \frac{2[3]_q(1-\beta)}{[2]_q^2}\right)c_1^4). \tag{2.11}$$

Taking the absolute value and applying the triangle inequality, this gives:

$$|H_{2,2}(f^{-1})| \leq \frac{(1-\beta)^2}{48[2]_q[3]_q^2[4]_q}(4[3]_q^2|c_1||c_3| - 3[2]_q[4]_q|c_2|^2 - \left((2-1)([3]_q^2 - \frac{3[2]_q[4]_q}{2}) - \frac{3[3]_q[4]_q}{[2]_q}\right)|c_1|^2|c_2|$$

$$+\frac{1}{16}(4[3]_q^2(1-2(1-\beta)^2)-3[2]_q[4]_q(2\beta+1)^2+\frac{12[[3]_q[4]_q}{[2]_q}\left((2\beta+1)+\frac{2[3]_q(1-\beta)}{[2]_q^2}\right)|c_1|^4].$$

Now applying Lemma 1.2, and comparing coefficients with simple simplification, this gives:

$$|H_{2,2}(f^{-1})|$$

$$\leq \frac{(1-\beta)^2}{48[2]_q[3]_q^2[4]_q} \quad [-3[2]_q[4]_q-\left((2\beta-1)([3]_q^2-\frac{3[2]_q[4]_q}{2})-\frac{3[3]_q[4]_q}{[2]_q}-4[3]_q^2-6[2]_q[4]_q\right)|c_1|^2$$

$$-4[3]_q^2|c_1|^3-\frac{1}{16}(3[2]_q[4]_q(2\beta+1)^2-4[3]_q^2(1-2(1-\beta)^2)+\frac{12[[3]_q[4]_q}{[2]_q}\left(\frac{2[3]_q(\beta-1)}{[2]_q^2}-(2\beta+1)\right)$$

$$+3[2]_q[4]_q+(2\beta-1)(\frac{3[2]_q[4]_q}{2}-[3]_q^2)+\frac{3[3]_q[4]_q}{[2]_q})|c_1|^4]. \qquad (2.12)$$

where,

$$\frac{1}{16}(3[2]_q[4]_q(2\beta+1)^2-4[3]_q^2(1-2(1-\beta)^2)+\frac{12[[3]_q[4]_q}{[2]_q}\left(\frac{2[3]_q(\beta-1)}{[2]_q^2}-(2\beta+1)\right)$$

$$+3[2]_q[4]_q+(2\beta-1)(\frac{3[2]_q[4]_q}{2}-[3]_q^2)+\frac{3[3]_q[4]_q}{[2]_q})>0$$

,

$$for\ 0\leq\beta<1.$$

This expression attains its maximum when $|c_1|=0$, thus:

$$|H_{2,2}(f^{-1})|\leq\frac{(1-\beta)^2}{\cdot}48[2]_q[3]_q^2[4]_q \quad |-3[2]_q[4]_q|$$

Lastly, this becomes

$$|H_{2,2}(f^{-1})|\leq\frac{1}{16[3]_q^2}\quad(1-\beta)^2$$

. The result is sharp for $w(z)=z^2$ in equation (2.7) as well as for $c_2=1$ and all other $c_i=0$ in equation (2.11).

Theorem 2.4: Let $f\in CL(\beta)$, and be of the form (1.1), then

$$H_{2,2}(f^{-1})<\frac{1}{4([3]_q-1)^2}(1-\beta)^2.$$

These results are sharp.

proof: Assume that (1.7) holds, following the proof of Theorem 2.3 then desired result suffices.

3. Conclusion

In the present study, the authors have successfully investigated and studied the sharp initial coefficient bounds for a new subclass of analytic functions involving q-derivative operator and these bounds were used to determined the second determinant and its inverses associated with q-difference operator.

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Olatunji Sunday Oluwafemi
Department of Mathematics, University of Lagos, Akoka, Lagos State, Nigeria.
E-mail address: `olatunjiso@futa.edu.ng`

Fagbemiro Olalekan *
Department of Mathematics, Federal University of Agriculture, Abeokuta, Abeokuta, Ogun State, Nigeria
E-mail address: `fagbemiroo@funaab.edu.ng`