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BOUNDARY BEHAVIOR OF UNIVALENT HARMONIC MAPPINGS ONTO BOUNDED CONVEX DOMAINS

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Abstract. *Many authors have examined various boundary behaviors of univalent harmonic mappings in the open unit disk. Building on the work of Laugesen, Bshouty and others, this paper extends earlier results on the boundary behavior of univalent harmonic mappings under different conditions. We determine the angular limits of the arguments and logarithms of the analytic and co-analytic parts of univalent harmonic mappings in terms of the derivative of the boundary function and the dilatation. Explicit formulas are obtained when this derivative is finite. We also show that the dilatation possesses a finite number of zeros within any Stolz angle provided the derivative of the boundary function tends to infinity. For mappings onto bounded convex domains, the complex derivative has no interior zeros in any Stolz angle. These results explore and complement earlier work and clarify the geometric role of the dilatation near the boundary.*

1. Introduction

Throughout this paper, GRM, $\mathbb{C}$, Ω, $\partial\Omega$, Δ, $\mathbb{T}$, ω and Φ denote the General Riemann Mapping Theorem, the complex plane, a region in the complex plane, the boundary of a region, the open unit disk, the unit circle, the dilatation and the boundary function respectively.

Let $f^*(e^{i\theta})$ be a Lebesgue integrable function on $\mathbb{T}$. Then the Poisson integral

$$f(z) = P[f^*] = \frac{1}{2\pi}\int_{-\pi}^{\pi} P(r, \varphi - \theta)\Phi(e^{i\varphi})d\varphi, z = re^{i\theta} \in \Delta,$$

where, $P(r, t)$ is the Poisson kernel of Δ, is a harmonic mapping of Δ whose unrestricted limit at every continuity point $e^{i\theta_0}$ of f^* is $f^*(e^{i\theta_0})$ [3].
The study of harmonic mappings is essentially an extension of analytic functions.

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These mappings have many applications in various areas of physics and other domains where Laplace's equation is prominent. To this effect, research on harmonic mappings is motivated both by their mathematical significance and their practical relevance to physical problems [7].
The investigation of univalent harmonic mappings gained considerable attention among complex analysts following the prestigious paper by Clunie and Sheil-Small [11] in 1984. In 1986, Hengartner and Schober [6] attempted to formulate an appropriate version of the Riemann Mapping Theorem for harmonic mappings, drawing on the theory of quasiconformal mappings. The investigation of these scholars together with their colleagues led to the emergence of different open problems, conjectures and unresolved questions. While some of these conjectures have been solved, many challenging questions demand further research.

A recent area of attention is the boundary values of univalent complex-valued harmonic mappings. Laugesen investigated conditions on the boundary behavior of f when its dilatation is an infinite Blaschke product. If f has an infinite piecewise constant boundary value (and maps onto a convex infinite-sided polygon), then ω is an infinite Blaschke product [7].

Bshouty et al in 2012, [3] studied the angular limits of $\arg h'(z)$ and $\arg g'(z)$ provided that $\frac{d\Phi}{d\theta}(e^{i\theta}) = 0$, Where, $f(z) = h(z) + \overline{g(z)}$ is the General Riemann Mapping from Δ onto a bounded convex domain. In addition, they proved that $\sqrt{\omega(e^{i\theta_0})}df^*(e^{i\theta_0}) \in \mathbb{R} \setminus \{0\}$ if and only if f^* is continuous at $e^{i\theta_0}$. Motivated by the work of these scholars, several natural problems arise concerning the boundary behavior of univalent harmonic mappings and the influence of the boundary function Φ on the analytic and co-analytic parts of f.

Thus, the objective of this paper is to address the following problems and others, which extend and complement those studied in [3] and the problem posed by Lyzzaik et al., [3, Problem 3.19]:

Problem 1: Determine the angular limits of $\log h'(z)$ and $\log \overline{g'(z)}$ for a univalent harmonic mapping $f(z) = h(z) + \overline{g(z)}$ from Δ onto a convex domain provided that $\frac{d\Phi}{d\theta}(e^{i\theta}) \neq 0$.

Problem 2: Let f be a GRM from the open unit disk onto itself and $\Phi(e^{i\theta})$ denotes the radial boundary values of f. Must the dilatation ω have only finitely many zeros in each Stolz angle S_{θ_0} with vertex at $e^{i\theta_0}$ if $\frac{d\Phi}{d\theta}(e^{i\theta_0})$ is $+\infty$?

Problem 3: Given a univalent harmonic mapping f with dilatation ω that is analytic across an interval $J \subset \mathbb{T}$ and satisfies $|\omega| = 1$ on J. Determine conditions under which the boundary function f^* is continuous or has infinitely many jumps.

Problem 4: For a univalent harmonic mapping f from Δ onto a bounded convex domain, determine how the argument of the dilatation ω controls the asymptotic direction of the radial derivative near the boundary and the tangent and normal vectors of the image curve.

Problem 2 is partially solved in the sense that the dilatation ω possesses only a finite set of zeros within any Stolz angle S_{θ_0} at $e^{i\theta_0}$ if $|\frac{d\Phi}{d\theta}(e^{i\theta_0})| \leq$c, where c is a specific constant.

The structure of this paper is as follows: In section 2, we review some foundational results that will be important for proving the key findings. In section 3, we state and prove the primary results of the paper. Theorem 3.1 formulates constraints on the boundary values of a complex-valued harmonic function f to determine the angular limits of the logarithms of analytic functions. Theorems 3.2 and 3.3 provide explicit formulas for the logarithm and the argument of analytic functions under the assumption that the boundary derivative of f is finite and nonzero. Theorem 3.4 shows that the dilatation ω contains only a finite set of zeros within any Stolz angle at $e^{i\theta_0}$ if $\frac{d\Phi}{d\theta}(e^{i\theta_0})$ is $+\infty$. Furthermore, we prove that there are no interior zeros of the complex derivative of $f(z)$ in a Stolz angle at $e^{i\theta_0}$ if the GRM maps Δ onto a bounded convex domain.

2. Preliminaries

In this section, we present key ideas and findings that will be essential for proving the main results. We start by presenting some classic results, along with essential definitions and theorems.

Definition 2.1. [5] Let $\Omega \subset \mathbb{C}$ be a simply connected domain. A complex-valued function f is called harmonic in Ω if it admits a decomposition

$$f(z) + \overline{g(z)},$$

where $h(z)$ and $g(z)$ are analytic in Ω.

Definition 2.2. [10] A complex-valued harmonic function $f(z) = h(z) + \overline{g(z)}$ is said to be orientation preserving at z_0 if $J_f(z_0) > 0$ and is orientation reversing at z_0 if $J_f(z_0) < 0$, where $J_f(z_0)$ is the Jacobian of f which is defined by

$$J_f(z_0) = |f_z(z_0)|^2 - |f_{\overline{z}}(z_0)|^2 = |h'(z_0)|^2 - |g'(z_0)|^2,$$

Its dilatation ω is given by $\omega(z) = \frac{g'(z)}{h'(z)}$. By Lewy's theorem, a harmonic mapping f in Δ is locally univalent if and only if $J_f(z) \neq 0$ for all $z \in \Delta$. Moreover, f is locally univalent and sense preserving in Δ if and only if $|\omega(z)| < 1$[8].

Definition 2.3. [9] A Stolz angle at a boundary point $e^{i\theta_0} \in \mathbb{T}$, is a non-tangential approach region in the open unit disk Δ. For $0 < \alpha < \infty$, it is defined by

$$S_\alpha(e^{i\theta_0}) = \left\{z \in \Delta : |z - e^{i\theta_0}| < \alpha(1 - |z|)\right\}.$$

Theorem 2.4. *(Hengartner and Schober* [6]*) Suppose Ω be a bounded , connected open set having a boundary that is locally connected. Assume that $\omega(\Delta) \subset \Delta$ and ω_0 is an invariant point of Ω. Then for $\bar{f}_{\overline{z}} = \omega(z) f_z$ there exists a solution*

satisfying:
(a) $f(0) = \omega_0$, $f_z(0) > 0$, *and* $f(\Delta) \subset \Omega$.
(b) There exists a countable set $A \subset \mathbb{T}$ *for which the unrestricted limits* $\Phi(e^{it}) = \lim_{z\to e^{it}} f(z)$ *exist on* $\mathbb{T}\backslash A$ *and belong to* $\partial\Omega$.
(c) $\Phi(e^{it-}) = esslim_{s\uparrow t}\Phi(e^{is})$ *and* $\Phi(e^{it+}) = esslim_{s\downarrow t}\Phi(e^{is})$ *exist on* $\mathbb{T}$, *belong to* $\partial\Omega$ *and are equal on* $\mathbb{T}\backslash A$.
(d) The cluster set of f *at* $e^{it} \in A$ *is the direct line segment connecting* $\Phi(e^{it-})$ *to* $\Phi(e^{it+})$.

Note: The mapping f is referred to as a GRM from Δ onto Ω.

Theorem 2.5. [3] *Let* Φ *be a real-valued function on* $\mathbb{T}$ *that is Lebesgue integrable. Then*
(i) If $(\frac{d\Phi}{d\theta})(e^{i\theta_0})$ *exists and is finite, then the angular limit*

$$\lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial\theta}(z) = \frac{d\Phi}{d\theta}(e^{i\theta_0})$$

exhibits uniform behavior in any Stolz angle with vertex at $e^{i\theta_0}$.
(ii) If $(\frac{d\Phi}{d\theta})(e^{i\theta_0}) = +\infty$, *then*

$$\lim_{r\to 1^-} \frac{\partial f}{\partial\theta}(re^{i\theta}) = +\infty.$$

Moreover, if Φ *is monotone increasing in a neighborhood of* θ_0, *then*

$$\lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial\theta}(z) = +\infty$$

exhibits uniform behavior in any Stolz sector with vertex located at $e^{i\theta_0}$.

Theorem 2.6. [3] *Assume* f *denotes an injective harmonic mapping of* Δ *onto a bounded convex domain* Ω *containing the origin and let* $f(0) = 0$. *Then*

$$|f_z|^2 + |f_{\bar{z}}|^2 \geq \frac{dist(0,\partial\Omega)^2}{16}.$$

3. Main Results

Theorem 3.1. *Suppose* $f(z) = h(z) + \overline{g(z)}$ *denotes the GRM from* Δ *onto a bounded convex set and* $(\frac{d\Phi}{d\theta})(e^{i\theta_0}) = 0$ *for some* $\theta_0 \in \mathbb{R}$. *If* $\lim_{z\to e^{i\theta_0}} arg\ \omega(z) = \alpha$, *then the following angular limits hold:*

(a) $\lim_{z\to e^{i\theta_0}} \log h'(z) = \left(-i(\theta_0 + \frac{\alpha}{2}) + \frac{1}{2}\log|\omega|\right) \pmod{\pi}$ *and*

(b) $\lim_{z\to e^{i\theta_0}} \log g'(z) = \left(i(\frac{\alpha}{2} - \theta_0) + \frac{3}{2}\log|\omega|\right) \pmod{\pi}$.

Proof. (a) By Theorem 2.5(i), since $(\frac{d\Phi}{d\theta})(e^{i\theta_0})$ exists and is finite,

$$\lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial\theta}(z) = \frac{d\Phi}{d\theta}(e^{i\theta_0})$$

exhibits uniform behavior in each Stolz angle with vertex at $e^{i\theta_0}$.
Using the Poisson integral formula, we get

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\Phi(e^{i\theta})\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}-z}\Phi(e^{i\theta})\,d\theta + \overline{\frac{1}{2\pi}\int_0^{2\pi} \frac{z}{e^{i\theta}-z}\overline{\Phi(e^{i\theta})\,d\theta}}.$$

Hence, f admits the canonical decomposition $f(z) = h(z) + \overline{g(z)}$.
Differentiating with respect to θ under the integral sign (which is justifiable), we obtain

$$f_\theta(z) = h_\theta(z) + \overline{g_\theta(z)} = izh'(z) + \overline{izg'(z)}.$$

It follows that the angular derivatives of the boundary function Φ is

$$\frac{d\Phi}{d\theta}(e^{i\theta_0}) = \lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial\theta}(z) = \lim_{z\to e^{i\theta_0}} i(zh'(z)) - \overline{zg'(z)}.$$

By Theorem 2.6, the analytic function $h'(z)$ does not vanish in Δ. Consequently, $-i\lim\limits_{z\to e^{i\theta_0}} \frac{\partial f}{\partial\theta}(z) = \lim\limits_{z\to e^{\theta_0}} zh'(z)(1 - \frac{\overline{zh'(z)}}{zh'(z)}\overline{\omega(z)}) = 0$.

This implies $\lim\limits_{z\to e^{i\theta_0}} (1 - \frac{\overline{zh'(z)}}{zh'(z)}\overline{\omega(z)}) = 0$.
Therefore, we get

$$\lim_{z\to e^{i\theta_0}} \log(1) = \lim_{z\to e^{i\theta_0}} \log(\overline{zh'(z)\omega(z)}) - \lim_{z\to e^{i\theta_0}} \log zh'(z).$$

Consequently, after rearranging terms we deduce that

$$\lim_{z\to e^{i\theta_0}} \log h'(z) = \left(-i(\theta_0 + \frac{\alpha}{2}) + \frac{\log|\omega|}{2}\right) \pmod{\pi}.$$

(b) Since $\lim\limits_{z\to z^{i\theta_0}} \arg\omega(z) = \alpha$, we have
$\lim\limits_{z\to e^{i\theta_0}} \log\omega(z) = \log|\omega(z)| + \mathrm{i}\alpha \pmod{\pi}$. Using $g'(z) = h'(z)\omega(z)$, it follows that:

$$\lim_{z\to e^{i\theta_0}} \log g'(z) = \lim_{z\to e^{i\theta_0}} \log h'(z) + \lim_{z\to e^{i\theta_0}} \log\omega(z).$$

By part (a), $\lim\limits_{z\to e^{i\theta_0}} \log\, h'(z) = \left(-i(\theta_0 + \frac{\alpha}{2}) + \frac{log|\omega|}{2}\right) \pmod{\pi}$.
Then, $\lim\limits_{z\to e^{i\theta_0}} \log\, g'(z) = \left(i(\frac{\alpha}{2} - \theta_0) + \frac{3log|\omega|}{2}\right) \pmod{\pi}$.

□

Theorem 3.2. *Assume $f(z) = h(z) + \overline{g(z)}$ denotes the GRM from the open unit disk onto a bounded convex set and $(\frac{d\Phi}{d\theta})(e^{i\theta_0}) = \gamma \neq 0, \infty$ for some $\theta_0, \gamma \in \mathbb{R}$. Then $\lim\limits_{z\to e^{i\theta_0}} \arg\omega(z) = 2\pi \pmod{\pi}$.*

Proof. Since $(\frac{d\Phi}{d\theta})(e^{i\theta_0}) = \gamma \neq 0, \infty$ exists and is finite, Theorem 2.5(i) implies that the angular limit

$$\lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial\theta}(z) = \frac{d\Phi}{d\theta}(e^{i\theta_0})$$

also exists.
Recall that

$$\frac{\partial f}{\partial \theta}(z) = izh'(z) + \overline{izg'(z)}.$$

This can be written as: $i\frac{d\Phi}{d\theta}(e^{i\theta_0}) = i\lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial \theta}(z) = -\lim_{z\to e^{i\theta_0}} zh'(z)(1 - \frac{\overline{zh'(z)}}{zh'(z)}\overline{\omega(z)}) = i\gamma \neq 0$.

Consequently, $h'(z) \neq 0$ in the neighborhood of $e^{i\theta_0}$.
Taking the arguments and passing to the limit gives:

$$\arg(-i\gamma) = \lim_{z\to e^{i\theta_0}} \left(\arg z + \arg h'(z) + \arg[1 - e^{-i(2\arg z + 2\arg h'(z) + \arg \omega(z))}]\right) \pmod{\pi}.$$

Using the identity $\arg\left(1 - e^{-i\theta}\right) = \frac{\pi-\theta}{2}$, we obtain
$\theta_0 + \lim_{z\to e^{i\theta_0}} \arg h'(z) + \frac{1}{2}(-2\theta_0 - 2\arg h'(z) - \arg \omega(z) + \pi) = \frac{-\pi}{2} \pmod{\pi}$.
Simplifying this equation leads to: $\lim_{z\to e^{i\theta_0}} \arg \omega(z) = 2\pi \pmod{\pi}$ which is as desired.

□

Theorem 3.3. *Suppose $f(z) = h(z) + \overline{g(z)}$ denotes the GRM from the open unit disk onto a bounded convex set and $(\frac{d\Phi}{d\theta})(e^{i\theta_0}) = \gamma \neq 0, \infty$ for some $\theta_0 \in \mathbb{R}$. If $\lim_{z\to e^{i\theta_0}} \arg \omega(z) = \alpha$, then*
Case I: when $\gamma \in \mathbb{R}$:

(a) $\lim_{z\to e^{i\theta_0}} \log h'(z) = \left(i(\beta + \frac{\alpha}{2} - \pi) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma}\right|\right) \pmod{\pi}$ *and*

(b) $\lim_{z\to e^{i\theta_0}} \log g'(z) = \left(i(\beta + \frac{3\alpha}{2} - \pi) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma\omega(z)}\right|\right) \pmod{\pi}$.

Case II: when $\gamma \in i\mathbb{R}$:

(c) $\lim_{z\to e^{i\theta_0}} \log h'(z) = \left(i(\beta + \frac{\alpha}{2} - 3\pi) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma}\right|\right) \pmod{\pi}$ *and*

(d) $\lim_{z\to e^{i\theta_0}} \log g'(z) = \left(i(\beta + \frac{3\alpha}{2} - 3\pi) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma\omega(z)}\right|\right) \pmod{\pi}$.

Proof. (a) Since $(\frac{d\Phi}{d\theta})(e^{i\theta_0}) = \gamma \neq 0, \infty$ exists and is finite, from Theorem 2.5(i) the following holds true.

$$\lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial \theta}(z) = \frac{d\Phi}{d\theta}(e^{i\theta_0}).$$

Recall that:

$$\frac{\partial f}{\partial \theta}(z) = izh'(z) + \overline{izg'(z)}.$$

It follows that:

$$-i\frac{d\Phi}{d\theta}(e^{i\theta_0}) = -i\lim_{z\to e^{i\theta_0}}\frac{\partial f}{\partial\theta}(z) = \lim_{z\to e^{i\theta_0}} zh'(z)\left(1-\frac{\overline{zh'(z)}}{zh'(z)}\overline{\omega(z)}\right) = -i\gamma \neq 0.$$

Consequently, $h'(z)\neq 0$ in the neighborhood of $e^{i\theta_0}$. Therefore, taking logarithms, we obtain

$$\log\,(-i\gamma) = \lim_{z\to e^{i\theta_0}}\left(\log z + \log h'(z) + \log[1-e^{-i(2\arg z+2\arg h'(z)+\arg\omega(z))}]\right).$$

It is known that $\log\,(1-e^{-i\theta}) = \log|1-e^{-i\theta}| + i\arg(1-e^{-i\theta}) = \log 2|sin\frac{\theta}{2}| + \frac{i(\pi-\theta)}{2}$. Since Ω convex and f is GRM, the angular limit

$$\lim_{z\to e^{i\theta_0}}\arg h'(z) = \beta$$

exists and is finite.
It follows that:

$$i\theta_0 + \lim_{z\to e^{i\theta_0}}\log h'(z) + \log 2|sin(\theta_0+\frac{\alpha}{2}+\beta)| + \frac{i\pi}{2} - i(\theta_0+\beta+\frac{\alpha}{2}) = \left(log|\gamma| - \frac{i\pi}{2}\right) \pmod{\pi}. \tag{3.1}$$

After simplification this gives:

$$\lim_{z\to e^{i\theta_0}}\log h'(z) = \left(i(\beta+\frac{\alpha}{2}-\pi) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma}\right|\right) \pmod{\pi}.$$

Hence proved.

(b) We know that the angular limit:

$$\lim_{z\to e^{i\theta_0}}\log g'(z) = \lim_{z\to e^{i\theta_0}}\log h'(z) + \lim_{z\to e^{i\theta_0}}\log\omega(z) = \lim_{z\to e^{i\theta_0}}\log h'(z) + \log|\omega(z)| + i\alpha.$$

Hence, using the result from (a) the result follows.
If $\gamma<0$, then we obtain:
$\lim_{z\to e^{i\theta_0}}\log\, h'(z) = \left(i(\beta+\frac{\alpha}{2}) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma}\right|\right) \pmod{\pi}$ and

$$\lim_{z\to e^{i\theta_0}}\log\, g'(z) = \left(i(\beta+\frac{3\alpha}{2}) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma\omega(z)}\right|\right) \pmod{\pi}.$$

(c) From equation (3.1) we have:
$\log\,(-i\gamma) = \lim_{z\to e^{i\theta_0}}\left(\log z + \log h'(z) + \log[1-e^{-i(2argz+2argh'(z)+arg\omega(z))}]\right).$
But since γ is purely imaginary, it follows that
$i\theta_0 + \lim_{z\to e^{i\theta_0}}\log h'(z) + \log 2|sin(\theta_0+\frac{\alpha}{2}+\beta)| + \frac{i\pi}{2} - i(\theta_0+\beta+\frac{\alpha}{2}) = (\log|\gamma| - i\pi) \pmod{\pi}$.
Hence, the result follows.
(d) Recall that the angular limit:

$$\lim_{z\to e^{i\theta_0}}\log g'(z) = \lim_{z\to e^{i\theta_0}}\log h'(z) + \lim_{z\to e^{i\theta_0}}\log\omega(z) = \lim_{z\to e^{i\theta_0}}\log h'(z) + \log|\omega(z)| + i\alpha.$$

Hence from (c)the result follows.
If $\gamma < 0$, then we have:
$\lim_{z\to e^{i\theta_0}} \log h'(z) = \left(i(\beta + \frac{\alpha}{2} + \pi) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma}\right|\right) \pmod{\pi}$ and

$$\lim_{z\to e^{i\theta_0}} \log g'(z) = \left(i(\beta + \frac{3\alpha}{2} + \pi) - \log 2\left|\frac{sin(\theta_0+\beta+\frac{\alpha}{2})}{\gamma\omega(z)}\right|\right) \pmod{\pi}.$$

□

Theorem 3.4. *Assume $f(z) = h(z) + \overline{g(z)}$ denotes the GRM from Δ onto Δ. If $\frac{d\Phi}{d\theta}(e^{i\theta_0})$ is $+\infty$ and Φ is monotone increasing in a neighborhood of θ_0. Then $\omega(z)$ contains a finite number of zeros in any given Stolz angle S_{θ_0} at $e^{i\theta_0}$.*

Proof. It is known from Theorem 2.5 (ii) that if $(\frac{d\Phi}{d\theta})(e^{i\theta_0}) = +\infty$ and Φ is monotone increasing in a neighborhood of θ_0, then

$$\lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial\theta}(re^{i\theta}) = +\infty$$

along any nontangential path. Since

$$f(z) = h(z) + \overline{g(z)}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1-|z|^2}{|e^{i\theta}-z|^2}\Phi(e^{i\theta})d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}}{e^{i\theta}-z}\Phi(e^{i\theta})d\theta + \overline{\frac{1}{2\pi}\int_0^{2\pi} \frac{z}{e^{i\theta}-z}\overline{\Phi(e^{i\theta})d\theta}}.$$

Differentiating with respect to θ, we get

$$\frac{d\Phi}{d\theta}(e^{i\theta_0}) = +\infty = \lim_{z\to e^{i\theta_0}} \frac{\partial f}{\partial\theta}(z)$$
$$= i\lim_{z\to e^{i\theta_0}} \left(zh'(z) - \overline{zg'(z)}\right).$$

Dividing both sides by $\bar{z} = e^{-i\theta}$, we get:

$$\lim_{z\to e^{i\theta_0}} |e^{2i\theta}h'(z) - \overline{g'(z)}| = +\infty.$$

This step implies:

$$\overline{\lim_{z\to e^{i\theta_0}}}(|h'(z)| - |g'(z)|) \le \infty.$$

so that

$$-\infty + \underline{lim}_{z\to e^{i\theta_0}}|h'(z)| \le \underline{lim}_{z\to e^{i\theta_0}}|g'(z)|.$$

Thus, for any finite value of $|h'(z)|$, $\lim_{z\to e^{i\theta_0}} |g'(z)| > 0$.

Consequently, we can observe that the accumulation points of $g'(z)$ at $e^{i\theta_0}$ do not contain zero, which implies at once that $g'(z)$ contains only a finite set of zeros within any Stolz angle S_{θ_0} at $e^{i\theta_0}$. Since $\omega(z) = \frac{g'(z)}{h'(z)}$, a finite number of these zeros are zeros of $\omega(z)$. Moreover, by Theorem 2.6 $h'(z)$ is different from zero. Consequently, $\omega(z)$ exhibits a finite number of zeros in any Stolz sector S_{θ_0} at $e^{i\theta_0}$.

□

Theorem 3.5. *Let $f(z) = h(z) + \overline{g(z)}$ be a GRM from Δ onto a bounded convex domain Ω whose dilatation $\omega(z)$ admits an analytic extension across an open interval:*

$$J = \left\{e^{it} : \alpha < t < \beta\right\}, \alpha < \beta < \alpha + 2\pi,$$

such that $|\omega(z)|$=1 on J. Then the following hold:

(a) The boundary function f^* is continuous at $e^{i\theta_0} \in J$ if and only if $\arg\left\{\sqrt{\omega(e^{i\theta_0})}df^*(e^{i\theta_0})\right\}$ $= \frac{\pi}{2} \pmod{\pi}$.

(b) $\Delta_J \arg\sqrt{\omega(z)} = -\infty$ if and only if f^* has infinitely many jumps on J.

Proof. (a) We show that $\sqrt{\omega(e^{i\theta_0})}df^*(e^{i\theta_0}) \in i\mathbb{R}$.
Choose an interval I $= [\theta_0 - \delta, \theta_0 + \delta] \subset J$ on which a continuous branch of $\sqrt{\omega(z)}$ is defined. Since $h(z)$ and $g(z)$ are analytic in a neighborhood of J, their derivatives $h'(z)$ and $g'(z)$ are continuous on I. By continuity on the compact set I both $h'(z)$ and $g'(z)$ are bounded there. Thus,

$$|df^*(e^{i\theta_0})| = |e^{i\theta_0}ih'(e^{i\theta_0}) - ie^{-i\theta_0}\overline{g'(e^{i\theta_0})}|$$

$$\leq |h'(e^{i\theta_0})| + |g'(e^{i\theta_0})| < \infty.$$

Therefore, $df^*(e^{i\theta_0})$ is finite and continuous.

Since $g'(z) = \omega(z)h'(z)$, we obtain
$df^*(e^{i\theta_0}) = i\left(e^{i\theta_0}h'(e^{i\theta_0}) - e^{-i\theta_0}\overline{\omega(e^{i\theta_0})h'(e^{i\theta_0})}\right)$.

Multiplying both sides by $\sqrt{\omega(e^{\theta_0})}$, we get

$$df^*(e^{i\theta_0}) = \frac{i\left(\sqrt{\omega(e^{i\theta_0})}e^{i\theta_0}h'(e^{i\theta_0}) - \overline{e^{i\theta_0}\sqrt{\omega(e^{i\theta_0})}h'(e^{i\theta_0})}\right)}{\sqrt{\omega(e^{i\theta_0})}}.$$

Let $A = e^{i\theta_0}h'(e^{i\theta_0})\sqrt{\omega(e^{i\theta_0})}$, and $A - \overline{A} = 2\mathrm{i}\Im A$.
$df^*(e^{i\theta_0}) = -2\overline{\sqrt{\omega(e^{i\theta_0})}}\Im A$, Since $\overline{\sqrt{\omega(e^{i\theta_0})}}\sqrt{\omega(e^{i\theta_0})} = 1$.
This implies $df^*(e^{i\theta_0})$ is perpendicular to $i\overline{\sqrt{\omega(e^{i\theta_0})}}$. Geometrically, this means that the tangent vector $df^*(e^{i\theta_0})$ is orthogonal to the normal direction determined by $\overline{\sqrt{\omega(e^{i\theta_0})}}$.
Equivalently, $\Re\left(df^*(e^{i\theta_0})\sqrt{\omega(e^{i\theta_0})}\right) = 0$.

Therefore, $\sqrt{\omega(e^{i\theta_0})}df^*(e^{i\theta_0}) \in i\mathbb{R}$ as desired.
To prove the converse, suppose that arg $\left(\sqrt{\omega(e^{i\theta_0})}df^*(e^{i\theta_0})\right) = \frac{\pi}{2} \pmod{\pi}$.
i.e., $u(\theta_0) = \sqrt{\omega(e^{i\theta_0})}df^*(e^{i\theta_0}) = ic$.
Since $|\omega| = 1$, we have $|df^*(e^{i\theta_0})| = |c| \neq 0$.
So the tangent is nonzero and has a fixed phase depends on $\omega(z)$ and the sign of c. Since $\omega(z)$, $h'(z)$ and $g'(z)$ are analytic across J, we may choose a continuous branch of $\sqrt{\omega(z)}$ in a small neighborhood of θ_0. Hence, $u(\theta)$ is continuous near θ_0. For any $\eta > 0$ there exist $\delta > 0$ such that $|u(\theta) - ic| < \eta$ whenever $|\theta - \theta_0| < \delta$.

$$f(e^{i\theta}) - f(e^{i\theta_0}) = \int_{\theta_0}^{\theta} df^*(e^{it})dt = \int_{\theta_0}^{\theta} \frac{u(t)}{\sqrt{\omega(e^{it})}}dt.$$

Let k = $\sup_{|t-\theta_0|\leq\delta} \left|\frac{1}{\sqrt{\omega(e^{it})}}\right| < \infty$. For $|\theta - \theta_0| < \delta$, we have

$$|f(e^{i\theta}) - f(e^{i\theta_0})| \leq k \int_{\theta_0}^{\theta} |u(t)|dt$$

$$\leq k \int_{\theta_0}^{\theta} (|ic| + |u(t) - ic|)dt$$

$$\leq k(|c| + \eta)|\theta - \theta_0|.$$

Letting $\theta \to \theta_0$, we conclude that f is continuous at $e^{i\theta_0}$. This completes the proof part of (a).
b. Assume that Δ_J arg $\sqrt{\omega(z)} = -\infty$.
Consider the continuous function $\Phi(z) =$ arg $(\sqrt{\omega(z)})$ + arg $(df^*(z))$ on J except at isolated zeros of $df^*(z)$. Since h'(z) and $\omega(z)$ extend analytically across J, $df^*(z)$ is continuous on J except possibly at isolated points where h'(z) or $df^*(z)$ may vanish. In particular, on any compact sub arc k of J away from those isolated zeros, $df^*(z)$ is continuous and arg $(df^*(z))$ is bounded (away from isolated zeros) by (a).
Zeros of an analytic function on J are isolated unless the function vanishes identically on J [1, Theorem 4.3]. In our GRM situation $df^*(z)$ cannot vanish identically on an open sub arc(otherwise it would be constant there), so zeros of $df^*(e^{i\theta})$ on J are isolated. By hypothesis arg $(\sqrt{\omega(z)})$ decreases without bound. Therefore, $\Phi(z)$ also decreases without bound as z runs across J. A continuous real function that decreases without bound must pass through the value $k\pi$ for infinitely many integers k. Concretely, for each integer k there is some $\theta_k \in$ J with

$$\Phi(\theta_k) = k\pi.$$

By [3, Lemma 4] each θ_k is a jump point of f. Since there are infinitely many integers k, there are infinitely many distinct jump points on J.

For the reverse implication, assume that f has infinitely many jump points $\{\theta_n\} \subset$ J . By [3, Lemma 4] each θ_n satisfies $\Phi(\theta_n) = \arg\ (\sqrt{\omega(e^{i\theta_n})}) + \arg\ (df^*(e^{i\theta_n})) = k_n\pi$ for each integer k_n. Thus, the set $\{k_n\}$ is infinite.
If the integers $\{k_n\}$ were bounded both above and below(i.e., there were only finitely many distinct integer values), then $\Phi(z)$ would take one of finitely many values. Therefore, the sequence $\{k_n\}$ are unbounded in at least one direction.
Note that arg $(df^*(z))$ is bounded on J away from isolated zeros of $df^*(e^{i\theta})$. If the sequence k_n were unbounded in the positive direction only, then

$$\arg(\sqrt{\omega(e^{i\theta_n})}) = k_n\pi - \arg(df^*(e^{i\theta_n}))$$

would be unbounded in the positive direction. Likewise, if $\{k_n\}$ is unbounded below, then $\arg(\sqrt{\omega(z)})$ is unbounded in the negative direction.

□

Theorem 3.6. *Let $f(z) = h(z) + \overline{g(z)}$ be a univalent harmonic mapping from Δ onto a bounded convex domain Ω. Suppose there exist a continuously differentiable angular function θ such that $z(r) = re^{i\theta(r)} \in \Delta$ and $\arg f(z) = \theta_0$ is constant. Then*

$$\theta(r) = \begin{cases} c + \sin^{-1}(\frac{c}{r}), & \textit{if } \arg f' \textit{ is constant} \\ \frac{-a(logr+1)}{r} + \frac{c}{r} - \theta_0, & \textit{otherwise,} \end{cases}$$

, where a and c are constants.

Proof. Consider the radial angular path $z(r) = re^{i\theta(r)}$ with $r \to 1^-$. Using the identity

$$\frac{d}{dr}\arg f(z(r)) = \Im\left(\frac{f'(z(r))z'(r)}{f(r)}\right).$$

Along the path, we have :

$$\frac{d}{dr}f(z(r)) = f'(z(r))\frac{dz}{dr} = f'(z)\left(e^{i\theta(r)} + ir\theta'(r)e^{i\theta(r)}\right).$$

But since $\arg f = \theta_0$ is constant, the imaginary part must vanish.
$\Im\left(Fe^{i\theta(r)}(i + ir\theta'(r))\right) = 0$, where $F(r) = \frac{f'(r)}{f(r)} = Re^{i\phi(r)}$.
Thus, the imaginary part becomes

$$R\Im\left(e^{i(\theta+\phi)}(1 + ir\theta'(r))\right).$$

We require the imaginary part to be zero. After computing and expanding, we obtain

$$\theta'(r) = -\frac{tan(\phi(r) + \theta(r))}{r}$$

Suppose $\arg F = \phi(r) = \theta_0 + \frac{a}{r}$ with $r \to 1^-$. Let $\Psi(r) = \phi(r) + \theta(r)$.
Consequently, we have

$$\Psi'(r) + \frac{tan\Psi}{r} = -\frac{a}{r^2}$$

For small value of Ψ, using Taylor series approximation $\tan\Psi \approx \Psi$. This reduces to

$$\frac{d}{dr}(r\Psi) = -\frac{a}{r}$$

. Integrating both sides yields the desired expression for $\theta(r)$. Thus, $\theta(r)$ describes a spiral path inside the unit disk whose angle depends logarithmically and rationally on r.
If argF is constant , then

$$\theta'(r) = \Psi'(r) = -\frac{tan\Psi}{r}$$

. This implies

$$\frac{d\Psi}{tan\Psi} = -\frac{dr}{r}.$$

After integrating and rearranging, the result follows immediately. □

Theorem 3.7. *Assume $f(z) = h(z) + \overline{g(z)}$ denotes the GRM from Δ onto a bounded convex domain Ω. If $\lim_{r\to 1^-} \arg\omega(re^{i\theta_0}) = \beta$, then*

a. $\lim_{r\to 1^-} \arg\left(\frac{\partial f}{\partial r}(re^{\theta_0})\right) = -\frac{\beta}{2} \pmod{\pi}$.

b. If $\lim_{z\to e^{i\theta_0}} |\omega(z)| = \lambda$, *where* $z \in S_{\theta_0}(\alpha)$ & $\lambda \in [0,1)$,
Then:

$$0 \leq \frac{m}{2} \leq |h'(z)| \leq \frac{M}{1-\lambda},$$

where $m = inf|f_z(z)| > 0$ *and* $M = sup|f_z(z)|$.
c. There are no zeros of the complex derivative of f inside a Stolz angle.

Proof. a. Given that $f(z) = h(z) + \overline{g(z)}$, where $z = re^{i\theta_0}$.

Let $h'(z) = R(z)e^{i\arg h'(z)}, \omega(z) = r(z)e^{i\arg\omega(z)}$.

The radial derivative of f is given by:

$$\frac{\partial f}{\partial r}(z) = e^{i\theta_0}h'(z) + \overline{e^{\theta_0}h'(z)\omega(z)}$$
$$= R(z)e^{i(\theta_0+\arg h'(z))}\left[1 + r(z)e^{-i(2\theta_0+2\arg h'(z)+\arg\omega(z))}\right].$$

Taking argument, we get
$\arg\left(\frac{\partial f}{\partial r}(re^{i\theta_0})\right) = \theta_0 + \arg h'(z) + \arg\left[1 + r(z)e^{-i(2\theta_0+2\arg h'(z)+\arg\omega(z))}\right]$.
For $\zeta \neq -\pi$, the identity $\arg\left[1 + e^{-i\zeta}\right] = -\frac{\zeta}{2} \pmod{\pi}$ holds, where

$$\zeta(z) = 2\theta_0 + 2\arg h'(z) + \arg\omega(z).$$

Since Ω is convex and f is the GRM, the radial limit

$$\lim_{r\to 1^-} \arg h'(z) = \alpha$$

exists and is finite. Hence,
$\lim_{r\to 1^-} \arg\left(\frac{\partial f}{\partial r}(re^{\theta_0})\right) = \theta_0 + \alpha - \left(\frac{2\theta_0+2\alpha+\beta}{2}\right) = -\frac{\beta}{2} \pmod{\pi}$.

This shows that the harmonic shear encoded entirely by $\omega(z)$, completely controls the direction of the tangent(or normal) vector to the image curve near $e^{i\theta_0}$. In particular, while h(z) governs the conformal stretching, it is the $\omega(z)$ that determines the asymptotic direction of the boundary image. This phenomenon highlights the dominant geometric role of the harmonic shear on the boundary behavior of univalent harmonic mapping onto convex domains.

Example 1. Consider the univalent harmonic function:

$$f(z) = h(z) + \overline{g(z)} = \Re\left[\frac{i}{2}\log\left(\frac{i+z}{i-z}\right)\right] + i\Im\left[\frac{1}{2}\log\left(\frac{i+z}{i-z}\right)\right]$$

Simplifying, we can write

$$f(z) = \frac{i-1}{2}\arg\left(\frac{i+z}{i-z}\right).$$

With the dilatation $\omega(z) = \frac{g'(z)}{h'(z)} = -z^2$ [2].

We observe that

$$\lim_{r\to 1^-} \arg \omega(re^{i\theta_0}) = -2\lim_{r\to 1^-} \arg(re^{i\theta_0}) \pmod{\pi}.$$

Then, it follows from part (a) of the proof that

$$\lim_{r\to 1^-} \arg\left(\frac{\partial f}{\partial r}(re^{\theta_0})\right) = \lim_{r\to 1^-} \arg(re^{i\theta_0}) \pmod{\pi}.$$

The function f maps Δ univalently and sense- preservingly, since $|\omega(z)| < 1$ [8]. The Mobius transform $\frac{i+z}{i-z}$ sends Δ to the upper half-plane, and the logarithm further it to a vertical strip.

The combination of real and imaginary parts rotates and scales this strip to produce a convex square region as shown, in Figure 1 below.

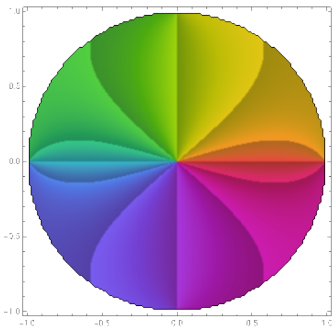

FIGURE 1. Image of f(Δ)

(b) Since $h'(z) = \frac{f_z(z) - \omega(z)\overline{f_{\bar{z}}}}{1-|\omega(z)|^2}$,
We obtain

$$|h'(z)| \geq \frac{|f_z(z)| - |\omega(z) f_{\bar{z}}|}{1-|\omega|^2}$$

$$\geq \frac{|f_z(z)| - |\omega(z) f_z|}{1-|\omega|^2} \geq \frac{m}{2},$$

Since $|f_z| > |f_{\bar{z}}|$.

Similarly, $|h'(z)| \leq \frac{|f_z(z)| + |\omega(z) f_z|}{1-|\omega|^2} \leq \frac{M}{1-\lambda}$.
The result follows immediately.

For (c), we know that

$$f'(z) = h'(z)\left(1 + \frac{\overline{\omega(z)h'(z)}}{h'(z)}\right).$$

Assume to the contrary that there is $z_0 \in \Delta$ with $f'(z_0) = 0$. This leads to

$$-1 = \frac{\overline{\omega(z_0)h'(z_0)}}{h'(z_0)}.$$

Since f is sense preserving, $h'(z_0) \neq 0$.
Thus, $|\omega(z_0)| = 1$. But f is a GRM and $|\omega(z_0)| < 1$, which is a contradiction.
Hence, $f'(z)$ has no zeros in Δ.

Therefore, there are no zeros of the complex derivative of f inside a Stolz angle.
Example 2. Consider

$$f(z) = z + \frac{\overline{kz^2}}{2}, 0 < k < 1.$$

We have $\omega(z) = kz$, $|\omega(z)| < 1$, for $|z| < 1$, showing that f is sense preserving.
Moreover, solving

$$f'(z) = 1 + k\bar{z} = 0.$$

This implies $|z| = \frac{1}{k} > 1$. This shows there are no zeros in the Δ. Hence, there are no zeros in any Stolz angle.

Since there are no zeros of f' in Δ, no folding occurs in Figure 2. The Light grey mesh(grid inside the image) represents the lines of the deformed grid, which come from vertical and horizontal lines of the disk being mapped by f. The red boundary curve is the image of the unit circle, which represents the outer shape of the image domain. Furthermore, the image is simply connected, convex domain without cusps or self intersection.

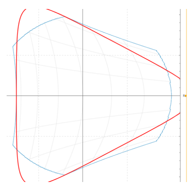

FIGURE 2. Image of $f(\Delta), k = 0.5$

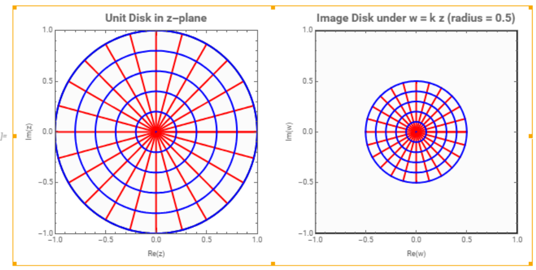

FIGURE 3. Image of Δ under $\omega(z) = kz, k = 0.5$

$\square$

Conclusion

In this paper, we set various conditions on the boundary behavior of complex-valued univalent harmonic mapping f to determine the angular limits of the argument and logarithm of analytic functions. We have examined that the dilatation ω possesses finite number of zeros in any Stolz angle if the complex derivative of f at the boundary tends to positive infinity. We have also shown that the complex derivative of f has no interior zeros within any Stolz angle at $e^{i\theta_0}$ provided that f maps the open unit disk onto a bounded convex domain. Our results establish the connection between dilatation, the boundary derivative and the occurrence of jumps in the boundary function. Furthermore, we have proved that the radial derivative of f near the boundary is determined by the dilatation which governs the asymptotic direction of the boundary image. These findings extend the work of Bshouty et al. and Laugesen, offering a more comprehensive understanding of the boundary behavior of univalent harmonic mappings and the influence of boundary regularity on the image domain.

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