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AN APPLICATION OF BANACH'S CONTRACTION PRINCIPLE TO THE NUMERICAL TREATMENT OF NONLINEAR VOLTERRA-FREDHOLM EQUATIONS IN HEALTH DOMAINS

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Abstract. This paper investigates the numerical approximations of nonlinear Volterra-Fredholm equations, focusing on their bounded solutions over specified regions. The research employed the Banach contraction principle to prove the existence of a unique solution in the space of continuous functions. The integral equations were formulated to model complex interactions in various applications, particularly infectious disease dynamics. Also, some key parameters, like the kernel functions and scalar multipliers were analyzed to ascertain that the contraction mappings conditions are satisfied. The Picard iteration was used to approximate solutions, proving convergence and stability results. The findings showed significance of these mathematical models in dynamic systems and optimizing treatment in healthcare. This work contributes to the existing literature on nonlinear integral equations.

1. Introduction

In the area of healthcare, mathematical modelling serves as a fundamental tool for analyzing and predicting complex biological phenomena. Among these models, nonlinear Volterra-Fredholm integral equations hold a special place ([1], [2], [3], [4]). These equations are critical for representing systems in which the current state is influenced not only by previous time-dependent interactions but also by spatial factors. They are particularly useful in describing dynamic biological processes, such as the spread of diseases, response to treatments, or progression of chronic conditions, scenarios where both memory and spatial considerations are vital.

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Despite their importance, these equations present significant challenges. Their nonlinear and integral nature [5] often makes it impossible to derive precise analytical solutions. This complexity necessitates the use of numerical approximation techniques, which offer a practical way to estimate solutions that cannot be determined exactly. These computational methods transform the problem into a form that can be solved using algorithms, providing results that are sufficiently accurate for practical purposes in research and clinical applications.

One powerful mathematical tool that helps in this approximation process is the Banach Contraction Principle [6]. This principle, rooted in functional analysis, ensures that under certain conditions, a sequence of approximations will converge to a unique and stable solution. By using this principle, mathematicians can guarantee that the numerical methods employed are not only efficient but also reliable. This assurance is helpful in healthcare sector, where the outcomes of mathematical models likely inform critical decisions involving are useful in diagnosis, treatment planning, and patient management.

The significance of nonlinear Volterra-Fredholm equations in healthcare cannot be overstated. They provide a robust framework for modelling time-delayed and spatially distributed systems, which are common in physiological and epidemiological studies ([7], [8]. For example, modelling the effects of a drug administered over time and across different organs involves understanding how earlier doses continue to influence the system while new doses are introduced, an ideal scenario for these types of equations.

In situations where exact solutions are unattainable, numerical approximations step in as indispensable tools. They enable scientists and healthcare professionals to simulate complex systems, test hypotheses, and make predictions with a high degree of confidence. These approximations, grounded in rigorous mathematical theory, bridge the gap between abstract mathematical constructs and real-world applications.

Moreover, the Banach Contraction Principle provides a theoretical foundation that supports the convergence and stability of these approximated solutions. This not only enhances the credibility of the results but also builds trust in the use of mathematical models in sensitive areas like healthcare, where accuracy and predictability are paramount.

By understanding these key concepts such as nonlinear Volterra-Fredholm equations [9], numerical approximations, and the Banach Contraction Principle ([10],[11],[12],[13],[14], [15]), one gains insight into how advanced mathematics contributes to healthcare innovation. These mathematical frameworks support better decision-making, enhance predictive accuracy, and ultimately contribute to improved patient outcomes. Bhat et al. (2024) developed an efficient discretization method for numerically solving nonlinear Volterra-Fredholm-Hammerstein equations that model various memory effects in control, physics, and population dynamics [16]. As we delve deeper into this area, we explore the powerful synergy between mathematical theory and medical practice ([17], [18], [19], [20]), highlighting how each informs and strengthens the other in solving real-world problems.

This research centers on examining the conditions under which the following nonlinear integral equations can be solved.

(i) **Problem (a):** Establishing the existence and approximation of the solution to the integral equation

$$u(x) = g(x) + \mu \iint_{\Omega(x)} k(t, y, u(y))\, dy\, dt, \quad x \in [a, b] \subset \mathbb{R},$$

where $g \in C[a, b]$ is a known continuous function, $\mu \in \mathbb{R}$ is a scalar parameter, and $k : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous kernel function, where $u : \mathbb{R} \to \mathbb{R}$ is the unknown function.

(ii) **Problem (b):** Proving the existence of a solution to the redefined nonlinear integral equation model for the viral load dynamics

$$u(x) = g(x) + \mu \int_0^x \int_0^T k\big(t, y, u(y), I(y), D(y)\big)\, dy\, dt, \quad x \in [0, T].$$

In both cases: $g \in C[a, b]$ is a given continuous function, $\mu \in \mathbb{R}$ is a scalar parameter, $k : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given kernel function. Viral dynamics is a field within applied mathematics that deals with modelling the progression and behaviour of viral infections inside a host. It employs mathematical models to describe how the number of infected cells and the viral load in the body evolve over time (Nowak & May, 2001).

To investigate these problems, the Banach contraction principle is used to demonstrate the solvability of Problem (a), while the generalized Banach Fixed Point Theorem is applied to confirm the solvability of Problem (b).

1.1. **Existence of Solution for a Nonlinear Integral Equation.** We consider the nonlinear integral equation

$$u(x) = g(x) + \mu \int_a^x \int_a^b k(t, y, u(y))\, dy\, dt, \quad \text{for } x \in [a, b] \subset \mathbb{R}.$$

Assumptions

Assume the following: $g \in C[a, b]$, that is; g is continuous on $[a, b]$, $k : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous, there exists a constant $L > 0$ such that for all $t, y \in [a, b]$ and $u_1, u_2 \in \mathbb{R}$,

$$|k(t, y, u_1) - k(t, y, u_2)| \leq L|u_1 - u_2|. \qquad \text{(Lipschitz condition)}$$

For the function space, let $X = C[a, b]$, the Banach space of continuous functions on $[a, b]$, equipped with the sup norm

$$\|u\| = \sup_{x \in [a,b]} |u(x)|.$$

For the operator definition, define an operator $T : X \to X$ by

$$(Tu)(x) = g(x) + \mu \int_a^x \int_a^b k(t, y, u(y))\, dy\, dt.$$

For the contraction mapping, let $u_1, u_2 \in X$, then for all $x \in [a, b]$,

$$\begin{aligned}
|(Tu_1)(x) - (Tu_2)(x)| &= \left| \mu \int_a^x \int_a^b [k(t, y, u_1(y)) - k(t, y, u_2(y))]\, dy\, dt \right| \\
&\leq |\mu| \int_a^x \int_a^b |k(t, y, u_1(y)) - k(t, y, u_2(y))|\, dy\, dt \\
&\leq |\mu| L \int_a^x \int_a^b |u_1(y) - u_2(y)|\, dy\, dt \\
&\leq |\mu| L \int_a^x \int_a^b \|u_1 - u_2\|\, dy\, dt \\
&= |\mu| L (x - a)(b - a) \|u_1 - u_2\|.
\end{aligned}$$

Taking the supremum over $x \in [a, b]$, we obtain:

$$\|Tu_1 - Tu_2\| \leq |\mu| L (b - a)^2 \|u_1 - u_2\|.$$

Let $\alpha = |\mu| L (b - a)^2$. If $\alpha < 1$, then T is a contraction.
In conclusion, by the Banach Fixed Point Theorem, if

$$|\mu| L (b - a)^2 < 1,$$

then the operator T has a unique fixed point $u^* \in C[a, b]$. Therefore, the integral equation

$$u(x) = g(x) + \mu \int_a^x \int_a^b k(t, y, u(y))\, dy\, dt$$

has a unique continuous solution on $[a, b]$.

2. Materials and Methods

In this section, some nolinear integral equations and their numerical datasets applications are presented.

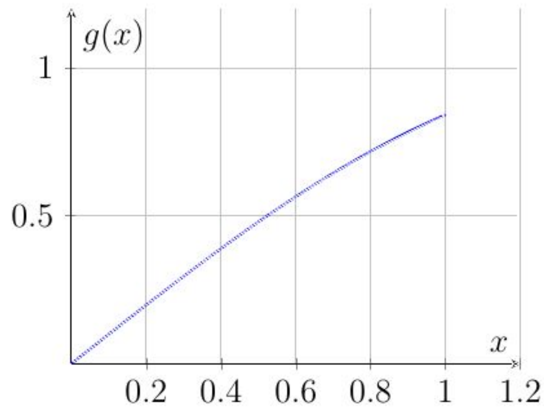
We consider the nonlinear integral equation

$$u(x) = g(x) + \mu \int_a^x \int_a^b k(t, y, u(y))\, dy\, dt, \quad x \in [a, b].$$

We consider two numerical datasets satisfying the conditions for the Banach contraction mapping principle.

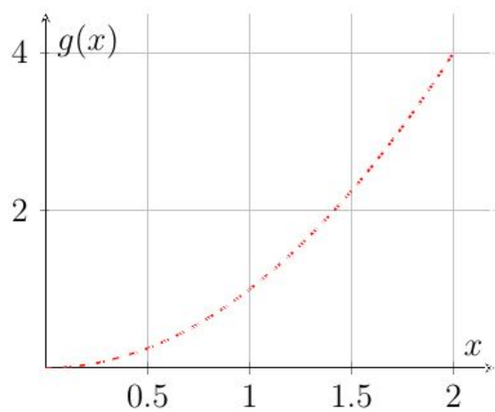
2.1. **Dataset 1.** Consider the dataset 1 with the parameters, Interval: $[a, b] = [0, 1]$, $\mu = 0.5$, $g(x) = \sin(x) \in C[0, 1]$, $k(t, y, u) = y + \frac{1}{2}u$, Lipschitz constant: $L = \frac{1}{2}$, and $|\mu|L(b - a)^2 = 0.5 \cdot 0.5 \cdot 1 = 0.25 < 1$

Given another dataset 2 with the parameters, interval: $[a, b] = [0, 2]$, $\mu = 0.1$, $g(x) = x^2 \in C[0, 2]$, $k(t, y, u) = \cos(t + y) + u$, Lipschitz constant: $L = \frac{1}{5}$ and $|\mu|L(b - a)^2 = 0.1 \cdot \frac{1}{5} \cdot 4 = 0.08 < 1$ The plots of $g(x)$ are given:



(a) Plot of $g(x) = \sin(x)$ on $[0, 1]$

$g(x) = x^2 \in C[0, 2]$



(b) Plot of $g(x) = x^2$ on $[0, 2]$

FIGURE 1. Separate plots of two functions on different intervals

2.2. **Convergence and Stability.** Consider the integral equation:

$$u(x) = g(x) + \mu \int_a^x \int_a^b k(t, y, u(y))\, dy\, dt, \quad \text{for } x \in [a, b].$$

Define the operator:

$$(Tu)(x) := g(x) + \mu \int_a^x \int_a^b k(t, y, u(y))\, dy\, dt.$$

Let $X = C[a,b]$ be the space of continuous functions with the uniform norm $\|u\| = \sup_{x\in[a,b]} |u(x)|$.

To ensure convergence and stability, we check if T is a contraction:

$$\|Tu - Tv\| \leq |\mu| L(b-a)^2 \|u - v\| < \|u - v\|.$$

2.3. **Preliminaries on Numerical Dataset Applications.** Interval: $[a,b] = [0,1]$, $\mu = 0.5$, Kernel: $k(t,y,u) = y + \frac{1}{2}u$, Lipschitz constant in u: $L = \frac{1}{2}$, and Evaluation: $|\mu|L(b-a)^2 = 0.5 \cdot 0.5 \cdot 1 = 0.25 < 1$

The operator T is a contraction, a unique solution exists, Picard iteration:

$$u_{n+1}(x) = \sin(x) + 0.5 \int_0^x \int_0^1 \left(y + \frac{1}{2} u_n(y) \right) dy\, dt$$

and the method is convergent and stable.

2.4. **First Picard Iteration for the Dataset 1.** Initial approximation:

$$u_0(x) = \sin(x)$$

Compute:

$$u_1(x) = \sin(x) + 0.5 \int_0^x \int_0^1 \left(y + \frac{1}{2} \sin(y) \right) dy\, dt$$

Inner integral:

$$\int_0^1 \left(y + \frac{1}{2}\sin(y) \right) dy = \int_0^1 y\, dy + \frac{1}{2} \int_0^1 \sin(y)\, dy$$

$$= \frac{1}{2} + \frac{1}{2}(1 - \cos(1)) = 1 - \frac{1}{2}\cos(1)$$

Outer integral:

$$\int_0^x \left(1 - \frac{1}{2}\cos(1) \right) dt = \left(1 - \frac{1}{2}\cos(1) \right) x$$

Thus:

$$u_1(x) = \sin(x) + \left(\frac{1}{2} - \frac{1}{4}\cos(1) \right) x$$

Simplify numerically:

$$\cos(1) \approx 0.5403\ (radian)$$

$$\frac{1}{2} - \frac{1}{4} \cdot \cos(1) = \frac{1}{2} - \frac{1}{4} \cdot 0.5403 \approx 0.500 - 0.1351 = 0.3649$$

Thus,

$$u_1(x) \approx \sin(x) + 0.3649x$$

2.5. **Second Picard Iteration for the Dataset 1.**

$$u_2(x) = \sin(x) + 0.5 \int_0^x \int_0^1 \left(y + \frac{1}{2}u_1(y) \right) \, dy \, dt$$

Substitute $u_1(y) = \sin(y) + \left(\frac{1}{2} - \frac{1}{4}\cos(1)\right) y$, then:

$$y + \frac{1}{2}u_1(y) = y + \left(\frac{1}{4} - \frac{1}{8}\cos(1) \right) + \frac{1}{2}\sin(y)$$

Evaluate the inner integral:

$$\int_0^1 \left[y + \left(\frac{1}{4} - \frac{1}{8}\cos(1) \right) + \frac{1}{2}\sin(y) \right] dy = \frac{5}{4} - \frac{5}{8}\cos(1)$$

Evaluate the outer integral:

$$\int_0^x \left(\frac{5}{4} - \frac{5}{8}\cos(1) \right) dt = \left(\frac{5}{4} - \frac{5}{8}\cos(1) \right) x$$

So:

$$u_2(x) = \sin(x) + \left(\frac{5}{8} - \frac{5}{16}\cos(1) \right) x$$

Simplify numerically:

$$\cos(1) \approx 0.5403 \; (radian)$$

$$\frac{5}{8} - \frac{5}{16} \cdot \cos(1) = \frac{5}{8} - \frac{5}{16} \cdot 0.5403 \approx 0.625 - 0.1688 = 0.4562$$

Thus,

$$u_2(x) \approx \sin(x) + 0.4562x$$

$$\vdots$$

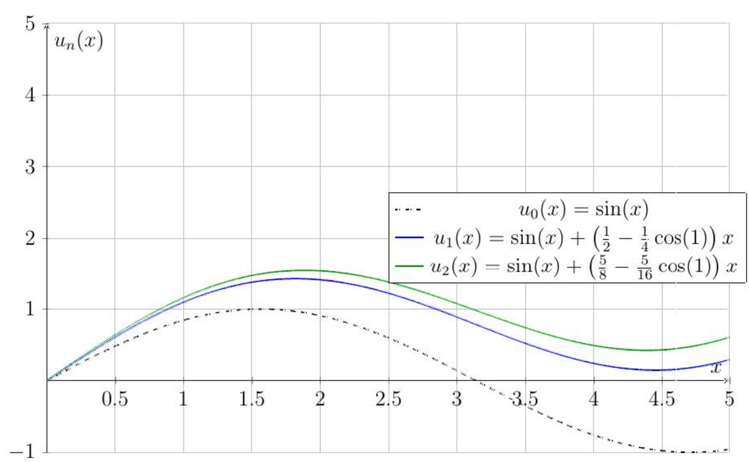


FIGURE 2. Successive approximations $u_n(x)$ behaviour

2.6. **Stability and Convergence for the Dataset 1.** Define:

$$(Tu)(x) = \sin(x) + 0.5 \int_0^x \int_0^1 \left(y + \frac{1}{2}u(y) \right) dy\, dt$$

Let $\|\cdot\|$ be the uniform norm on $C[0,1]$. Then:

$$\|Tu - Tv\| \leq 0.25\|u - v\|, \quad \text{(Lipschitz constant } \lambda = 0.25 < 1)$$

Hence: the operator T is a contraction, the sequence $\{u_n\}$ converges uniformly to a unique fixed point and the iteration is stable under small perturbations.

2.7. **Stability and Convergence for the Dataset 1.** Since the operator is a contraction with factor $\lambda = 0.25 < 1$, we have: the sequence $\{u_n\}$ converges uniformly, the numerical scheme is stable and the error decreases geometrically: $\|u_{n+1} - u_n\| \leq \lambda \|u_n - u_{n-1}\|$

We consider the integral equation

$$u(x) = g(x) + \mu \int_a^x \int_a^b k(t, y, u(y))\, dy\, dt, \quad x \in [a,b] \subset \mathbb{R}$$

Assumptions

Suppose $g \in C([a,b])$, $k(t,y,u)$ continuous on $[a,b]^2 \times \mathbb{R}$ and k is Lipschitz in u: there exists $L > 0$ such that

$$|k(t,y,u_1) - k(t,y,u_2)| \leq L|u_1 - u_2|$$

For Picard iteration: define a sequence of approximations:

$$u_0(x) = g(x)$$

$$u_{n+1}(x) = g(x) + \mu \int_a^x \int_a^b k(t, y, u_n(y))\, dy\, dt$$

For convergence: let $\|\cdot\|_\infty$ be the sup norm on $C([a,b])$, then:

$$\|u_{n+1} - u_n\|_\infty \leq |\mu|(b-a)^2 L \cdot \|u_n - u_{n-1}\|_\infty$$

Let $\lambda = |\mu|(b-a)^2 L$. If $\lambda < 1$, we obtain:

$$\|u_{n+1} - u_n\|_\infty \leq \lambda^n \|u_1 - u_0\|_\infty$$

Hence, $\{u_n\}$ converges uniformly to a function $u \in C([a,b])$ that solves the integral equation.

For stability: let $u, \tilde{u}$ be solutions corresponding to $g, \tilde{g}$, respectively. Then:

$$\begin{aligned} |u(x) - \tilde{u}(x)| &\leq |g(x) - \tilde{g}(x)| + |\mu| \int_a^x \int_a^b |k(t,y,u(y)) - k(t,y,\tilde{u}(y))|\, dy\, dt \\ &\leq \|g - \tilde{g}\|_\infty + \lambda \|u - \tilde{u}\|_\infty \end{aligned}$$

Hence:

$$\|u - \tilde{u}\|_\infty \leq \frac{1}{1-\lambda} \|g - \tilde{g}\|_\infty$$

This proves the solution depends continuously on g, i.e., the method is stable.

2.8. **Dataset 2 Analysis.** Interval: $[a, b] = [0, 2]$, $\mu = 0.1$, Kernel: $k(t, y, u) = \cos(t + y) + u$, Lipschitz constant in u: $L = \frac{1}{5}$ and Evaluation: $|\mu|L(b - a)^2 = 0.1 \cdot \frac{1}{5} \cdot 4 = 0.08 < 1$. The operator T is a contraction, a unique solution exists, Picard iteration:

$$u_{n+1}(x) = x^2 + 0.1 \int_0^x \int_0^2 (\cos(t + y) + u_n(y))\, dy\, dt$$

and the method is convergent and stable.

2.8.1. *First Picard Iteration for the Dataset 2.* Initial Approximation:

$$u_0(x) = x^2$$

Iteration 1: Compute $u_1(x)$

$$u_1(x) = x^2 + 0.1 \int_0^x \int_0^2 [\cos(t + y) + u_0(y)]\, dy\, dt$$

Substitute $u_0(y) = y^2$:

$$u_1(x) = x^2 + 0.1 \int_0^x \int_0^2 \left[\cos(t + y) + y^2\right] dy\, dt$$

Separate the integrals:

$$u_1(x) = x^2 + 0.1 \left(\int_0^x \int_0^2 \cos(t + y)\, dy\, dt + \int_0^x \int_0^2 y^2\, dy\, dt \right)$$

First integral:

$$\int_0^x \int_0^2 \cos(t + y)\, dy\, dt = \int_0^x [\sin(t + 2) - \sin(t)]\, dt$$

Now integrate:

$$\int_0^x [\sin(t + 2) - \sin(t)]\, dt = [-\cos(t + 2) + \cos(t)]_0^x = -\cos(x+2)+\cos(2)+\cos(x)-\cos(0)$$

So:

$$\int_0^x \int_0^2 \cos(t+y)\, dy\, dt = [-\cos(t + 2) + \cos(t)]_0^x = -\cos(x+2)+\cos(2)+\cos(x)-\cos(0)$$

Second integral:

$$\int_0^x \int_0^2 y^2\, dy\, dt = \int_0^x \left[\frac{y^3}{3}\right]_0^2 dt = \int_0^x \frac{8}{3} dt = \frac{8}{3}x$$

Final expression for $u_1(x)$:

$$u_1(x) = x^2 + 0.1 \left[-\cos(x + 2) + \cos(x) + \cos(2) - 1 + \frac{8}{3}x \right]$$

Simplify numerically:

$$\cos(0) = 1, \quad \cos(2) \approx -0.4161(radian)$$

$$\cos(2) - \cos(0) \approx 1 - 0.4161 = -1.4161$$

Thus,

$$u_1(x) \approx x^2 + 0.1 \left[-\cos(x+2) + \cos(x) + \frac{8}{3}x - 1.4161 \right]$$

Table 1: A Table showing the Contraction, Convergence and Stability for the Datasets

Dataset	$[a,b]$	μ	L	$\lambda = \|\mu\|L(b-a)^2$	Contraction?	Convergence/Stability
1	[0, 1]	0.5	0.5	0.25	Yes	Yes
2	[0, 2]	0.1	0.2	0.08	Yes	Yes

3. Result

In this section, existence of solution to the integral equation problem are investigated.

Let's consider the integral equation

$$u(x) = g(x) + \mu \int_a^b \int_a^x k(t, y, u(y))\, dy\, dt, \quad x \in [a,b].$$

Assumptions:
Suppose that $g : [a,b] \to \mathbb{R}$ is continuous and bounded, $k : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies a Lipschitz condition in the third variable; i.e., there exists a constant $L > 0$ such that for all $t, y \in [a,b]$ and all $u, v \in \mathbb{R}$,

$$|k(t,y,u) - k(t,y,v)| \leq L|u - v|.$$

and $\mu \in \mathbb{R}$ is a fixed parameter.

Define the operator $T : C([a,b]) \to C([a,b])$ by

$$(Tu)(x) := g(x) + \mu \int_a^b \int_a^x k(t, y, u(y))\, dy\, dt.$$

Step 1: T maps $C([a,b])$ into itself.
Since g and k are continuous and u is continuous, the integral

$$x \mapsto \int_a^b \int_a^x k(t, y, u(y))\, dy\, dt$$

defines a continuous function on $[a,b]$. Therefore, $Tu \in C([a,b])$ for all $u \in C([a,b])$.

Step 2: T is a contraction for sufficiently small $|\mu|$.
For $u, v \in C([a,b])$ and any $x \in [a,b]$, we have

$$\begin{aligned}|(Tu)(x) - (Tv)(x)| &= \left|\mu \int_a^b \int_a^x \big[k(t,y,u(y)) - k(t,y,v(y))\big]\,dy\,dt\right| \\ &\le |\mu| \int_a^b \int_a^x |k(t,y,u(y)) - k(t,y,v(y))|dy\,dt \\ &\le |\mu| L \int_a^b \int_a^x |u(y) - v(y)|dy\,dt \\ &\le |\mu| L \|u - v\| \int_a^b \int_a^x dy\,dt,\end{aligned}$$

where

$$\|u - v\| := \sup_{y \in [a,b]} |u(y) - v(y)|.$$

Note that

$$\int_a^b \int_a^x dy\,dt = \int_a^b (x-a)dt = (b-a)(x-a) \le (b-a)^2.$$

Hence,

$$|(Tu)(x) - (Tv)(x)| \le |\mu| L (b-a)^2 \|u - v\|.$$

Taking the supremum over $x \in [a,b]$, we get

$$\|Tu - Tv\| \le q\|u - v\|, \quad \text{where} \quad q := |\mu| L (b-a)^2.$$

If

$$q < 1,$$

then T is a contraction.

Step 3: Existence and uniqueness by Banach Fixed Point Theorem
Since $C([a,b])$ with the sup norm is a complete metric space, the Banach fixed point theorem guarantees the existence of a unique $u^* \in C([a,b])$ such that

$$T(u^*) = u^*,$$

i.e., u^* is a continuous solution to the integral equation.

Remarks: If the contraction condition $q < 1$ does not hold globally, existence can still be obtained via other methods such as the Schauder fixed point theorem or by restricting to smaller intervals and applying continuation arguments.

3.1. **Stability and Convergence of the Integral Equation.** Consider the integral equation

$$u(x) = g(x) + \mu \int_a^b \int_a^x k(t,y,u(y))\,dy\,dt, \quad x \in [a,b].$$

Assumptions

Assume the kernel $k : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition

$$|k(t,y,u) - k(t,y,v)| \le L|u - v|, \quad \text{for all } u, v \in \mathbb{R},$$

for some constant $L > 0$.

For Stability, let u_1, u_2 be two solutions corresponding to data g_1, g_2 and kernels k_1, k_2. Define

$$w(x) := u_1(x) - u_2(x).$$

Then,

$$w(x) = g_1(x) - g_2(x) + \mu \int_a^b \int_a^x \left[k_1(t, y, u_1(y)) - k_2(t, y, u_2(y))\right] dy\, dt.$$

Using the triangle inequality and Lipschitz continuity,

$$\begin{aligned} |w(x)| \leq \|g_1 - g_2\|_\infty + |\mu| \int_a^b \int_a^x |k_1(t, y, u_1(y)) - k_1(t, y, u_2(y))|\, dy\, dt \\ + |\mu| \int_a^b \int_a^x |k_1(t, y, u_2(y)) - k_2(t, y, u_2(y))|\, dy\, dt \\ \leq \|g_1 - g_2\|_\infty + |\mu| L(b-a)^2 \|w\|_\infty + \epsilon, \end{aligned}$$

where ϵ accounts for kernel perturbations.

If we consider $k_1 = k_2 = k$, then

$$|w(x)| \leq \|g_1 - g_2\|_\infty + |\mu| L(b-a)^2 \|w\|_\infty.$$

Taking supremum over $x \in [a, b]$,

$$\|w\|_\infty \leq \|g_1 - g_2\|_\infty + |\mu| L(b-a)^2 \|w\|_\infty.$$

If

$$|\mu| L(b-a)^2 < 1,$$

then rearranging,

$$\|w\|_\infty \leq \frac{\|g_1 - g_2\|_\infty}{1 - |\mu| L(b-a)^2}.$$

This shows that small changes in g lead to small changes in the solution u, proving stability.

For convergence, define the successive approximations

$$u_0(x) := g(x),$$

and for $n \geq 0$,

$$u_{n+1}(x) := g(x) + \mu \int_a^b \int_a^x k(t, y, u_n(y))\, dy\, dt.$$

Consider the difference

$$\begin{aligned} |u_{n+1}(x) - u_n(x)| &= \left| \mu \int_a^b \int_a^x \big(k(t, y, u_n(y)) - k(t, y, u_{n-1}(y))\big) dy dt \right| \\ &\leq |\mu| \int_a^b \int_a^x |k(t, y, u_n(y)) - k(t, y, u_{n-1}(y))| dy dt \\ &\leq |\mu| L(b-a)^2 \|u_n - u_{n-1}\|_\infty. \end{aligned}$$

Taking supremum over $x \in [a,b]$,

$$\|u_{n+1} - u_n\|_\infty \leq q\|u_n - u_{n-1}\|_\infty,$$

where

$$q := |\mu| L(b-a)^2.$$

If

$$q < 1,$$

then $\{u_n\}$ is a contraction sequence and hence converges uniformly to a unique fixed point u^*, which is the unique solution to the integral equation.

Therefore, under the Lipschitz condition on k and the condition $|\mu|L(b-a)^2 < 1$, the integral equation has a unique, stable solution, and the successive approximations converge to this solution.

Theorem 3.1. *Suppose that:*
i. $g \in C[a,b]$,
ii. $k : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ *is continuous,*
iii. There exists a constant $L > 0$ *such that for all* $t, y \in [a,b]$ *and* $u_1, u_2 \in \mathbb{R}$,

$$|k(t,y,u_1) - k(t,y,u_2)| \leq L|u_1 - u_2|.$$

If the scalar parameter μ *satisfies*

$$|\mu| L(b-a)^2 < 1,$$

then the nonlinear Volterra-Fredholm integral equation

$$u(x) = g(x) + \mu \int_a^x \int_a^b k(t,y,u(y))\,dy\,dt, \quad x \in [a,b]$$

has a unique continuous solution $u^* \in C[a,b]$.

Moreover, the solution can be approximated by the Picard iteration

$$u_{n+1}(x) = g(x) + \mu \int_a^x \int_a^b k(t,y,u_n(y))\,dy\,dt,$$

which converges uniformly to the unique solution u^*.

Proof. Define the operator $T : C[a,b] \to C[a,b]$ by

$$(Tu)(x) := g(x) + \mu \int_a^x \int_a^b k(t,y,u(y))\,dy\,dt.$$

Using the Lipschitz condition on k and the definition of the sup norm, for any $u_1, u_2 \in C[a,b]$,

$$\begin{aligned}\|Tu_1 - Tu_2\| &= \sup_{x\in[a,b]} |(Tu_1)(x) - (Tu_2)(x)| \\ &\leq |\mu| L(b-a)^2 \|u_1 - u_2\|.\end{aligned}$$

Since $|\mu|L(b-a)^2 < 1$, the operator T is a contraction on the complete metric space $(C[a,b], \|\cdot\|)$. By the Banach contraction principle, T has a unique fixed point u^*, which is the unique solution to the integral equation. The Picard iteration sequence defined by $u_{n+1} = Tu_n$ converges to u. □

This section has examined the existence, uniqueness, and numerical approximation of solutions to nonlinear Volterra-Fredholm integral equations on a real interval. It considers two integral problems characterized by a continuous kernel and a scalar parameter. By employing the Banach contraction principle and its extensions, the study establishes sufficient conditions for the existence of solutions within the Banach space of continuous functions. These conditions hinge on the continuity of the kernel and the Lipschitz continuity of the nonlinear term. The associated integral operators are shown to be contractions when the product of the parameter, the Lipschitz constant, and the square of the interval length is less than one, ensuring a unique fixed-point solution. To support the theoretical findings, two numerical examples are presented with specific kernel functions and parameters that meet the contraction criteria. The Picard iteration method is used to approximate the solutions, demonstrating convergence and numerical stability. This work contributes to the theory of nonlinear integral equations by offering clear criteria for solution existence and uniqueness, along with effective iterative methods for approximation tools that are highly relevant to applied mathematics and engineering problems.

3.2. **Establishing the existence and approximation of the solution to the integral equation and the nonlinear integral equation model viral load.** In this section, we shall establish the existence and approximation of the solution to the integral equation for Problem (a) and the nonlinear integral equation model viral load for Problem (b).

3.3. **Establishing the existence and approximation of the solution to the integral equation for Problem (a).** Consider the nonlinear integral equation

$$u(x) = g(x) + \mu \iint_{\Omega(x)} k(t, y, u(y))\, dy\, dt, \quad x \in [a, b],$$

where $g \in C[a, b]$ is a known continuous function, $\mu \in \mathbb{R}$ is a scalar parameter, and $k : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ is a given continuous kernel function.

Define the Banach space

$$X := \{u : [a, b] \to \mathbb{R} \mid u \text{ is continuous}\},$$

equipped with the supremum norm

$$\|u\| := \sup_{x\in[a,b]} |u(x)|.$$

Then, the integral operator

$$(Tu)(x) := g(x) + \mu \iint_{\Omega(x)} k(t, y, u(y))\, dy\, dt$$

is a contraction mapping on a closed subset of X.

For the Kernel function, the following conditions hold:

(A1) $k(t, y, u)$ is continuous in $(t, y) \in [a, b]^2$ for each fixed u.

(A2) There exists a constant $L > 0$ such that for all $t, y \in [a, b]$ and for all $u, v \in \mathbb{R}$,

$$|k(t, y, u) - k(t, y, v)| \leq L|u - v|.$$

That is, k is Lipschitz continuous in the last argument with Lipschitz constant L.

(A3) The double integral over the region $\Omega(x)$ satisfies

$$M := \sup_{x\in[a,b]} \iint_{\Omega(x)} dt\, dy < \infty.$$

For the contraction mapping property, for any $u, v \in X$, we have

$$|(Tu)(x) - (Tv)(x)| = \left| \mu \iint_{\Omega(x)} \big(k(t, y, u(y)) - k(t, y, v(y))\big) dy\, dt \right|.$$

Using the Lipschitz condition (A2),

$$\leq |\mu| \iint_{\Omega(x)} L|u(y) - v(y)|\, dy\, dt \leq |\mu|L\|u - v\| \iint_{\Omega(x)} dy\, dt.$$

Taking supremum over $x \in [a, b]$,

$$\|Tu - Tv\| \leq |\mu|LM\|u - v\|.$$

For the existence and uniqueness, if

$$q := |\mu|LM < 1,$$

then T is a contraction on X, and by the Banach Fixed Point Theorem, there exists a unique fixed point $u^* \in X$ such that

$$T(u^*) = u^*,$$

that is, u^* is the unique continuous solution to the integral equation.

For the approximation of the solution, we start from an initial guess $u_0 \in X$ (e.g., $u_0(x) = g(x)$), define the iterative sequence

$$u_{n+1} = Tu_n.$$

Then $u_n \to u^*$ in the supremum norm with the error estimate

$$\|u_n - u^*\| \leq \frac{q^n}{1-q}\|u_1 - u_0\|.$$

Thus, the solution can be approximated to any desired accuracy by successive iterations of the integral operator. This section also showed that: under suitable Lipschitz and boundedness conditions, the nonlinear integral operator is a contraction, Banach's fixed point theorem guarantees the existence and uniqueness of a continuous solution and the solution can be approximated numerically by iterative application of the integral operator.

Theorem 3.2. ***Existence and Uniqueness of a Continuous Solution to a Nonlinear Integral Equation***

Let $g \in C[a, b]$, and let $k : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function (kernel) satisfying the following assumptions:

(A1) *For each fixed* $u \in \mathbb{R}$*, the function* $(t, y) \mapsto k(t, y, u)$ *is continuous on* $[a, b]^2$*.*
(A2) *There exists a constant* $L > 0$ *such that for all* $t, y \in [a, b]$ *and for all* $u, v \in \mathbb{R}$*,*

$$|k(t, y, u) - k(t, y, v)| \leq L|u - v|.$$

(A3) *The integration domain* $\Omega(x) \subset [a, b]^2$ *satisfies*

$$M := \sup_{x\in[a,b]} \iint_{\Omega(x)} dt\, dy < \infty.$$

Let $X := C[a, b]$ *be the Banach space of continuous functions on* $[a, b]$ *equipped with the supremum norm*

$$\|u\| := \sup_{x\in[a,b]} |u(x)|.$$

Define the integral operator $T : X \to X$ *by*

$$(Tu)(x) := g(x) + \mu \iint_{\Omega(x)} k(t, y, u(y))\, dy\, dt.$$

Then, if

$$q := |\mu|LM < 1,$$

the operator T *is a contraction on* X*. Consequently, by the Banach Fixed Point Theorem, there exists a unique function* $u^* \in X$ *such that*

$$Tu^* = u^*,$$

i.e., u^* *is the unique continuous solution of the nonlinear integral equation*

$$u(x) = g(x) + \mu \iint_{\Omega(x)} k(t, y, u(y))\, dy\, dt.$$

Proof. (1) For any $u, v \in X$, using assumption (A2), we estimate:

$$\begin{aligned} |(Tu)(x) - (Tv)(x)| &= \left|\mu \iint_{\Omega(x)} [k(t, y, u(y)) - k(t, y, v(y))]\, dy\, dt\right| \\ &\leq |\mu| \iint_{\Omega(x)} L|u(y) - v(y)|\, dy\, dt \\ &\leq |\mu|L\|u - v\| \iint_{\Omega(x)} dy\, dt. \end{aligned}$$

(2) Taking supremum over $x \in [a, b]$, and using assumption (A3):

$$\|Tu - Tv\| \leq |\mu|LM\|u - v\| = q\|u - v\|.$$

(3) Since $q < 1$, T is a contraction.

By the Banach Fixed Point Theorem, T has a unique fixed point $u^* \in X$, which is the unique solution of the integral equation. □

3.3.1. *Convergence of Iterative Approximation.* Define the Picard iteration sequence:

$$u_0(x) := g(x), \quad u_{n+1} := Tu_n.$$

Then the sequence $\{u_n\} \subset X$ converges uniformly to u^*, and the error satisfies:

$$\|u_n - u^*\| \leq \frac{q^n}{1-q}\|u_1 - u_0\|.$$

3.4. **Nonlinear Integral Equation Model - Viral Load for Problem (b).** The viral load $u(x)$ satisfies

$$u(x) = g(x) + \mu \int_0^x \int_0^T k\big(t, y, u(y), I(y), D(y)\big)\, dy\, dt, \quad x \in [0, T].$$

For the kernel structure, we assume the kernel decomposes as

$$k(t, y, u, I, D) = r(t, y) \cdot f_1(u) \cdot f_2(I) \cdot f_3(D),$$

where:
i. $r(t, y)$ models time-dependent replication and clearance rates,
ii. $f_1(u) = \frac{u}{1+\alpha u}$ models viral replication saturation,
iii. $f_2(I) = \frac{1}{1+\beta I}$ models viral suppression by immune cells (with $\beta > 0$),
iv. $f_3(D) = \frac{1}{1+\gamma D}$ models drug inhibitory effects (with $\gamma > 0$).

Assume $g \in C([0, T])$, and $I, D \in C([0, T])$ are known continuous functions. We aim to prove the existence and uniqueness of a continuous solution u and to provide an iterative approximation scheme.

Defining the Banach space:

$$X := \{u : [0, T] \to \mathbb{R} \mid u \text{ is continuous}\},$$

with the supremum norm:

$$\|u\| := \sup_{x \in [0,T]} |u(x)|.$$

Define the operator $T : X \to X$ as:

$$(Tu)(x) := g(x) + \mu \int_0^x \int_0^T r(t, y) f_1(u(y)) f_2(I(y)) f_3(D(y))\, dy\, dt.$$

For the mapping into X, since $g \in C([0, T])$, and the integrand is a product of continuous functions, the integral defines a continuous function. Hence, T maps X into itself.

For the Lipschitz property of the nonlinearity, for $f_1(u) = \dfrac{u}{1+\alpha u}$, the derivative is:

$$f_1'(u) = \frac{1}{(1+\alpha u)^2},$$

which is bounded above by 1 for $u \in [0, R]$. Thus, f_1 is Lipschitz with constant $L_1 \leq 1$ on bounded subsets.

The functions $f_2(I)$, $f_3(D)$ are continuous and bounded since $I, D \in C([0,T])$, with:

$$0 < f_2(I(y)), f_3(D(y)) \leq 1.$$

Define $K(y) := f_2(I(y))f_3(D(y)) \in C([0,T])$. Let:

$$L := \sup_{(t,y)\in[0,T]^2} |r(t,y)|, \quad M := \sup_{x\in[0,T]} \int_0^x \int_0^T dy\,dt = T^2.$$

For the contraction mapping, let $u, v \in X$. Then:

$$\begin{aligned}|(Tu)(x) - (Tv)(x)| &= \left|\mu \int_0^x \int_0^T r(t,y)\big(f_1(u(y)) - f_1(v(y))\big)K(y)\,dy\,dt\right| \\ &\leq |\mu| \cdot L \cdot \int_0^x \int_0^T |f_1(u(y)) - f_1(v(y))| \cdot |K(y)|\,dy\,dt \\ &\leq |\mu| \cdot L \cdot L_1 \cdot \|u - v\| \cdot \int_0^x \int_0^T |K(y)|\,dy\,dt \\ &\leq |\mu| L L_1 T^2 \|u - v\|.\end{aligned}$$

Thus,

$$\|Tu - Tv\| \leq q\|u - v\|, \quad \text{with } q := |\mu| L L_1 T^2.$$

For an application of the Banach fixed point theorem, if $q < 1$, then T is a contraction on X, and by the Banach Fixed Point Theorem:
i. There exists a unique $u^* \in X$ such that $T(u^*) = u^*$,
ii. Hence, u^* is the unique continuous solution to the integral equation.

For an iterative approximation, starting from $u_0(x) := g(x)$, define:

$$u_{n+1}(x) := Tu_n(x), \quad n \geq 0.$$

Then $u_n \to u^*$ in the supremum norm with the error estimate:

$$\|u_n - u^*\| \leq \frac{q^n}{1-q}\|u_1 - u_0\|.$$

In this section, under the assumptions:
i. $r(t,y)$, $I(y)$, $D(y)$ are continuous on $[0,T]$,
ii. $f_1(u)$ is Lipschitz on bounded intervals,
iii. $|\mu| L L_1 T^2 < 1$,
the viral load equation has a unique continuous solution u^*, and this solution can be approximated via an iterative scheme with guaranteed convergence.

Theorem 3.3. ***Existence, Uniqueness, and Iterative Approximation of Viral Load Equation***

Let $T > 0$ be fixed, and define the Banach space

$$X := \{u : [0,T] \to \mathbb{R} \mid u \text{ is continuous}\}$$

with the supremum norm $\|u\| := \sup_{x\in[0,T]} |u(x)|$.

Consider the integral equation for the viral load $u(x)$:

$$u(x) = g(x) + \mu \int_0^x \int_0^T k\big(t, y, u(y), I(y), D(y)\big)\, dy\, dt, \quad x \in [0, T],$$

where the kernel k has the separable structure

$$k(t, y, u, I, D) = r(t, y) \cdot f_1(u) \cdot f_2(I) \cdot f_3(D),$$

with the following properties:
i. $g \in C([0, T])$ *is a given continuous function.*
ii. $r \in C([0, T]^2)$ *is a continuous replication and clearance rate.*
iii. $I, D \in C([0, T])$ *are known continuous immune and drug concentration functions.*
iv. $f_1(u) = \frac{u}{1+\alpha u}$, *for some* $\alpha > 0$, *models viral saturation.*
v. $f_2(I) = \frac{1}{1+\beta I}$, $f_3(D) = \frac{1}{1+\gamma D}$, *with* $\beta, \gamma > 0$, *model immune and drug suppression respectively.*

Assume the following:

(1) f_1 *is Lipschitz continuous on bounded sets with Lipschitz constant* $L_1 < 1$,
(2) $K(y) := f_2(I(y)) f_3(D(y)) \in C([0, T])$, *and* $0 < K(y) < 1$,
(3) $L := \sup_{(t,y)\in[0,T]^2} |r(t, y)| < \infty$,
(4) *The contraction condition holds:*

$$|\mu| \cdot L \cdot L_1 \cdot T^2 < 1.$$

Then the following conclusions hold:

i. *Existence and Uniqueness: There exists a unique function* $u^* \in X$ *that satisfies the integral equation.*
ii. *Iterative Approximation: The sequence defined by*

$$u_0(x) := g(x), \quad u_{n+1}(x) := g(x) + \mu \int_0^x \int_0^T r(t, y) f_1(u_n(y)) f_2(I(y)) f_3(D(y))\, dy\, dt,$$

converges uniformly to $u^* \in X$.
iii. *Error Estimate: The convergence rate satisfies the inequality*

$$\|u_n - u^*\| \leq \frac{q^n}{1-q} \|u_1 - u_0\|, \quad \text{where } q := |\mu| L L_1 T^2 < 1.$$

4. Discussion

The nonlinear integral equation system that is derived in this work presents a new approach for modelling the coupled behaviour of viral load, immune response, and drug pharmacokinetics that is of great biological significance. With regard to the issue of control, while the theoretic aspect is ensured through fixed point theorems, the integral equation system does manage to cover the key aspects of temporal interactions. This is promising for further refinements to the approach.

The work also showed the importance and reliability of mixing Banach's Contraction Principle and Picard iterations in solving the various aspects of nonlinear integral equations in the medical and healthcare sector. One of the major advantages of the presented work is the clear derivation and establishment of the conditions of contraction in relevant biological parameters, kernel, and intervals, which is essential in proving the reliability of the model. Furthermore, the numerical methods provide assurance that the whole process is effective in solving the integral equations if the conditions of contraction are met. Moreover, the work highlights the development and extension of the fixed-point scheme to solve equations involving the dynamics of the viral load, which is essential in validating the reliability of the computations in the presence of some biologically relevant nonlinearities, including viral saturation, immunological suppression, and chemotherapeutic inhibition. In all, the work is important in improving the reliability and relevance of fixed-point theorems and their applications in various medical and healthcare mathematical model computations.

For future research, one can relax the contraction conditions by using wider ranges of parameters with different fixed-point tools, such as Schauder or Krasnosel'skii theorems. Stochastic effects and fractional-order kernels will provide realism in modelling uncertainty and memory. Further work could be done on data-driven parameter estimation and its subsequent validation against clinical datasets for better prediction capability. Also, extensions to coupled systems and spatially distributed models that is PDE-integral hybrids are worthy directions for further research. Finally, the development of efficient numerical algorithms and their software realization will enable large-scale simulations and real-time decision-making in healthcare applications.

5. Conclusion

This study used the Banach Contraction Principle in demonstrating the existence, uniqueness, and numerical solution of nonlinear Volterra-Fredholm integral equations, particularly in healthcare modelling problems. The nonlinear integral equations were presented as an integral operator in the Banach space of continuous functions, and sufficient conditions, particularly the Lipschitz continuous condition on the kernel as well as bounds on the model parameters, were presented through nonlinear integral equations, as stated in the study. Furthermore, the Picard iteration technique was employed in demonstrating the numerical solution, together with explicit examples that mirrored nonlinear integral equations in healthcare problems. Lastly, the study was able to demonstrate a nonlinear model on the viral load, immunity, and drug response, particularly in healthcare modelling, as a means of showcasing the generalization on a nonlinear integral operator that satisfied the requirements of a contraction operator. The general use of nonlinear integral equations in healthcare problems was emphasized as a generalization in the study.

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