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## C\*-ALGEBRA-VALUED $b$ -METRIC-LIKE SPACE AND SOME FIXED POINT THEOREMS

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ABSTRACT. In this article, we initiate the notion of  $C^*$ -algebra valued  $b$ -metric-like space. The Banach and Kannan fixed point results are examined in such space. Contrasting examples are formulated to show that the proposed ideas herein are novel and extend some important significant results in the literature.

### 1. INTRODUCTION

The exploration of novel spaces and their applications has formed a fascinating field of research within mathematics. On this development, the concept of quasi-metric space with relaxed triangle inequality is currently gaining a lot of attention due to its enormous applicability in nonlinear analysis, differential equations and integral equations where standard metrics fail. In this direction, Bakhtin [5] launched the idea of contraction mapping principle in quasi-metric space and subsequently, Czerwik [7] extended the notion to  $b$ -metric space. Since then, fixed point theory for single-valued and multi-valued operators in  $b$ -metric space has been the subject of numerous publications. On the other hand, the idea of dislocated metric space, where the metric is said to have nonzero self distance was launched by Hitzler and Seda [9] in the sense of logic programming. In line with this, Matthew [16] brought-up the frame of partial metric space and established the Banach contraction principle in this setting. As a refinement of partial metric space, Amini-Harandi [4] presented an extension of the concept of partial metric space known as a metric-like space and proved some fixed point theorem in this domain. Following [4, 7] Alghamdi et al. [3] initiated the body of  $b$ -metric-like space and explored the existence and uniqueness of fixed point in a  $b$ -metric-like space and partially ordered  $b$ -metric-like space. There after, some topological structures and some fixed point results in this space were put forth by Hussain et al.[11]. In literature, metric-like space are referred to as dislocated spaces.

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The concept of  $C^*$ -algebra (see [18]) is an interesting area of study due to vast utility in quantum theory and statistical mechanics. Along this lane, Ma et al. [12] proposed the concept of a  $C^*$ -algebra-valued metric space and established novel fixed point theorems for contractive and non-expansive mappings within a complete  $C^*$ -algebra-valued metric space. As an improvement of [12], Ma and Jiang [13] developed the premises of  $C^*$ -algebra-valued  $b$ -metric space and proved some fixed point results with application to an integral-type operator. Later, Chandok et al. [6] offered a novel generalisation of partial metrics in  $C^*$ -algebras and demonstrated some fixed point results with  $C_*$ -class functions. Mlaiki et al. [17] enlarged the class of  $C^*$ -algebra-valued partial metric space as well as the class of  $C^*$ -algebra-valued  $b$ -metric space by introducing the class of  $C^*$ -algebra-valued partial  $b$ -metric space and presented some fixed point results in this context. Following this, Maheswari et al. [15] proved some common coupled fixed point theorems on complete  $C^*$ -algebra-valued partial  $b$ -metric space. Recently, Ahmad et al. [1] initiated the framework of  $C^*$ -algebra-valued metric-like space, some topological features of the novel space and discussed the Banach and Kannan-type fixed point results with an application. Furthermore, Madadi et al. [14] brought up the frame of  $C^*$ -algebra-valued fuzzy metric space and studied some topological features in such a space. For a monograph on various development in fixed point results of  $C^*$ -algebra-valued metric space, consult [2].

It is observed from an overview of the current literature that little or probably no work has been conducted in the context of  $C^*$ -algebra-valued  $b$ -metric-like space. To reduce this disparity, this study commences the idea of  $C^*$ -algebra-valued  $b$ -metric-like space. Some fixed point results are discussed in such space. Examples are formulated to show how the obtained results enhance a few pertinent ones in the literature. It is worthy of note that metric-like space are significant in fixed point theory as they expand the applicability of contraction principles and fixed point theorems to contexts where standard metric axioms are not effective. This promotes the effectiveness of fixed point analysis in chaotic systems, computational models and generalised spaces. Moreover, the rationale for examining  $b$ -metric space is to improve the limitations of traditional metric space by incorporating a scaling constant into the triangle inequality. This enables the extension of fixed point theory and convergence results to more general and realistic settings, with extensive applications in mathematics, computer science, and applied sciences. Consequently, an integration of these concepts, as suggested in this work, represents a compelling topic of research.

## 2. PRELIMINARIES

In this manuscript,  $\mathbb{A}$  refers to a unital  $C^*$ -algebra with a unit  $I$  and take  $\mathbb{A}_h = \{\alpha \in \mathbb{A} : \alpha = \alpha^*\}$ . An element  $\alpha \in \mathbb{A}$  is said to be a positive element, denoted by  $\alpha \succeq \theta$ , if  $\alpha \in \mathbb{A}_h$  and  $\sigma(\alpha) \subset \mathbb{R}_+ = [0, \infty)$ , where  $\sigma(\alpha)$  is the spectrum of  $\alpha$ . Using positive elements, a partial ordering  $\preceq$  on  $\mathbb{A}_h$  is defined as follows:  $\alpha \preceq \beta$  if and only if  $\beta - \alpha \succeq \theta$ , where  $\theta$  denoted the zero element in  $\mathbb{A}$ . Henceforth,  $\mathbb{A}_+$  denotes the set  $\{\alpha \in \mathbb{A} : \alpha \succeq \theta\}$  and  $|\alpha| = (\alpha\alpha^*)^{\frac{1}{2}}$ .

Ma et. al [12] introduced the idea of  $C^*$ -algebra-valued metric space as follows.

**Definition 2.1.** [12] Let  $X$  be a nonempty set. Suppose that the mapping  $d : X \times X \longrightarrow \mathbb{A}$  satisfies:

- 1)  $\theta \preceq d(x, y)$  and  $d(x, y) = \theta \Leftrightarrow x = y$  for all  $x, y \in X$ ;
- 2)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- 3)  $d(x, y) \preceq d(x, z) + d(z, y)$  for all  $x, y, z \in X$ .

Then  $d$  is termed as a  $C^*$ -algebra-valued metric on  $X$  and  $(X, \mathbb{A}, d)$  is termed as a  $C^*$ -algebra-valued metric space.

**Definition 2.2.** [12] Suppose that  $(X, \mathbb{A}, d)$  is a  $C^*$ -algebra-valued metric space. We call a mapping  $T : X \longrightarrow X$  a  $C^*$ -algebra-valued contractive mapping on  $X$ , if there exists an  $A \in \mathbb{A}$  with  $\|A\| < 1$  such that

$$d(Tx, Ty) \preceq A^*d(x, y)A, \quad \forall x, y \in X. \quad (2.1)$$

**Lemma 2.3.** [8] Suppose that  $\mathbb{A}$  is a unital  $C^*$ -algebra with a unit  $I$ .

- i) If  $\alpha \in \mathbb{A}_+$  with  $\|\alpha\| < \frac{1}{2}$ , then  $I - \alpha$  is invertible and  $\|\alpha(I - \alpha)^{-1}\| < 1$ .
- ii) Suppose that  $\alpha, \beta \in \mathbb{A}$  with  $\alpha, \beta \succeq \theta$  and  $\alpha\beta = \beta\alpha$ , then  $\alpha\beta \succeq \theta$ .
- iii) By  $\mathbb{A}'$  we denote the set  $\{\alpha \in \mathbb{A} : \alpha\beta = \beta\alpha, \forall \beta \in \mathbb{A}\}$ . Let  $\alpha \in \mathbb{A}$ , if  $\beta, \gamma \in \mathbb{A}$  with  $\beta \succeq \gamma \succeq \theta$  and  $I - \alpha \in \mathbb{A}'_+$  is invertible operator, then  $(I - \alpha)^{-1}\beta \succeq (I - \alpha)^{-1}\gamma$ .

In 2015, Ma and Jiang [13] initiated the concept of  $C^*$ -algebra-valued  $b$ -metric space as follows.

**Definition 2.4.** [13] Let  $X$  be a nonempty set and  $S \in \mathbb{A}'$  such that  $S \succeq I$ . Suppose the mapping  $d : X \times X \longrightarrow \mathbb{A}$  satisfies:

- ( $d_1$ )  $\theta \preceq d(x, y)$  for all  $x, y \in X$  and  $d(x, y) = \theta \Leftrightarrow x = y$ ;
- ( $d_2$ )  $d(x, y) = d(y, x)$  for all  $x, y \in X$ ;
- ( $d_3$ )  $d(x, y) \preceq S(d(x, z) + d(z, y))$  for all  $x, y, z \in X$ .

Then  $d$  is called a  $C^*$ -algebra-valued  $b$ -metric on  $X$  and  $(X, \mathbb{A}, d)$  is a  $C^*$ -algebra-valued  $b$ -metric space.

The notion of partial metric was extended to  $C^*$ -algebra by Chandok et al. [6]. Thereafter, [17] presented the idea of  $C^*$ -algebra-valued partial  $b$ -metric space as follows;

**Definition 2.5.** [17] Let  $X \neq \emptyset$  and  $S \in \mathbb{A}$  such that  $S \succ I$ . A mapping  $\wp : X \times X \longrightarrow \mathbb{A}$  is called a  $C^*$ -algebra-valued partial  $b$ -metric on  $X$ , if it satisfies the following for all  $x, y, z \in X$ :

- (i)  $\wp(x, y) \succeq \theta$  and  $x = y \Leftrightarrow \wp(x, x) = \wp(y, y) = \wp(x, y)$ ;
- (ii)  $\wp(x, x) \preceq \wp(y, x)$ ;
- (iii)  $\wp(x, y) = \wp(y, x)$ ;
- (iv)  $\wp(x, y) \preceq S[\wp(x, z) + \wp(z, y)] - \wp(z, z)$ .

The triplet  $(X, \mathbb{A}, d)$  is called a  $C^*$ -algebra-valued partial  $b$ -metric space.

Following [4, 6], Ahmad et al. [1] launched the concept of  $C^*$ -algebra-valued metric-like space as follows.

**Definition 2.6.** [1] Let  $X$  be a nonempty set. A mapping  $d : X \times X \longrightarrow \mathbb{A}$  is called as a  $C^*$ -algebra-valued metric-like if for all  $x, y, z \in X$ ;

- (i)  $d(x, y) = \theta \Rightarrow x = y$ ;
- (ii)  $d(x, y) = d(y, x)$ ;
- (iii)  $d(x, y) \preceq d(x, z) + d(z, y)$ .

Then the pair  $(X, \mathbb{A}, d)$  is said to be a  $C^*$ -algebra-valued metric-like space .

**Definition 2.7.** [1] Let  $(X, \mathbb{A}, d)$  be a  $C^*$ -algebra-valued metric-like space . A sequence  $\{x_n\}_n$  in  $X$  is said to be  $d$ -convergent to  $x$  if it converges to  $x$  in the  $C^*$ -algebra-valued metric-like topology  $y_d$ , i.e., for every  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$  such that

$$\|d(x_n, x) - d(x, x)\| < \epsilon \quad \forall n \geq N \quad \text{or} \quad \lim_{n \rightarrow \infty} d(x_n, x) = d(x, x).$$

The analogue of Banach and Kannan-type fixed point results was presented by [1] as follows.

**Theorem 2.8.** [1] Let  $(X, \mathbb{A}, d)$  be a  $C^*$ -algebra-valued metric-like space and  $T$  be a contractive mapping on  $X$ , that is, if there exists an  $A \in \mathbb{A}$  with  $\|A\| < 1$  such that

$$d(Tx, Ty) \preceq A^*d(x, y)A, \quad \forall x, y \in X.$$

Then, there exists a unique fixed point of  $T$  in  $X$ .

**Theorem 2.9.** [1] Let  $(X, \mathbb{A}, d)$  be a complete  $C^*$ -algebra-valued metric-like space and  $T : X \longrightarrow X$  be a self map satisfying

$$d(Tx, Ty) \preceq A[d(x, Tx) + d(y, Ty)], \quad x, y \in X, \quad (2.2)$$

with  $\|A\| < \frac{1}{2}$ ,  $A \in \mathbb{A}'_+$ . Then  $T$  has a unique fixed point in  $X$ .

### 3. MAIN RESULTS

In this section, we propose the idea of  $C^*$ -algebra-valued  $b$ -metric-like space by introducing a constant operator  $S \succeq I$  to the triangle inequality of  $C^*$ -algebra-valued metric-like space in the sense of [1]. The present study improves and generalizes the notion of metric-like space due to Amini-Harandi [4] and the principal findings of [1, 13]. We adapt the idea of the following results from [1].

**Definition 3.1.** Let  $X$  be a nonempty set and  $S \in \mathbb{A}'$  such that  $S \succeq I$ . A mapping  $d : X \times X \longrightarrow \mathbb{A}$  is called a  $C^*$ -algebra-valued  $b$ -metric-like if for all  $x, y, z \in X$ ;

- $(d_{b1})$   $d(x, y) = \theta \Rightarrow x = y$ ;
- $(d_{b2})$   $d(x, y) = d(y, x)$ ;
- $(d_{b3})$   $d(x, y) \preceq S(d(x, z) + d(z, y))$ .

Then, the pair  $(X, \mathbb{A}, d)$  is called a  $C^*$ -algebra-valued  $b$ -metric-like space .

*Remark 3.2.*

- (i) A  $C^*$ -algebra-valued  $b$ -metric-like  $d$  on  $X$  fulfills all the requirements of a  $C^*$ -algebra-valued  $b$ -metric except that  $d(x, x)$  may be positive for  $x \in X$ .

- (ii) If  $S = I$ , then  $(X, \mathbb{A}, d)$  becomes a  $C^*$ -algebra-valued metric-like space on  $X$ .
- (iii) If  $\mathbb{A} = \mathbb{R}$  and  $S = I$ , the  $C^*$ -algebra-valued  $b$ -metric-like space is just the ordinary metric-like space in the sense of [4].

In the following examples, we illustrate that a  $C^*$ -algebra-valued metric-like space is not necessary a  $C^*$ -algebra-valued  $b$ -metric-like.

Recall that Jensen's inequality is given by  $\left(\frac{\alpha+\beta}{2}\right)^p \leq \left(\frac{\alpha^p+\beta^p}{2}\right)$  for all  $p \geq 1$ .

**Example 3.3.** Let  $X = \mathbb{R}^+$ ,  $\mathbb{A} = \mathbb{M}_2(\mathbb{R}^+)$  with  $\|A\| = \max\{a_1, a_2, a_3, a_4\}$ . Define  $d : X \times X \rightarrow \mathbb{A}$  by

$$d(x, y) = \begin{bmatrix} |x+y|^p & 0 \\ 0 & |x+y|^p \end{bmatrix}, \text{ for } p > 1.$$

A natural ordering on  $\mathbb{A}$  is given by

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \eta \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \Leftrightarrow \alpha = w, \beta = x, \gamma = y, \eta = z.$$

Then we show that  $(X, \mathbb{A}, d)$  is a  $C^*$ -algebra-valued  $b$ -metric-like space.

If  $d(x, y) = \theta$ , then  $|x+y|^p = 0 \Rightarrow x = y = 0$ .

If  $x = y$ , then  $d(x, y) = d(x, x) = \begin{bmatrix} |x+x|^p & 0 \\ 0 & |x+x|^p \end{bmatrix} = \begin{bmatrix} |2x|^p & 0 \\ 0 & |2x|^p \end{bmatrix} \neq \theta$ , that

is  $(d_{b1})$  holds. To show  $(d_{b3})$ , we have

$$\begin{aligned} d(x, y) &= \begin{bmatrix} |x+y|^p & 0 \\ 0 & |x+y|^p \end{bmatrix} = \begin{bmatrix} |x+z-z+y|^p & 0 \\ 0 & |x+z-z+y|^p \end{bmatrix} \\ &= \begin{bmatrix} |x+z+y-z|^p & 0 \\ 0 & |x+z+y-z|^p \end{bmatrix}. \end{aligned}$$

By Jensen's inequality, we have

$$\begin{aligned} d(x, y) &\preceq \begin{bmatrix} 2^{p-1} |(x+z)^p + (y-z)^p| & 0 \\ 0 & 2^{p-1} |(x+z)^p + (y-z)^p| \end{bmatrix} \\ &\preceq \begin{bmatrix} 2^{p-1} (|x+z|^p + |y-z|^p) & 0 \\ 0 & 2^{p-1} (|x+z|^p + |y-z|^p) \end{bmatrix} \\ &= \begin{bmatrix} 2^{p-1} & 0 \\ 0 & 2^{p-1} \end{bmatrix} \begin{bmatrix} |x+z|^p + |y-z|^p & 0 \\ 0 & |x+z|^p + |y-z|^p \end{bmatrix} \\ &= \begin{bmatrix} 2^{p-1} & 0 \\ 0 & 2^{p-1} \end{bmatrix} \left( \begin{bmatrix} |x+z|^p & 0 \\ 0 & |x+z|^p \end{bmatrix} + \begin{bmatrix} |y-z|^p & 0 \\ 0 & |y-z|^p \end{bmatrix} \right) \\ &= S(d(x, z) + d(z, y)). \end{aligned}$$

Hence,  $d$  is a  $C^*$ -algebra-valued  $b$ -metric-like on  $X$  with  $S = \begin{bmatrix} 2^{p-1} & 0 \\ 0 & 2^{p-1} \end{bmatrix}$  and not a  $C^*$ -algebra-valued partial  $b$ -metric since,  $d(3, 3) \succ d(2, 3)$ .

**Example 3.4.** Let  $X = \mathbb{R}$ ,  $\mathbb{A} = \mathbb{R}^2$  and  $d : X \times X \rightarrow \mathbb{A}$  be a mapping defined by  $d(x, y) = ((\max\{|x|, |y|\})^2, 0)$ . Then  $(X, \mathbb{A}, d)$  is a  $C^*$ -algebra-valued  $b$ -metric-like space. To see this, take  $d(x, y) = \theta$ , then  $((\max\{|x|, |y|\})^2, 0) = 0 \Rightarrow \max\{|x|, |y|\} = 0 \Rightarrow |x| = |y| = 0 \Rightarrow x = y = 0$ .

If  $x = y$ , then  $d(x, y) = d(x, x) = ((\max\{|x|, |x|\})^2, 0) = (|x|^2, 0) \neq \theta$  which gives  $(d_{b1})$ . Now,

$$\begin{aligned} d(x, y) &= ((\max\{|x|, |y|\})^2, 0) \preceq ((|x| + |y|)^2, 0) \\ &\preceq (2(|x|^2 + |y|^2), 0) \\ &\preceq (2(\max\{|x|^2, |z|^2\} + \max\{|y|^2, |z|^2\}), 0) \\ &\preceq (2((\max\{|x|, |z|\})^2 + (\max\{|y|, |z|\})^2), 0) \\ &= (2, 2) \left( ((\max\{|x|, |z|\})^2, 0) + ((\max\{|y|, |z|\})^2, 0) \right) \\ &= S(d(x, z) + d(z, y)), \end{aligned}$$

proving  $(d_{b3})$ . Hence,  $d$  is a  $C^*$ -algebra-valued  $b$ -metric-like on  $X$  with  $S = (2, 2)$ .

**Example 3.5.** Let  $X = L^\infty(\Omega)$  and  $\mathcal{H} = L^2(\Omega)$ , where  $\Omega$  is a Lebesgue measurable set,  $\mathcal{B}(\mathcal{H})$  is the set of all bounded linear operators in  $\mathcal{H}$ . Clearly,  $\mathcal{B}(\mathcal{H})$  is a  $C^*$ -algebra. Define  $d : X \times X \rightarrow \mathcal{B}(\mathcal{H})$  by

$$d(x, y) = \pi_{|x+y|^2}, \quad \forall x, y \in X.$$

$\pi_h : \mathcal{H} \rightarrow \mathcal{H}$  is a multiplication operator defined by  $\pi_h(\varpi) = h \cdot \varpi$ ,  $\varpi \in \mathcal{H}$ .

Then  $d$  is a  $C^*$ -algebra-valued  $b$ -metric-like.

**Proposition 3.6.** [1] *Let  $(X, \mathbb{A}, d)$  be a  $C^*$ -algebra-valued  $b$ -metric-like space .*

- i) *If  $d(x, y) = \theta$ , then  $d(x, x) = d(y, y) = \theta$ ;*
- ii) *if  $x \neq y$ , then  $d(x, y) \succ \theta$ ;*
- iii) *if  $\{x_n\}$  is a sequence in  $X$  such that  $\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \theta$  then,  $\lim_{n \rightarrow \infty} d(x_n, x_n) = \lim_{n \rightarrow \infty} d(x_{n+1}, x_{n+1}) = \theta$ ;*
- iv)  *$d(x, x) \preceq \frac{2S}{q} \sum_{i=1}^q d(x, x_i)$  holds for all  $x_i, x \in X$ , where  $1 \leq i \leq q$ .*
- v) *If  $\{x_n\}$  is a sequence in  $X$  such that  $x_n \rightarrow x$  as  $n \rightarrow \infty$  and  $d(x, x) = \theta$ , then  $\lim_{n \rightarrow \infty} d(x_n, y) = Sd(x, y) \forall y \in X$ .*

*Proof.* For the proof of (i) and (ii) see [1].

iii) By triangle inequality,

$$\begin{aligned} d(x_n, x_n) &\preceq S(d(x_n, x_{n+1}) + d(x_{n+1}, x_n)) \\ &= 2Sd(x_{n+1}, x_n). \end{aligned} \tag{3.1}$$

Taking the limit as  $n \rightarrow \infty$  in (3.1),

$$\theta \preceq \lim_{n \rightarrow \infty} d(x_n, x_n) \preceq 2S \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \theta.$$

Similarly,

$$\theta \preceq \lim_{n \rightarrow \infty} d(x_{n+1}, x_{n+1}) \preceq 2S \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \theta.$$

iv) By  $(d_{b3})$ ,  $d(x, x) \preceq 2Sd(x, x_i)$ , where  $1 \leq i \leq q$ . Then,

$$\sum_{i=1}^q d(x, x) \preceq 2S \sum_{i=1}^q d(x, x_i),$$

from which we have  $d(x, x) \preceq \frac{2S}{q} \sum_{i=1}^q d(x, x_i)$ .

v) From  $(d_{b3})$ ,  $d(x_n, y) \preceq S(d(x_n, x) + d(x, y))$ . This implies that

$$d(x_n, y) - Sd(x, y) \preceq Sd(x_n, x). \quad (3.2)$$

By (iv) and letting  $q \rightarrow \infty$ , the expression (3.2) gives  $\lim_{n \rightarrow \infty} d(x_n, y) = Sd(x, y)$ .

□

In what follows, we prove Banach-type and Kanan-type fixed point theorems in the context of  $C^*$ -algebra-valued  $b$ -metric-like space .

**Definition 3.7.** Let  $(X, \mathbb{A}, d)$  be a  $C^*$ -algebra-valued  $b$ -metric-like space . A mapping  $T : X \rightarrow X$  is a  $C^*$ -algebra-valued contractive mapping on  $X$ , if there exists an  $A \in \mathbb{A}$  with  $\|A\| < 1$  such that  $\forall x, y \in X$ :

$$d(Tx, Ty) \preceq A^*d(x, y)A. \quad (3.3)$$

**Theorem 3.8.** Let  $(X, \mathbb{A}, d)$  be a complete  $C^*$ -algebra-valued  $b$ -metric-like space with  $S \succeq I$  and  $T$  be a contractive mapping on  $X$ . Hence,  $T$  has a unique fixed point of  $T$  in  $X$ .

*Proof.* If  $A = \theta$ , then the result is trivial. Suppose  $A \neq \theta$ , then we choose  $x_0 \in X$  and set  $x_1 = Tx_0$ ,  $x_2 = Tx_1, \dots, x_{n+1} = Tx_n$ ,  $n = 1, 2, \dots$ . Recall that in a  $C^*$ -algebra, if  $\alpha, \beta \in \mathbb{A}_+$  and  $\alpha \preceq \beta$ , then for any  $x \in \mathbb{A}$ ,  $x^*\alpha x \preceq x^*\beta x$ . Thus, by (3.3),

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \preceq A^*d(x_n, x_{n-1})A \preceq (A^*)^2d(x_{n-1}, x_{n-2})A^2 \\ &\preceq \dots \preceq (A^*)^n d(x_1, x_0)A^n. \end{aligned}$$

For  $p > 1$  and denoting  $R_0 = d(x_1, x_0)$ , it follows that

$$\begin{aligned} d(x_{n+p}, x_n) &\preceq S(d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_n)) \\ &\preceq Sd(x_{n+p}, x_{n+p-1}) + S^2(d(x_{n+p-1}, x_{n+p-2}) + d(x_{n+p-2}, x_n)) \\ &\preceq Sd(x_{n+p}, x_{n+p-1}) + S^2d(x_{n+p-1}, x_{n+p-2}) + \\ &\quad S^3d(x_{n+p-2}, x_{n+p-3}) + S^3d(x_{n+p-3}, x_n) \\ &\preceq Sd(x_{n+p}, x_{n+p-1}) + S^2d(x_{n+p-1}, x_{n+p-2}) + \dots \\ &\quad + S^{p-1}d(x_{n+2}, x_{n+1}) + S^{p-1}d(x_{n+1}, x_n). \end{aligned}$$

This demonstrates that

$$\begin{aligned} d(x_{n+p}, x_n) &\preceq S(A^*)^{n+p-1}R_0A^{n+p-1} + S^2(A^*)^{n+p-2}R_0A^{n+p-2} + \dots \\ &\quad + S^{p-1}(A^*)^{n+1}R_0A^{n+1} + S^{p-1}(A^*)^nR_0A^n \end{aligned} \quad (3.4)$$

and

$$\begin{aligned} d(x_n, x_n) &\preceq S(d(x_n, x_{n+1}) + d(x_{n+1}, x_n)) \\ &= S(d(Tx_{n-1}, Tx_n) + d(Tx_n, x_{n-1})) \\ &\preceq 2SA^*d(x_{n-1}, x_n)A \\ &\preceq \dots \\ &\preceq 2S(A^*)^nR_0A^n. \end{aligned}$$

This gives

$$\frac{1}{2}d(x_n, x_n) \preceq S(A^*)^n R_0 A^n. \quad (3.5)$$

Similarly,

$$\begin{aligned} d(x_{n+p}, x_{n+p}) &\preceq S\left(d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p})\right) \\ &= S\left(d(Tx_{n+p-1}, Tx_{n+p-2}) + d(Tx_{n+p-2}, x_{n+p-1})\right) \\ &\preceq 2S(A^*)^{n+p-1} R_0 A^{n+p-1}, \end{aligned}$$

that is,

$$\frac{1}{2}d(x_{n+p}, x_{n+p}) \preceq S(A^*)^{n+p-1} R_0 A^{n+p-1}. \quad (3.6)$$

Using (3.4), (3.5) and (3.6), we obtain

$$\left\| d(x_n, x_{n+p}) - \frac{1}{2}d(x_n, x_n) - \frac{1}{2}d(x_{n+p}, x_{n+p}) \right\|$$

$$\begin{aligned}
&\leq \left\| S^2(A^*)^{n+p-2}R_0A^{n+p-2} + S^3(A^*)^{n+p-3}R_0A^{n+p-3} + \dots \right. \\
&\quad \left. + S^{p-1}(A^*)^{n+1}R_0A^{n+1} + (A^*)^nR_0A^n(S^{p-1} - S) \right\| \\
&= \left\| \sum_{k=2}^{p-1} S^k(A^*)^{n+p-k}R_0A^{n+p-k} + (A^*)^nR_0A^n(S^{p-1} - S) \right\| \\
&\preceq \left\| \sum_{k=2}^{p-1} \left( S^{\frac{k}{2}}A^{n+p-k}R_0^{\frac{1}{2}} \right)^* \left( S^{\frac{k}{2}}A^{n+p-k}R_0^{\frac{1}{2}} \right) \right\| + \\
&\quad \left\| \left( A^nR_0^{\frac{1}{2}}(S^{p-1} - S)^{\frac{1}{2}} \right)^* \left( A^nR_0^{\frac{1}{2}}(S^{p-1} - S)^{\frac{1}{2}} \right) \right\| \\
&= \left\| \sum_{k=2}^{p-1} \left| S^{\frac{k}{2}}A^{n+p-k}R_0^{\frac{1}{2}} \right|^2 \right\| + \left\| \left| A^nR_0^{\frac{1}{2}}(S^{p-1} - S)^{\frac{1}{2}} \right|^2 \right\| \\
&\preceq \sum_{k=2}^{p-1} \left\| S^{\frac{k}{2}}A^{n+p-k}R_0^{\frac{1}{2}} \right\|^2 I + \left\| A^nR_0^{\frac{1}{2}}(S^{p-1} - S)^{\frac{1}{2}} \right\|^2 I \\
&\preceq \|R_0^{\frac{1}{2}}\|^2 \sum_{k=2}^{p-1} \|S^k\| \|A\|^{2(n+p-k)} + \|R_0^{\frac{1}{2}}\|^n \|A\|^{2n} \|(S^{p-1} - S)\| \\
&= \|R_0\| \|A\|^{2(n+p)} \sum_{k=2}^{p-1} \left( \frac{\|S\|}{\|A\|^2} \right)^k + \|R_0\| \|A\|^{2n} \|(S^{p-1} - S)\| \\
&= \|R_0\| \|A\|^{2(n+p)} \frac{\|S\|^2 (\|S\|^{p-1} - \|A\|^{2(p-1)})}{\|A\|^{2p}\|S\| - \|A\|^{2(p+1)}} + \|R_0\| \|A\|^{2n} \|(S^{p-1} - S)\| \\
&= \|R_0\| \|A\|^{2n} \frac{(\|S\|^{p+1} - \|S\|^2\|A\|^{2(p-1)})}{\|S\| - \|A\|^2} + \|R_0\| \|A\|^{2n} \|(S^{p-1} - S)\| \\
&\longrightarrow \infty \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Therefore,  $\{x_n\}$  is a Cauchy sequence with respect to  $\mathbb{A}$ . By the completeness of  $(X, \mathbb{A}, d)$ , there exists an  $x \in X$  such that

$$\lim_{n \rightarrow \infty} d(x_n, x) = d(Tx_{n-1}, x) = d(x, x).$$

Now, we show that  $x$  is a fixed point of  $T$ . For this, consider:

$$\begin{aligned}
d(Tx, x) &\preceq S(d(Tx, Tx_{n-1}) + d(Tx_{n-1}, x)) \\
&\preceq SA^*d(x, x_{n-1})A + Sd(x_n, x).
\end{aligned}$$

As  $n \rightarrow \infty$  and by Proposition 3.6(iv), we have

$$\begin{aligned}
d(Tx, x) &\preceq Sd(x, x) + Sd(x, x) = 2Sd(x, x) \\
&\preceq 2S \frac{2S}{q} \sum_{i=1}^q d(x, x_i).
\end{aligned}$$

Letting  $q \rightarrow \infty$ , we obtain

$$d(Tx, x) \preceq \theta \Rightarrow d(Tx, x) = \theta \Rightarrow x = Tx.$$

Hence,  $T$  has a fixed point in  $X$ .

To show uniqueness, suppose  $y (\neq x)$  is another fixed point of  $T$ . Then,

$$d(x, y) = d(Tx, Ty) \preceq A^*d(x, y)A.$$

Taking the norm on both sides, we get

$$\begin{aligned} 0 \leq \|d(x, y)\| &= \|d(Tx, Ty)\| \leq \|A^*d(x, y)A\| \\ &\leq \|A^*\| \|d(x, y)\| \|A\| \\ &= \|A\|^2 \|d(x, y)\| < \|d(x, y)\|, \end{aligned}$$

a contradiction, so  $d(x, y) = \theta$  and  $x = y$ , hence uniqueness holds.  $\square$

**Theorem 3.9.** *Let  $(X, \mathbb{A}, d)$  be a complete  $C^*$ -algebra-valued  $b$ -metric-like space with  $S \succeq I$  and  $T : X \rightarrow X$  be a self map satisfying:*

$$d(Tx, Ty) \preceq A[d(x, Tx) + d(y, Ty)], \quad x, y \in X, \quad (3.7)$$

with  $\|A\| < \frac{1}{2}$ ,  $A \in \mathbb{A}'_+$ . Then  $T$  possesses a unique fixed point in  $X$ .

*Proof.* If  $A = \theta$ , then the result is obvious. Suppose  $A \neq \theta$ , we choose  $x_0 \in X$  and set  $x_1 = Tx_0$ ,  $x_2 = Tx_1, \dots, x_{n+1} = Tx_n$ ,  $n = 1, 2, \dots$ . Since  $A \in \mathbb{A}'_+$ , then  $A[d(x, Tx) + d(y, Ty)]$  is also a positive element. So,

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n+1}) &\preceq A[d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})] \\ &= A[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]. \end{aligned}$$

$$\text{This implies } (I - A)d(x_{n+1}, x_n) \preceq Ad(x_{n-1}, x_n).$$

Since  $A \in \mathbb{A}'_+$  with  $\|A\| < \frac{1}{2}$ , then  $(I - A)^{-1} \in \mathbb{A}'_+$  and  $A(I - A)^{-1} \in \mathbb{A}'_+$  with  $\|A(I - A)^{-1}\| < 1$ . Then, taking  $\mathcal{D} = A(I - A)^{-1}$ , we have

$$\begin{aligned} d(x_{n+1}, x_n) &\preceq A(I - A)^{-1}d(x_n, x_{n-1}) \\ &= \mathcal{D}d(x_n, x_{n+1}) \preceq \mathcal{D}^2d(x_{n-2}, x_{n-1}) \\ &\preceq \dots \preceq \mathcal{D}^n d(x_0, x_1). \end{aligned}$$

$$\begin{aligned} d(x_{n+p}, x_n) &\preceq S(d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_n)) \\ &\preceq Sd(x_{n+p}, x_{n+p-1}) + S^2(d(x_{n+p-1}, x_{n+p-2}) + d(x_{n+p-2}, x_n)) \\ &\preceq Sd(x_{n+p}, x_{n+p-1}) + S^2d(x_{n+p-1}, x_{n+p-2}) + \\ &\quad S^3d(x_{n+p-2}, x_{n+p-3}) + S^3d(x_{n+p-3}, x_n) \\ &\preceq Sd(x_{n+p}, x_{n+p-1}) + S^2d(x_{n+p-1}, x_{n+p-2}) + \dots + \\ &\quad S^{p-1}d(x_{n+2}, x_{n+1}) + S^{p-1}d(x_{n+1}, x_n), \end{aligned}$$

that is,

$$\begin{aligned} d(x_{n+p}, x_n) &\preceq S\mathcal{D}^{n+p-1}R_0 + S^2\mathcal{D}^{n+p-2}R_0 + \dots + \\ &\quad S^{p-1}\mathcal{D}^{n+1}R_0 + S^{p-1}\mathcal{D}^nR_0. \end{aligned} \quad (3.8)$$

Also,

$$\begin{aligned}
d(x_n, x_n) &\preceq S(d(x_n, x_{n+1}) + d(x_{n+1}, x_n)) \\
&= S(d(Tx_{n-1}, Tx_n) + d(Tx_n, x_{n-1})) \\
&\preceq SA(d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)) + \\
&\quad SA(d(x_n, Tx_n) + d(x_{n-1}, Tx_{n-1})) \\
&= 2SAd(x_{n-1}, x_n) + 2SAd(x_n, x_{n+1}) \\
&\preceq 2SAD^{n-1}R_0 + 2SAD^n R_0,
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{1}{2}d(x_n, x_n) &\preceq SAR_0\mathcal{D}^{n-1} + SAR_0\mathcal{D}^n \\
&\preceq SR_0\mathcal{D}^{n-1} + SR_0\mathcal{D}^n.
\end{aligned} \tag{3.9}$$

Similarly,

$$\begin{aligned}
d(x_{n+p}, x_{n+p}) &\preceq S(d(x_{n+p}, x_{n+p-1}) + d(x_{n+p-1}, x_{n+p})) \\
&= S(d(Tx_{n+p-1}, Tx_{n+p-2}) + d(Tx_{n+p-2}, x_{n+p-1})) \\
&\preceq SA(d(x_{n+p-1}, Tx_{n+p-1}) + d(x_{n+p-2}, Tx_{n+p-2})) \\
&\quad + SA(d(x_{n+p-2}, Tx_{n+p-2}) + d(x_{n+p-1}, Tx_{n+p-1})) \\
&= 2SAd(x_{n+p-1}, x_{n+p}) + 2SAd(x_{n+p-2}, x_{n+p-1}) \\
&\preceq 2SAD^{n+p-1}R_0 + 2SAD^{n+p-2}R_0,
\end{aligned}$$

which gives

$$\begin{aligned}
\frac{1}{2}d(x_{n+p}, x_{n+p}) &\preceq SAR_0\mathcal{D}^{n+p-1} + SAR_0\mathcal{D}^{n+p-2} \\
&\preceq SR_0\mathcal{D}^{n+p-1} + S^2R_0\mathcal{D}^{n+p-2}.
\end{aligned} \tag{3.10}$$

Using (3.8),(3.9) and (3.10), we obtain

$$\left\| d(x_{n+p}, x_n) - \frac{1}{2}d(x_n, x_n) - \frac{1}{2}d(x_{n+p}, x_{n+p}) \right\|$$

$$\begin{aligned}
&\leq \left\| S^3 \mathcal{D}^{n+p-3} R_0 + S^4 \mathcal{D}^{n+p-4} R_0 + \cdots + S^{p-1} \mathcal{D}^{n+1} R_0 \right. \\
&\quad \left. + \mathcal{D}^n R_0 (S^{p-1} - S) + S R_0 \mathcal{D}^{n-1} \right\| \\
&= \left\| \sum_{k=3}^{p-1} S^k \mathcal{D}^{n+p-k} R_0 + \mathcal{D}^n R_0 (S^{p-1} - S) + S R_0 \mathcal{D}^{n-1} \right\| \\
&\leq \left\| \sum_{k=3}^{p-1} S^k \mathcal{D}^{n+p-k} R_0 \right\| + \left\| \mathcal{D}^n R_0 (S^{p-1} - S) \right\| + \left\| S R_0 \mathcal{D}^{n-1} \right\| \\
&= \|R_0\| \|\mathcal{D}\|^{n+p} \sum_{k=3}^{p-1} \left( \frac{\|S\|}{\|\mathcal{D}\|} \right)^k + \|\mathcal{D}\|^n \|R_0\| \|(S^{p-1} - S)\| + \|S\| \|R_0\| \|\mathcal{D}\|^{n-1} \\
&= \|R_0\| \|\mathcal{D}\|^{n+p} \frac{\|S\|^3 (\|S\|^{p-1} - \|\mathcal{D}\|^{p-1})}{\|\mathcal{D}\|^{p+1} (\|S\| - \|\mathcal{D}\|)} + \|\mathcal{D}\|^n \|R_0\| \|(S^{p-1} - S)\| + \|S\| \|R_0\| \|\mathcal{D}\|^{n-1} \\
&= \|R_0\| \|\mathcal{D}\|^{n-1} \frac{\|S\|^3 (\|S\|^{p-1} - \|\mathcal{D}\|^{p-1})}{\|S\| - \|\mathcal{D}\|} + \|\mathcal{D}\|^n \|R_0\| \|(S^{p-1} - S)\| + \|S\| \|R_0\| \|\mathcal{D}\|^{n-1} \\
&\quad \longrightarrow \infty \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Therefore,  $\{x_n\}$  is a Cauchy sequence with respect to  $\mathbb{A}$ . By the completeness of  $(X, \mathbb{A}, d)$ , there exists an  $x \in X$  such that

$$\lim_{n \rightarrow \infty} d(x_n, x) = d(Tx_{n-1}, x) = d(x, x).$$

Now, we show that  $x$  is a fixed point of  $T$ . For this,

$$\begin{aligned}
d(Tx, x) &\preceq S(d(Tx, Tx_{n-1}) + d(Tx_{n-1}, x)) \\
&\preceq SA[d(x, Tx) + d(x_{n-1}, Tx_{n-1})] + Sd(x_n, x) \\
&= SA[d(x, Tx) + d(x_{n-1}, x_n)] + Sd(x_n, x). \tag{3.11}
\end{aligned}$$

As  $n \rightarrow \infty$  in (3.11), we get

$$\begin{aligned}
d(Tx, x) &\preceq SA[d(x, Tx) + d(x, x)] + Sd(x, x) \\
(I - SA)d(Tx, x) &\preceq (I + SA)d(x, x).
\end{aligned}$$

This gives,

$$d(Tx, x) \preceq (I - SA)^{-1}(I + SA)d(x, x).$$

and by Proposition 3.6(iv), we have

$$d(Tx, x) \preceq (I - SA)^{-1}(I + SA) \frac{2S}{q} \sum_{i=1}^q d(x, x_i).$$

Letting  $q \rightarrow \infty$ , we obtain:

$$d(Tx, x) \preceq \theta, \text{ which implies } d(Tx, x) = \theta.$$

Hence,  $x = Tx$ .

Now, suppose  $y(\neq x)$  is another fixed point of  $T$ . Then,

$$\begin{aligned}\theta \preceq d(x, y) = d(Tx, Ty) &\preceq A(d(x, Tx) + d(y, Ty)) \\ &= A(d(x, x) + d(y, y)).\end{aligned}$$

Taking the norm on both side and using Proposition 3.6(iv), we obtain

$$\begin{aligned}0 &\leq \|d(x, y)\| \leq \|d(Tx, Ty)\| \\ &\leq \left\| A \frac{2S}{q} \sum_{i=1}^q d(x, x_i) + A \frac{2S}{q} \sum_{i=1}^q d(y, y_i) \right\|. \quad (3.12)\end{aligned}$$

Letting  $q \rightarrow \infty$  in (3.12) yields  $\|d(x, y)\| \leq 0$ , and this implies that  $x = y$ .  $\square$

We provide the subsequent example to support the assumptions of Theorem 3.8.

**Example 3.10.** Let  $(X, \mathbb{A}, d)$  be a  $C^*$ -algebra-valued  $b$ -metric-like space as in Example 3.3. Define a self mapping  $T : X \rightarrow X$  by  $Tx = \frac{x}{2}$ . Then,

$$\begin{aligned}d(Tx, Ty) = d\left(\frac{x}{2}, \frac{y}{2}\right) &= \begin{bmatrix} |\frac{x}{2} + \frac{y}{2}|^p & 0 \\ 0 & \frac{x}{2} + \frac{y}{2}|^p \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2^p}|x + y|^p & 0 \\ 0 & \frac{1}{2^p}|x + y|^p \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2^p} & 0 \\ 0 & \frac{1}{2^p} \end{bmatrix} \begin{bmatrix} |x + y|^p & 0 \\ 0 & |x + y|^p \end{bmatrix} \\ &\preceq Sd(x, y).\end{aligned}$$

This implies that  $\|d(Tx, Ty)\| \preceq \frac{1}{2^p} \|d(x, y)\|$ , that is,  $\|S\| \leq \frac{1}{2^p}$ .

Hence,  $T$  satisfies the contractive condition (3.7), and by the hypotheses of Theorem 3.8,  $T$  has a unique fixed point at  $x = \theta$ , i.e.  $T\theta = \theta$ .

#### 4. CONCLUSION

The concept of  $C^*$ -algebra unifies analysis, topology and algebra to create an efficient structure for studying noncommutative spaces, operators, and quantum systems. They are applicable to both applied and pure sciences such as quantum physics, signal processing, control theory, analysis, topology and geometry. Therefore, we further developed these domains, by presenting the idea of  $C^*$ -algebra-valued  $b$ -metric-like space, which generalized the idea of  $C^*$ -algebra-valued metric-like space (see [1]). We proved the analogue of Banach and Kannan-type fixed point theorems in such spaces. Few examples were constructed to showcase the significance of the obtained results. The major difference between this space and the existing ones is the relaxation of the triangle inequality within the frame of  $C^*$ -algebra. The primary notion of this study can be improved with more investigations. For instance, some fixed point results of generalized contractions, quasi-metric space, fuzzy metric space and the hybrid of these frames can be examined.

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