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DEVELOPMENT OF A CONTINUOUS MULTI-STEP ONE-FOURTH STEP HYBRID SCHEME FOR THIRD-ORDER IVP IN ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. This study introduces a hybrid block approach for the approximate solution of initial value problems involving third-order ordinary differential equations. The formulation of the method employs orthogonal and Chebyshev polynomials as basis functions, with its performance enhanced through the incorporation of off-step points. This modification is aimed at achieving zero-stability while maintaining high computational accuracy. Several numerical experiments are carried out to demonstrate the methods effectiveness, and the results confirm its reliability and efficiency in solving such problems.

1. Introduction

In the realms of science and engineering, mathematical models are essential tools for interpreting and predicting the behavior of various physical systems. The formulation of such models frequently produces equations containing derivatives of an unknown function, which may depend on one or more variables. These equations are collectively known as differential equations. Beyond physical sciences, differential equations are also fundamental in specialization such as economics, psychology, medicine, biology,operations research and anthropology.

1.1. Preliminaries.

In many real-world cases, the differential equations derived from these models did not have analytic solution. This limitation necessitates the use of numerical methods to obtain approximate solutions. Over time, several numerical strategies such as the finite difference method, finite element method, and weighted residual method have been developed, each tailored to specific categories of differential equations.

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When the dependent variable is a function of two or more independent variables, the equation is classified as a Partial Differential Equation (PDE). Conversely, if the dependent variable depends on single independent variable, the equation is known as an Ordinary Differential Equation (ODE). The present study focuses specifically on numerical approaches for solving ODEs.

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1.2. **Literature Review.** Numerical scheme that can be used to address ODEs have been extensively discuss by many author. Notable contributions include those by [4],[3] and [2]. Significant collaborative works such as [6] along with research by [1], have further advanced the field.[5] introduced a continuous five-step block method based on a multistep collocation framework, resulting in a family of eight discrete schemes. More recently,[7] investigated a hybrid continuous multistep technique for second-order ODEs. Building on these developments, the current research aims to construct higher-step numerical schemes utilizing orthogonal polynomials as basis functions.

2. Methodology

In this research, we use Chebyshev polynomial and orthogonal polynomial to be the basis function for obtaining the numerical solution. Also, we try to obtain linear multi step methods for the approximate solution of the IVP in ODE;

$$y''(x) = f(x, y, y') \qquad y'(x_0) = z_0, y(x_0) = y_0, x \in [a, b] \tag{2.1}$$

This lead us to the construction of one-four step length with some grids point. The method is then used to obtained the approximate block method we are seeking for.

We shall obtained the multi-step collocation method of the form

$$\sum_{j=0}^{n} \alpha_i y_{n+j} = h \sum_{j=0}^{n} \beta_j f_{n+j} + h\beta_v f_{n+\alpha} \tag{2.2}$$

α_j and β_j are the continuous coefficients.

For us to derived approximate value which will be compared with the exact value $y(x)$ with the continuous form:

$$y(x) = \sum_{i}^{j} a_i T_i(x) + \sum_{j+1}^{p+q-1} a_{j+1}\alpha_{j+1}(x) \equiv y(x) = \sum_{j=0}^{p+q-1} a_i \varphi_n(x) \tag{2.3}$$

This $T_i(x)$ represent the chebyshev polynomial, a_i are unknown coefficients, $\alpha_n(x)$ denote the orthogonal polynomial of degree $p + q - 1$, where p represent the number of interpolation point and q is standing for the number of collocation point respectively which is to satisfy $1 \leq p \leq k$ and $q > 0$. The integer $k \geq 1$ stand for the step number of the method.

2.1. **One-fourth Step hybrid Method.** Consider the step number $k = \frac{1}{24}, \frac{1}{12}, \frac{1}{8}, \frac{1}{6}, \frac{5}{24}$ and $\frac{1}{4}$ as the off-step points. Following the same procedure, the desired method is obtained as

$$
\begin{aligned}
y_{n+\frac{1}{24}} &= y_{n+\frac{1}{24}} hy'_n + \frac{1}{288}h^2y''_n + h^3\bigg(\frac{6887}{195955200}f_n + \frac{1499}{16329600}f_{n+\frac{1}{24}} - \frac{233}{3732480}f_{n+\frac{1}{12}} \\
&+ \frac{13}{244944}f_{n+\frac{1}{8}} - \frac{379}{13063680}f_{n+\frac{1}{6}} + \frac{149}{16329600}f_{n+\frac{5}{24}} - \frac{491}{391910400}f_{n+\frac{1}{4}}\bigg) \\
y_{n+\frac{1}{12}} &= y_{n+\frac{1}{12}} hy'_n + \frac{1}{1152}h^2y''_n + h^3\bigg(\frac{343801}{50164531200}f_n + \frac{6031}{597196800}f_{n+\frac{1}{24}} - \frac{32981}{3344302080}f_{n+\frac{1}{12}} \\
&+ \frac{5177}{244944}f_{n+\frac{1}{8}} - \frac{15107}{627056640}f_{n+\frac{1}{6}} + \frac{5947}{4180377600}f_{n+\frac{5}{24}} - \frac{9809}{50164531200}f_{n+\frac{1}{4}}\bigg) \\
y_{n+\frac{1}{8}} &= y_{n+\frac{1}{8}} hy'_n + \frac{1}{128}h^2y''_n + h^3\bigg(\frac{1959}{22937600}f_n + \frac{1599}{5734400}f_{n+\frac{1}{24}} - \frac{537}{4587520}f_{n+\frac{1}{12}} \\
&+ \frac{1}{7680}f_{n+\frac{1}{8}} - \frac{327}{4587520}f_{n+\frac{1}{6}} + \frac{129}{5734400}f_{n+\frac{5}{24}} - \frac{71}{22937600}f_{n+\frac{1}{4}}\bigg) \\
y_{n+\frac{1}{6}} &= y_{n+\frac{1}{6}} hy'_n + \frac{1}{72}h^2y''_n + h^3\bigg(\frac{3863}{24494400}f_n + \frac{583}{1020600}f_{n+\frac{1}{24}} - \frac{113}{816480}f_{n+\frac{1}{12}} \\
&+ \frac{17}{61236}f_{n+\frac{1}{8}} - \frac{31}{233280}f_{n+\frac{1}{6}} + \frac{43}{1020600}f_{n+\frac{5}{24}} - \frac{71}{12247200}f_{n+\frac{1}{4}}\bigg) \\
y_{n+\frac{5}{24}} &= y_{n+\frac{5}{24}} hy'_n + \frac{25}{1152}h^2y''_n + h^3\bigg(\frac{505625}{2006581248}f_n + \frac{162125}{167215104}f_{n+\frac{1}{24}} - \frac{85625}{668860416}f_{n+\frac{1}{12}} \\
&+ \frac{66875}{125411328}f_{n+\frac{1}{8}} - \frac{119375}{668860416}f_{n+\frac{1}{6}} + \frac{1625}{23887872}f_{n+\frac{5}{24}} - \frac{18625}{2006581248}f_{n+\frac{1}{4}}\bigg) \\
y_{n+\frac{1}{4}} &= y_{n+\frac{1}{4}} hy'_n + \frac{1}{32}h^2y''_n + h^3\bigg(\frac{33}{89600}f_n + \frac{33}{22400}f_{n+\frac{1}{24}} - \frac{3}{35840}f_{n+\frac{1}{12}} \\
&+ \frac{1}{1120}f_{n+\frac{1}{8}} - \frac{1}{17920}f_{n+\frac{1}{6}} + \frac{3}{22400}f_{n+\frac{5}{24}} - \frac{1}{76800}f_{n+\frac{1}{4}}\bigg)
\end{aligned} \tag{2.4}
$$

$$y'_{n+\frac{1}{24}} = y'_n + \frac{1}{24}hy''_n + h^2\left(\frac{28549}{69672960}f_n + \frac{275}{2985984}f_{n+\frac{1}{24}} - \frac{5717}{7741440}f_{n+\frac{1}{12}} + \frac{10621}{17418240}f_{n+\frac{1}{8}} - \frac{7703}{23224320}f_{n+\frac{1}{6}} + \frac{403}{3870720}f_{n+\frac{5}{24}} - \frac{199}{13934592}f_{n+\frac{1}{4}}\right)$$

$$y'_{n+\frac{1}{12}} = y'_n + \frac{1}{12}hy''_n + h^2\left(\frac{1027}{1088640}f_n + \frac{97}{30240}f_{n+\frac{1}{24}} - \frac{1}{648}f_{n+\frac{1}{12}} + \frac{197}{136080}f_{n+\frac{1}{8}} - \frac{97}{120960}f_{n+\frac{1}{6}} + \frac{23}{90720}f_{n+\frac{5}{24}} - \frac{19}{544320}f_{n+\frac{1}{4}}\right)$$

$$y'_{n+\frac{1}{8}} = y'_n + \frac{1}{8}hy''_n + h^2\left(\frac{253}{172032}f_n + \frac{165}{28672}f_{n+\frac{1}{24}} - \frac{267}{286720}f_{n+\frac{1}{12}} + \frac{5}{2048}f_{n+\frac{1}{8}} - \frac{363}{286720}f_{n+\frac{1}{6}} + \frac{57}{28672}f_{n+\frac{5}{24}} - \frac{47}{860160}f_{n+\frac{1}{4}}\right)$$

$$y'_{n+\frac{1}{6}} = y'_n + \frac{1}{6}hy''_n + h^2\left(\frac{17}{8505}f_n + \frac{47}{5670}f_{n+\frac{1}{24}} - \frac{1}{7560}f_{n+\frac{1}{12}} + \frac{41}{8505}f_{n+\frac{1}{8}} - \frac{1}{648}f_{n+\frac{1}{6}} + \frac{1}{1890}f_{n+\frac{5}{24}} - \frac{1}{13608}f_{n+\frac{1}{4}}\right)$$

$$y'_{n+\frac{5}{24}} = y'_n + \frac{5}{24}hy''_n + h^2\left(\frac{35225}{13934592}f_n + \frac{8375}{774144}f_{n+\frac{1}{24}} - \frac{3125}{4644864}f_{n+\frac{1}{12}} + \frac{25625}{3483648}f_{n+\frac{1}{8}} - \frac{625}{1548288}f_{n+\frac{1}{6}} + \frac{275}{331776}f_{n+\frac{5}{24}} - \frac{1375}{13934592}f_{n+\frac{1}{4}}\right)$$

$$y'_{n+\frac{1}{4}} = y'_n + \frac{1}{4}hy''_n + h^2\left(\frac{41}{13440}f_n + \frac{3}{224}f_{n+\frac{1}{24}} - \frac{3}{2240}f_{n+\frac{1}{12}} + \frac{17}{1680}f_{n+\frac{1}{8}} - \frac{3}{4480}f_{n+\frac{1}{6}} + \frac{3}{1120}f_{n+\frac{5}{24}}\right) \tag{2.5}$$

$$y''_{n+\frac{1}{24}} = y''_n + h\Bigg(\frac{19087}{1451520}f_n + \frac{2713}{60480}f_{n+\frac{1}{24}} - \frac{15487}{483840}f_{n+\frac{1}{12}} + \frac{293}{11340}f_{n+\frac{1}{8}} - \frac{6737}{483840}f_{n+\frac{1}{6}}$$
$$+ \frac{263}{60480}f_{n+\frac{5}{24}} - \frac{863}{1451520}f_{n+\frac{1}{4}}\Bigg)$$
$$y''_{n+\frac{1}{12}} = y''_n + h\Bigg(\frac{1139}{90720}f_n + \frac{47}{756}f_{n+\frac{1}{24}} - \frac{11}{30240}f_{n+\frac{1}{12}} + \frac{83}{5670}f_{n+\frac{1}{8}} - \frac{269}{30240}f_{n+\frac{1}{6}}$$
$$+ \frac{11}{3780}f_{n+\frac{5}{24}} - \frac{37}{90720}f_{n+\frac{1}{4}}\Bigg)$$
$$y''_{n+\frac{1}{8}} = y''_n + h^2\Bigg(\frac{137}{10752}f_n + \frac{27}{448}f_{n+\frac{1}{24}} - \frac{387}{17920}f_{n+\frac{1}{12}} + \frac{17}{420}f_{n+\frac{1}{8}} - \frac{243}{17920}f_{n+\frac{1}{6}}$$
$$+ \frac{9}{2240}f_{n+\frac{5}{24}} - \frac{29}{53760}f_{n+\frac{1}{4}}\Bigg)$$
$$y''_{n+\frac{1}{6}} = y''_n + h^2(\frac{143}{11340}f_n + \frac{58}{945}f_{n+\frac{1}{24}} - \frac{16}{945}f_{n+\frac{9}{4}5}f_{n+\frac{1}{12}} + \frac{188}{2835}f_{n+\frac{1}{8}} - \frac{29}{3780}f_{n+\frac{1}{6}}$$
$$+ \frac{2}{524} - \frac{1}{2835}f_{n+\frac{1}{4}}\Bigg)$$
$$y''_{n+\frac{5}{24}} = y''_n + h^2(\frac{3715}{290304}f_n + \frac{725}{12096}f_{n+\frac{1}{24}} - \frac{2125}{96768}f_{n+\frac{1}{12}} + \frac{125}{226}f_{n+\frac{1}{8}} - \frac{3875}{96768}f_{n+\frac{1}{6}}$$
$$+ \frac{235}{12096}f_{n+\frac{5}{24}} - \frac{275}{290304}f_{n+\frac{1}{4}}\Bigg)$$
$$y''_{n+\frac{1}{4}} = y''_n + h^2(\frac{41}{3360}f_n + \frac{9}{140}f_{n+\frac{1}{24}} - \frac{9}{1120}f_{n+\frac{1}{12}} + \frac{17}{210}f_{n+\frac{1}{8}} - \frac{9}{1120}f_{n+\frac{1}{6}}$$
$$+ \frac{9}{140}f_{n+\frac{5}{24}} - \frac{41}{3360}f_{n+\frac{1}{4}}\Bigg) \tag{2.6}$$

2.2. **Analysis of the Derived Method.** The Taylor series expansion of equations (6) will give $(7, 7, 7, 7, 7, 7, 7)$ as the order and error constants to be

$$\Bigg(\frac{4001}{115039094423578214400}, \frac{199}{898742925184204800}, \frac{29}{52601323467571200}, \frac{29}{28085716412006400},$$
$$\frac{7625}{4601563776943128576}, \frac{1}{410947839590400}\Bigg)$$

2.2.1. *Zero stability of Method.* For us to obtain the zero stability of this method, we used the matrix difference equation writhe below:

$$p^0 Y_{m+1} = p^{'} y_m + h^2[Q^0 F_{m+1} + Q^{'} F_m + hR^{'}] \tag{2.7}$$

The coefficients of equation above is the matrices $p^0, p^{'}, Q^0, Q^{'}$ and R^0.which are defined as:

$$p^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

$$Q^{'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.9}$$

$$Q^0 = \begin{bmatrix} \frac{1499}{16329600} & -\frac{233}{3732480} & \frac{13}{244944} & -\frac{379}{1363680} & \frac{146}{16329600} & -\frac{146}{391910400} \\ \frac{6031}{567196800} & -\frac{32981}{3344302080} & \frac{5177}{244944} & -\frac{15107}{62756640} & \frac{5947}{4180377600} & -\frac{9809}{5164531200} \\ \frac{1599}{5734400} & -\frac{537}{4587520} & \frac{1}{7680} & -\frac{327}{4587520} & \frac{129}{5734400} & -\frac{71}{22937600} \\ \frac{583}{1020600} & -\frac{113}{816480} & \frac{17}{61236} & -\frac{31}{233280} & \frac{43}{1020600} & -\frac{71}{12247200} \\ \frac{162125}{167215104} & -\frac{85625}{668860416} & \frac{66875}{125411328} & -\frac{119375}{668860416} & \frac{1625}{23887872} & -\frac{18625}{2006581248} \\ \frac{33}{22400} & -\frac{3}{35840} & \frac{1}{1120} & -\frac{1}{17920} & \frac{3}{22400} & -\frac{1}{76800} \end{bmatrix}, \tag{2.10}$$

$$Q^{'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{6887}{195955200} \\ 0 & 0 & 0 & 0 & 0 & \frac{343810}{50164531200} \\ 0 & 0 & 0 & 0 & 0 & \frac{1959}{2237600} \\ 0 & 0 & 0 & 0 & 0 & \frac{3863}{24494400} \\ 0 & 0 & 0 & 0 & 0 & \frac{505625}{2006581248} \\ 0 & 0 & 0 & 0 & 0 & \frac{33}{89600} \end{bmatrix} R^{'} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{1}{24} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{24} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \tag{2.11}$$

Now,

$$\left|[\lambda p^0 - p^{'}]\right| = \left|\lambda\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\end{bmatrix} - \begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\right| = \begin{vmatrix}\lambda&0&0&0&0&-1\\0&\lambda&0&0&0&-1\\0&0&\lambda&0&0&-1\\0&0&0&\lambda&0&-1\\0&0&0&0&\lambda&-1\\0&0&0&0&0&\lambda-1\end{vmatrix} = 0 \qquad (2.12)$$

That is $\lambda^4 - \lambda^3 = 0$, This gives $\lambda = 0, 0, 0, 1$
Since the roots have moduli less than or equal to one and are simple, the method is zero-stable.

3. Numerical Examples

3.1. **Third Order Problems.** The following example will be solved using the new method derived and compared it with the exact solution.

Problem 1. Yakusak *et al*. (2016) solved this Problem
$y''' = y'' - y' + y$, $y(0) = 1$, $y'(0) = 1$, $y''(0) = -1$, $h = 0.01$
Analytical solution: $y(x) = cosx$

Problem 2. Muhammed and Adeniyi (2014) solved this Problem
$y''' = -5y'' - 7y' - 3y$, $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$, $h = 0.1$
Analytical solution: $y(x) = e^{-x} + xe^{-x}$

Problem 3. Oladode (2013) solved this Problem
$y''' = 3sinx$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = -2$, $h = 0.1$
Analytical solution: $y(x) = 3cosx + \frac{x^2}{2} - 2$

Problem 4. Muhammed and Adeniyi,(2014) solved this Problem
$y''' = x - 4y'$, $y(0) = 0$, $y'(0) = 1$,$y''(0) = 2$, $h = 0.1$
Analytical solution:$y(x) = 2(1 - cosx) + sinx$

Problem 5. Yakusak *et al*.,(2016) solved this Problem
$y''' = e^x$, $y(0) = 3$, $y'(0) = 1$, $y''(0) = 5$, $h = 0.1$,
Analytical solution: $y(x) = 2 + 2x^2 + e^x$

Table: 1a. Numerical Results for Problem 1.

x	Exact	$\frac{1}{4}HBMT$
0.0100	0.999950000417	0.9999500004166653
0.0200	0.999800006667	0.9998000066665778
0.0300	0.999550033749	0.9995500337489875
0.0400	0.999200106661	0.9992001066609779
0.0500	0.998750260395	0.9987502603949663
0.0600	0.998200539935	0.9982005399352042
0.0700	0.997551000253	0.9975510002532796
0.0800	0.996801706303	0.9968017063026194
0.0900	0.995952733012	0.9959527330119943
0.1000	0.995004165278	0.9950041652780258

Table:1b. Errors of Methods for Problem 1.

x	$\frac{1}{4}EHBMT$	Yakusak *et al.* (2016)
0.0100	$0.0000E-00$	$0.0000E-00$
0.0200	$0.0000E-00$	$0.0000E-00$
0.0300	$0.0000E-00$	$0.0000E-00$
0.0400	$5.5511E-16$	$1.0000E-10$
0.0500	$1.5543E-15$	$1.0000E-10$
0.0600	$3.9968E-15$	NA
0.0700	$9.2149E-15$	NA
0.0800	$1.8541E-14$	NA
0.0900	$3.3751E-14$	NA
0.1000	$5.7843E-14$	NA

Table:2a. Numerical Results for Problem 2.

x	Exact	$\frac{1}{4}HBMT$
0.1000	0.9953211598395556	0.995321159845
0.2000	0.9824769036935782	0.982476903767
0.3000	0.9630636868862332	0.963063687215
0.4000	0.9384480644498952	0.938448065373
0.5000	0.9097959895689508	0.909795991572
0.6000	0.8780986177504422	0.878098621448
0.7000	0.8441950164453953	0.844195022549
0.8000	0.8087921354109972	0.808792144702
0.9000	0.7724823535071357	0.772482366802
1.0000	0.7357588823428811	0.735758900467

Table:2b Errors of Methods for Problem 2.

x	$\frac{1}{4}EHBMT$	Muhammed and Adeniyi,(2014)
0.1000	$5.1306E-12$	$1.0000E-10$
0.2000	$7.3145E-11$	$3.0000E-10$
0.3000	$3.2882E-10$	$7.0000E-10$
0.4000	$9.2302E-10$	$7.0000E-10$
0.5000	$2.0033E-09$	NA
0.6000	$3.6973E-09$	NA
0.7000	$6.1038E-09$	NA
0.8000	$9.2905E-09$	NA
0.9000	$1.3294E-08$	NA
1.0000	$1.8124E-08$	NA

Table:3a. Numerical Results for Problem 3.

x	Exact	$\frac{1}{4}HBMT$
0.1000	0.990012495834	0.9900124958340775
0.2000	0.960199733524	0.9601997335237251
0.3000	0.911009467377	0.9110094673768181
0.4000	0.843182982009	0.8431829820086554
0.5000	0.757747685671	0.7577476856711183
0.6000	0.656006844729	0.6560068447290348
0.7000	0.539526561853	0.5395265618534646
0.8000	0.410120128041	0.4101201280414952
0.9000	0.269829904812	0.2698299048119921
1.0000	0.120906917604	0.1209069176044171

Table:3b. Errors of Methods for Problem 3.

x	$\frac{1}{4}EHBMT$	Olabode(2013)
0.1000	$5.5511E-16$	$1.65922E-10$
0.2000	$2.2204E-16$	$4.76275E-10$
0.3000	$1.1102E-16$	$6.23182E-10$
0.4000	$2.2204E-16$	$1.99134E-09$
0.5000	$2.2204E-16$	$3.28882E-10$
0.6000	$4.4409E-16$	$1.27096E-09$
0.7000	$2.2204E-16$	$4.84653E-09$
0.8000	$5.5511E-17$	$1.09585E-08$
0.9000	$7.2164E-16$	$2.0188E-08$
1.0000	$1.0547E-15$	$3.53956E-08$

Table:4a. Numerical Results for Problem 4.

x	Exact	$\frac{1}{4}HBMT$
0.1000	0.0049875166547672	0.004987516654
0.2000	0.019801063619	0.0198010636244591
0.3000	0.043999572185	0.0439995722044353
0.4000	0.0768674919974061	0.076867491952
0.5000	0.1174433176497229	0.117443317565
0.6000	0.1645579210356239	0.164557920896
0.7000	0.2168811607062063	0.216881160495
0.8000	0.2729749104314945	0.272974910136
0.9000	0.3313503927549581	0.331350392365
1.0000	0.3905275318525949	0.390527531361

Table:4b. Errors of Methods for Problem 4.

x	$\frac{1}{4}EHBMT$	Muhammed and Adeniyi,(2014)
0.1000	$3.7840E-13$	$9.61000E-10$
0.2000	$3.0140E-12$	$6.50000E-09$
0.3000	$9.9970E-12$	$1.59700E-08$
0.4000	$2.3060E-11$	$1.6640E-08$
0.5000	$4.3450E-11$	$2.03000E-08$
0.6000	$7.1770E-11$	$2.66000E-08$
0.7000	$1.0800E-10$	$2.67000E-08$
0.8000	$1.5120E-10$	$2.71000E-08$
0.9000	$1.9980E-10$	$2.7700E-08$
1.0000	$2.5180E-1$	$2.72000E-08$

Table:5a. Numerical Results for Problem 5.

x	Exact	$\frac{1}{4}HBMT$
0.1000	3.1251709180756477	3.125170918076
0.2000	3.3014027581601697	3.301402758160
0.3000	3.5298588075760033	3.529858807576
0.4000	3.8118246976412702	3.811824697641
0.5000	4.1487212707001282	4.1487212707001291
0.6000	4.5421188003905097	4.5421188003905124
0.7000	4.9937527074704793	4.9937527074704819
0.8000	5.5055409284924721	5.5055409284924748
0.9000	6.0796031111569562	6.0796031111569571
1.0000	6.7182818284590544	6.7182818284590518

Table:5b. Errors of Methods for Problem 5.

x	$\frac{1}{4}EHBMT$	Yakusak *et al.* (2016)
0.1000	$0.00000000E+00$	$0.00000000E+00$
0.2000	$4.44090000E-16$	$0.00000000E+00$
0.3000	$0.00000000E+00$	$0.00000000E+00$
0.4000	$8.88180000E-16$	$0.00000000E+00$
0.5000	$8.88180000E-16$	$0.00000000E+00$
0.6000	$8.88180000E-16$	$1.00000000E-10$
0.7000	$8.88180000E-16$	$1.00000000E-10$
0.8000	$1.77640000E-15$	$1.00000000E-10$
0.9000	$2.66450000E-15$	$1.00000000E-10$
1.0000	$4.44090000E-15$	$1.00000000E-10$

4. Discussion

The table shows the comparison between the result obtained using the proposed method and is preferable to those reported by Olabode(2016) and Yakusak *et al.* (2016).

5. Conclusion

In this paper, we present a class of numerical schemes that utilize fractional step lengths for solving Third-order ordinary differential equations. The method is consistent and zero-stable, ensuring convergence. Additionally, it exhibits a wide region of absolute stability. The derived methods effectively solve third-order initial value problems, yielding accurate results. The results of our method are shown in the table above, where we compare it with some existing methods. Our new method demonstrates superior performance in comparison.

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