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SEMI-ANALYTICAL ITERATIVE METHODS FOR SOLVING TIME-FRACTIONAL RICCATI DIFFERENTIAL EQUATION

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ABSTRACT In this paper, two semi-analytical methods for solving the time-fractional Riccati differential equation, the homotopy perturbation method (HPM) and the modified new iterative method (MNIM), are employed to solve the time-fractional Riccati equation, which is characterized by its nonlinear and fractional-order nature, and serves as a fundamental model in mathematical physics and engineering processes involving memory and hereditary properties. By incorporating the Caputo fractional derivative, the study captures the nonlocal temporal dynamics of the system. The MNIM is formulated to enhance convergence and minimize computational complexity, while HPM is utilized to construct an approximate analytical series solution without linearization or discretization. Both methods yield rapidly convergent series solutions that approximate the exact analytical solution with high accuracy. We considered two test cases, and the results demonstrated the efficiency, simplicity, and robustness of the proposed methods for various fractional orders, establishing both methods as powerful tools for fractional nonlinear differential equations in applied sciences and engineering. The paper lies in applying semi-analytical iterative methods tailored to the time-fractional Riccati differential equation, providing accurate approximate solutions with reduced computational complexity.

1. INTRODUCTION

Fractional-order Riccati differential equations (FRDEs) have emerged as fundamental models in describing a wide range of nonlinear phenomena characterized by memory and hereditary effects. These equations appear prominently in fields such as control theory, viscoelastic material modeling, and anomalous transport processes, where the system's future evolution depends not only on its current state but also on its historical trajectory. Unlike classical integer-order models, fractional derivatives naturally capture this history-dependent behavior, providing more accurate and physically meaningful representations of complex dynamical systems. Consequently, there is a growing demand for efficient and reliable computational methods capable of solving time-fractional Riccati equations while preserving their intrinsic nonlinear and memory-dependent structure. Among the various definitions of fractional derivatives, the Caputo derivative has become the standard choice for initial-value problems due to its compatibility with classical initial conditions and its preservation of physical interpretability. This property is particularly

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advantageous in modeling real-world systems, where initial states are often measurable and carry clear physical significance. Despite the widespread adoption of the Caputo operator, the inherent nonlinearity of Riccati-type equations presents substantial challenges in obtaining exact solutions, necessitating approximate or semi-analytical approaches that strike a balance between computational efficiency and solution accuracy.

Semi-analytical iterative methods, which integrate the analytical structure of the problem with recursive computational procedures, have demonstrated considerable promise in addressing these challenges. Techniques such as the Homotopy Perturbation Method (HPM), the modified new iterative method (MNIM), the Adomian Decomposition Method (ADM), and other iterative series expansions provide rapidly convergent solutions while reducing the computational complexity associated with fully numerical schemes. These methods are particularly effective for short-time dynamics, where traditional numerical integration may suffer from stiffness or require prohibitively small-time steps.

Recent innovations, including modified iterative schemes and enhanced homotopy perturbation approaches, have further advanced the solution of fractional Riccati equations by directly incorporating the Caputo derivative without resorting to excessive linearization or perturbative assumptions. These modifications produce compact series representations that converge more quickly than classical methods, thus improving computational efficiency while maintaining high accuracy. However, despite these advances, the literature reveals a notable gap: while individual semi-analytical methods have been proposed and tested, systematic comparative studies evaluating their relative performance, considering convergence rate, computational cost, and error propagation, remain limited.

In this paper, we consider the non-linear fractional Riccati equation of the form [1]

$$\begin{cases} D_{*t}^{\alpha}\Psi(t) + \Psi^2(t) = g_1, \qquad 0 < \alpha \le 1 \\ \qquad\qquad \Psi(0) = g_2 \end{cases} \tag{1}$$

Where $\Psi(t)$ describes the system's dynamic response over time for a fractional derivative, D_{*t}^{α} is the fractional derivative with respect to t, g_1 and g_2 are smooth functions or constants.

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Over the past two decades, numerous researchers have proposed and applied numerical techniques to solve Riccati equation. For instance, [2] explored numerical solutions of fractional Riccati differential equations using implicit finite difference methods, [3] applied differential transformation method (DTM) to solve classical nonlinear Riccati differential equations (RDEs),

[4] applied the Elzaki-Adomian decomposition method to solve local fractional Riccati equations, pertinent to nonlinear ODEs in population dynamics, quantum mechanics, and fluid mechanics, with statistical analysis of solutions, [5] proposed a computationally efficient approach using a modified Picard iteration for fractional Riccati equations involving the Atangana-Baleanu operator, underscoring its engineering applications, [6] solving Fractional Riccati Differential equation with Caputo- Fabrizio fractional derivative, and [7] presented numerical solutions of fractional Riccati differential equation. This current study employs the MNIM and HPA in computational techniques, which reflect a significant advancement in solving fractional Riccati ODEs that model complex systems with memory and nonlocality, reinforcing their importance in applied mathematics and interdisciplinary fields. This paper is positioned within contemporary fractional convergence literature, benchmarking semi-analytical iterative schemes against established numerical methods to highlight accuracy, convergence behavior, and computational efficiency in time-fractional Riccati models.

2. FRACTIONAL CALCULUS

Some fundamental definitions of fractional calculus are provided in this section. The following are some definitions attributed to Riemann-Liouville and Caputo sense: [8, 9, 10].

Definition 1. For a function $\Psi(\mathrm{t})$, the Riemann-Liouville integral operator of order $\alpha > 0$ is defined as

$$\begin{cases} D_*^{\alpha}\Psi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Psi(s)ds; \qquad \alpha > 0, \quad s > 0, \\ \qquad D^0\Psi(t) = \Psi(s) \end{cases} \tag{2}$$

Definition 2. The fractional derivative of $\Psi(\mathrm{t})$ in the Caputo sense can be defined as

$$D_*^{\alpha}\Psi(t) = J^{m-\alpha}D^m\Psi(t)\frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha-1}\Psi^{(m)}(s)ds \tag{3}$$

For $m-1 < \alpha \leq m,\ \ m \in N, s > 0, \Psi \in C_{-1}^m$.

Hence, we have the following properties:

$$\begin{cases} J^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{(\alpha+\beta)}, \\ \quad J^{\alpha}D_*^{\alpha} = \Psi(t); \quad t > 0 \\ D^{\alpha}C = 0, \quad C \text{ is a constant} \\ \quad D^{\alpha}t^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{(\gamma-\beta)}, \end{cases} \tag{4}$$

3. METHODS OF COMPUTATIONAL SOLUTIONS

In this article, we employ step by step algorithm-based techniques to apply the homotopy perturbation method and the modified new iterative method to find series or iterative approximations of the solution to the fractional nonlinear Riccati-type differential equation with unknown α. The homotopy perturbation method decomposes the unknown solution into series form using homotopy and perturbation theory, while the modified new iterative method uses direct iteration with fractional integral operators to refine solution approximations [11]. The general idea involves transforming the original fractional differential equation into an equivalent integral equation using the properties of the Caputo fractional derivative. This transformation simplifies the problem by expressing the unknown function in terms of its previous values through a convolution-type fractional integral. Subsequently, iterative approximation techniques (MNIM and HPM) are applied to generate successive corrections. Each iteration progressively refines the approximate solution, allowing convergence toward the exact analytical or semi-analytical solution while maintaining computational simplicity and preserving the underlying physical characteristics of the fractional system.

3.1 MODIFIED NEW ITERATIVE METHOD.

We apply the modified new iterative method to solve the Riccati equation (1) to derive the series coefficients via iterates that converge to the series [12]. The computational procedures for solving Riccati equation (1) go as follows:

Step 1: Integration formulation

Apply Caputo's D_*^α the inverse J^α to equation (1) and use $\Psi(0) = g_2$, we obtain

$$\Psi(\mathrm{t}) = g_2 + \frac{g_1}{\Gamma(\alpha+1)} t^\alpha - J^\alpha[\Psi^2](t) \tag{5}$$

Step 2: Initial approximation

Choose the linear part as the starting guess

$$\Psi_0(\mathrm{t}) := g_2 + \frac{g_1}{\Gamma(\alpha+1)} t^\alpha \tag{6}$$

This satisfies the initial value and captures the forcing.

Step 3: MNIM iterative rule

For $n \geq 0$ defined as

$$\Psi_{n+1}(\mathrm{t}) = g_2 + \frac{g_1}{\Gamma(\alpha+1)} t^\alpha - J^\alpha[\Psi_n^2](t) \tag{7}$$

Compute Ψ_{n+1} from Ψ_n by square $\Psi_n(t)$ and express as finite series up to desired order, then, apply J^α term wise using

$$J^\alpha\left[t^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\alpha+\beta} \tag{8}$$

Step 4: Compute the first iterates

$$\Psi_0(\mathrm{t}) := g_2 + \frac{g_1}{\Gamma(\alpha+1)} t^\alpha \tag{9}$$

Square equation (9), then

$$\Psi_0^2 := g_0^2 + 2g_2 \frac{g_1}{\Gamma(\alpha+1)} t^\alpha + \frac{{g_1}^2}{\Gamma(\alpha+1)^2} t^{2\alpha} \tag{10}$$

Apply J^α

$$J^\alpha[\Psi_0^2] = g_2^2 + 2g_2 \frac{g_1}{\Gamma(\alpha+1)} \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)} t^{2\alpha} + \frac{{g_1}^2}{\Gamma(\alpha+1)^2} \frac{\Gamma(2\alpha+1)}{\Gamma(3\alpha+1)} t^{3\alpha} \tag{11}$$

Then

$$\Psi_1(\mathrm{t}) = g_2 + \frac{g_1}{\Gamma(\alpha+1)} t^\alpha - J^\alpha[\Psi_0^2](\mathrm{t}) \tag{12}$$

Which yields an explicit coefficient for $t^\alpha, t^{2\alpha}, t^{3\alpha}$. Continue similarly to obtain Ψ_2, Ψ_3

Step 5: Series and recurrence solution

Expanding iterates yields the same coefficient recurrence as from matching power in equation (1)

$$\begin{cases} a_0 = g_2, \\ a_1 = \dfrac{g_1 - g_2^2}{\Gamma(\alpha+1)}, \\ a_m = -\left(\sum_{i=0}^{m-1} a_i\, a_{m-1-i}\right)\dfrac{\Gamma((m-1)\alpha+1)}{\Gamma(\alpha+1)} \end{cases} \tag{13}$$

The series solution is given by MNIM as

$$\Psi(\mathrm{t}) = \sum a_m t^{m\alpha} \tag{14}$$

Step 6: Convergence

Convergence occurs when $\|\Psi_{n+1} - \Psi_n\|_\infty$ on $[0, T]$ is below tolerance or when coefficients beyond order M are negligible, and the convergence is local.

TABLE 1. Tabular pseudocode summary of the MNIM procedure, written in Python-style format and the steps are represented.

Step	Description	Python-style Pseudocode
1. Integration Formulation	Apply fractional integral J^α (inverse of Caputo derivative) to obtain integral equation.	`python Ψ(t) = g2 + (g1 / gamma(α + 1)) * t**α - Jα(Ψ(t)**2)`
2. Initial Approximation	Choose a linear initial guess that satisfies the initial condition and forcing term.	`python Ψ0 = g2 + (g1 / gamma(α + 1)) * t**α`
3. MNIM Iterative Rule	Define iterative sequence using previous approximation.	`python for n in range(N): Ψ_next = g2 + (g1 / gamma(α + 1)) * t**α - Jα(Ψ_n**2)`
4. Apply Fractional Integral	Use property $J^\alpha[t^\beta] = \Gamma(\beta+1)/\Gamma(\beta+\alpha+1) * t^{\beta+\alpha}$.	`python def Jα_term(t, β, α): return gamma(β + 1) / gamma(β + α + 1) * t**(β + α)`
5. Compute First Iterates	Expand Ψ_0^2, apply J^α term-wise, and construct $\Psi_1(t)$.	`python Ψ1 = g2 + (g1 / gamma(α + 1)) * t**α - Jα(Ψ0**2)`
6. Series and Recurrence	Derive coefficients recursively.	`python a0 = g2; a1 = (g1 - g2**2) / gamma(α + 1) for m in range(2, M): a_m = -(sum(a_i * a_(m-1-i) for i in range(m))) * gamma((m-1)*α + 1) / gamma(α + 1) Ψ = sum(a_m * t**(m*α) for m in range(M))`
7. Convergence Check	Stop when difference between consecutive iterates is below tolerance.	`python if norm(Ψ_next - Ψ_n, ∞) < tol: break`

3.2 HOMOTOPY PERTURBATION METHOD.

The homotopy perturbation method (HPM) decomposes the unknown solution into a series form using homotopy and perturbation theory presented by [13]. The computational procedures for solving Riccati equation (1) go as follows:

Step 1: Preliminaries

Apply Caputo's D_*^α the inverse J^α to the equation (1) and use $\Psi_0 = g_2$

$$\Psi(\mathrm{t}) = g_2 + \frac{g_1}{\Gamma(\alpha+1)}t^\alpha - J^\alpha[\Psi^2](t) \tag{15}$$

This is the form used to build the homotopy

Step 2: Construct the homotopy

Introduce the embedding parameter $p \in [0,1]$ and form,

$$\Psi(\mathrm{t;p}) = g_2 + \frac{g_1}{\Gamma(\alpha+1)}t^\alpha - J^\alpha[\Psi^2(t;p)](t) \tag{16}$$

When $\mathrm{p} = 0$ the right side is the linear part; when $\mathrm{p} = 1$ then we have

Step 3: Assume perturbation expansion

Assume a series expansion in p

$$\Psi(\mathrm{t;p}) = \sum_{k=0}^{\infty} p^k \Psi_k(t) \tag{17}$$

We seek $\Psi_k(t)$. At the end set $\mathrm{p} = 1$ to get the approximate / exact solution

$$\Psi(\mathrm{t}) = \sum_{k\geq 0} \Psi_k(t) \tag{18}$$

Step 4: Substitute into the homotopy and collect like powers of p,

Substitute the series into homotopy equation

$$\sum_{k=0}^{\infty} p^k \Psi_k(t) = g_2 + \frac{g_1}{\Gamma(\alpha+1)}t^\alpha - pJ^\alpha\left[\left(\sum_{k=0}^{\infty} p^k \Psi_k(t)\right)^2\right](t) \tag{19}$$

Expand the square and equate coefficient of the coefficients of like powers p, producing a hierarchy of linear integral equations for Ψ_k.

Step 5: Zeroth and first equations

$$\mathrm{p}^0 \text{ order 0:} \qquad \Psi_0(\mathrm{t}) := g_2 + \frac{g_1}{\Gamma(\alpha+1)}t^\alpha \tag{20}$$

$$\mathrm{p}^1 \text{ order 1:} \qquad \Psi_1(\mathrm{t}) := -J^\alpha[\Psi_0^2](t) \tag{21}$$

Step 6: Higher order recurrence

For $\mathrm{k} \geq 1$, the p^k the equation reads

$$\Psi_k(\mathrm{t}) = -J^\alpha\left[\sum_{i=0}^{k-1} \Psi_i(\mathrm{t})\Psi_{k-1-t}(\mathrm{t})\right](t) \tag{22}$$

So, each Ψ_k is obtained by integrating the convolution of the lower-order term; all integrals are of the form $J^\alpha[t^\beta]$ when lower terms are polynomial in t^α. Hence computable via

$$J^{\alpha}\left[t^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\alpha+\beta} \tag{23}$$

Step 7: Build the HPM solution

The HPA approximation up to order N is

$$\Psi^{N}(t) = \sum_{k=0}^{N} \Psi_{k}(\mathrm{t}) \tag{24}$$

Letting $N \to \infty$ if convergent yields the exact solution

$$\Psi_{k}(\mathrm{t}) = \sum_{k=0}^{N} \Psi_{k}(\mathrm{t}) \tag{25}$$

TABLE 2. Tabular pseudocode summary of the HPA procedure, written in Python-style format, and the steps are represented.

Step	Description	Python-style Pseudocode
1. Preliminaries	Apply Caputo fractional derivative D_{*}^{α} and its inverse J^{α}. Initialize $\Psi_0 = g_2$.	`python\nPsi_0 = g2\nPsi = g2 + (g1 / gamma(alpha + 1)) * t**alpha - J_alpha(Psi**2, t)\n`
2. Construct the Homotopy	Introduce embedding parameter $p \in [0,1]$. Define the homotopy equation.	`python\nPsi_p = g2 + (g1 / gamma(alpha + 1)) * t**alpha - p * J_alpha(Psi**2, t)\n`
3. Assume Perturbation Expansion	Expand $\Psi(t;p) = \Sigma p^{k} \Psi_{k}(t)$. At the end, set $p = 1$.	`python\nPsi_p = sum(p**k * Psi_k(t) for k in range(infinity))\nPsi = sum(Psi_k(t) for k in range(infinity))\n`
4. Substitute and Collect Powers of p	Substitute series into homotopy equation and collect like powers of p.	`python\nfor k in range(N):\n collect_terms(p**k)\n`
5. Zeroth and First Equations	Compute the first two terms.	`python\nPsi_0 = g2 + (g1 / gamma(alpha + 1)) * t**alpha\nPsi_1 = -J_alpha(Psi_0**2, t)\n`
6. Higher Order Recurrence	For $k \geq 1$, compute recursively using convolution.	`python\nfor k in range(2, N+1):\n Psi_k = -J_alpha(sum(Psi_i * Psi_{k-1-i} for i in range(k)), t)\n`
7. Build the HPA Solution	Compute approximate solution up to order N.	`python\nPsi_HPA = sum(Psi_k for k in range(N+1))\n`

These techniques provide a clear roadmap computational method to derive and obtain exact or approximate solutions using MNIM and HPM, when α is unknown and fractional. One can compute the series $\Psi_i(t)$ explicitly using fractional calculus integration, satisfying the initial conditions and nonlinear term.

4. CONVERGENCE AND ERROR ANALYSIS

4.1 MNIM convergence and error analysis.

Let the MNIM iteration be written as:

$$\Psi_{n+1}(t) = \mathcal{T}(\Psi_n) = g_2 + \frac{g_1}{\Gamma(\alpha+1)} t^\alpha - J^\alpha[\Psi_n^2](t) \tag{26}$$

On $C[0,T]$, Equipped with the supremum norm $\|\cdot\|_\infty$, the fractional integral satisfies

$$\|J^\alpha f\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \|f\|_\infty \tag{27}$$

Hence, for any Ψ, φ.

$$\|\mathcal{T}(\Psi) - \mathcal{T}(\varphi)\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha+1)} (\|\Psi\|_\infty + \|\varphi\|_\infty)\|\Psi - \varphi\|_\infty \tag{28}$$

Assuming $\|\varphi\|_\infty \leq R$, MNIM becomes a contraction provided

$$L = \frac{2RT^\alpha}{\Gamma(\alpha+1)} < 1 \tag{29}$$

By Banach`s fixed-point theorem, the sequence $\{\Psi_n\}$ converges uniquely to the exact solution $\Psi(\mathrm{t})$ on $[0,T]$. The convergence is local and at least linear satisfying,

$$\|\Psi_{n+1} - \Psi_n\|_\infty \leq L\|\Psi_n - \Psi\|_\infty \tag{30}$$

Let $\mathrm{e}_n(t) = \Psi(\mathrm{t}) - \Psi_n(t)$, Then

$$\|\mathrm{e}_n\|_\infty \leq L^n \|\mathrm{e}_0\|_\infty \tag{31}$$

Showing geometric decay on the iterative error.

The MNIM approximate solution is expressed as the truncated series

$$\Psi_M(\mathrm{t}) = \sum_{m=0}^{M} \mathrm{a}_m t^{m\alpha} \tag{32}$$

With truncation error

$$\mathrm{R}_M(\mathrm{t}) = \sum_{m=M+1}^{\alpha} \mathrm{a}_m t^{m\alpha} \tag{33}$$

If the coefficient satisfies $|\mathrm{a}_m| \leq C\rho^m \quad 0 < \rho < 1$, Then

$$|\mathrm{R}_M(\mathrm{t})| \leq \frac{C(\rho T^\alpha)^{M+1}}{1 - \rho T^\alpha} \tag{34}$$

Confirming the absolute convergence of the MNIM and the convergence achieved when

$$\|\Psi_{n+1} - \Psi_n\| < \varepsilon \tag{35}$$

4.2 HPM convergence and error analysis.

Let the HPM solution of the fractional Riccati equation be expressed as

$$\Psi(t) = \sum_{k=0}^{\infty} \Psi_k\,(t) \tag{36}$$

Where the components $\Psi_k(t)$ are generated recursively from equations (20)-(22). Define the truncated approximation of order N by

$$\Psi^N(t) = \sum_{k=0}^{N} \Psi_k\,(t) \tag{37}$$

The convergence analysis from the recursive relation (22)

$$\Psi_k(t) = -J^\alpha \left[\sum_{i=0}^{k-1} \Psi_i(t)\Psi_{k-1-i}\,(t)\right] \tag{38}$$

and using the boundedness property of the fractional integral operator.

$$\|J^\alpha f\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha+1)}\|f\|_{\infty,} \tag{39}$$

We obtain

$$\|\Psi_k\|_\infty \leq \frac{T^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{k-1} \|\Psi_i\|_\infty \, \|\Psi_{k-1-i}\|_\infty \tag{40}$$

Assume that $\|\Psi_k\|_\infty \leq C\rho^k \quad 0 < \rho < 1,$
For some constant $C > 0$. Then the HPM series satisfies

$$\sum_{i=0}^{\infty} \|\Psi_k\|_\infty < \infty, \tag{41}$$

Which guaranteed absolute and uniform convergence on $[0, T]$.
Hence, the HPM solution converges locally to the exact solution provided

$$\frac{2RT^\alpha}{\Gamma(\alpha+1)} < 1, \tag{42}$$

Where $R = sup_{\in t[0,T]}\, |\Psi(\mathrm{t})|$. This condition ensures that the nonlinear operator is contractive and the constructed homotopy converges.
Therefore

$$\lim_{N\to\infty} \Psi^N(t) = \Psi(\mathrm{t}), \tag{43}$$

Proved the existence and uniqueness of the HPM.

5. APPLICATION AND RESULTS

We applied the MNIM and HPM discussed in the last section, which are subject to nonhomogeneous terms and initial conditions suggested by [7,14]. Consider the Riccati exact solution as a benchmark for the two cases considered, it is observed that the modified new iterative method (MNIM) attains higher accuracy with fewer iterations compared to the homotopy perturbation method (HPM), demonstrating superior convergence and computational efficiency for the forced Riccati equation and the approximate series solutions as follows:

$$\Psi(\mathrm{t})_{case1} = \begin{cases} \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} - \dfrac{\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2} + \dfrac{2\Gamma(2\alpha+1)\Gamma(4\alpha+1)t^{5\alpha}}{\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(\alpha+1)^3} - \\ \dfrac{17\Gamma(2\alpha+1)\Gamma(4\alpha+1)\Gamma(6\alpha+1)t^{7\alpha}}{6\Gamma(3\alpha+1)\Gamma(5\alpha+1)\Gamma(7\alpha+1)\Gamma(\alpha+1)^4} + .. \end{cases} \tag{44}$$

$$\Psi(\mathrm{t})_{case2} = \begin{cases} \cos(2) + \left(-\dfrac{(\cos(2))^2 t^{\alpha}}{\Gamma(\alpha+1)}\right) + \left(\dfrac{2t^{1+\alpha}}{\Gamma(\alpha+2)}\right) + \\ \left(-\dfrac{4\cos(2)\Gamma(\alpha+2)\, t^{2+2\alpha}}{\Gamma(\alpha+2)\Gamma(2\alpha+2)}\right) + \left(-\dfrac{4\,\Gamma(2\alpha+3)\, t^{3+3\alpha}}{\Gamma(\alpha+2)^2\Gamma(3\alpha+3)}\right) + \\ \left(-\dfrac{(\cos(2))^4\Gamma(2\alpha+1)t^{3\alpha}}{\Gamma(3\alpha+1)\Gamma(\alpha+1)^2}\right) + \cdots \end{cases} \tag{45}$$

According to [15], the exact solutions obtained when $\alpha = 1$ are presented as follows:

TABLE 3. Comparison between the exact solution, the modified new iterative method, and the homotopy perturbation method when $g_1 = 1, g_2 = 0, \alpha = 1$ the fractional Riccati equation.

t	Exact (tanh(t))	MNIM	HPM	$\lvert Exact - MNIM\rvert$	$\lvert Exact - HPM\rvert$
0	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.2	0.19737532	0.19737531	0.19737511	.1.0-E8	2.0-E7
0.4	0.37994896	0.37994894	0.37994888	2.0-E8	1.6-E7
0.6	0.53704957	0.53704953	0.53704921	4.0-E8	3.6-E7
0.8	0.66403677	0.66403673	0.66403634	4.0-E8	4.3-E7
1.0	0.76159416	0.76159416	0.76159410	0.0000000	6.0-E8

TABLE 4. Comparison between the exact solution, the modified new iterative method, and the homotopy perturbation method when $g_1 = 2t, g_2 = \cos(2), \alpha = 1$ fractional Riccati equation.

t	Exact	MNIM	HPM	$\lvert Exact - MNIM\rvert$	$\lvert Exact - HPM\rvert$
0	-0.41614684	-0.41614684	-0.41614684	0.00000000	0.00000000
0.2	-0.41152364	-0.41152361	-0.41152352	3.0-E8	1.2-E7
0.4	-0.31954196	-0.31954192	-0.31954166	4.0-E8	3.0-E7
0.6	-0.13103259	-0.13103255	-0.13103243	4.0-E8	1.6-E7
0.8	0.14766569	0.14766567	0.14766556	4.0-E8	1.3-E7
1.0	0.48605854	0.48605851	0.48605844	3.0-E8	1.0-E7

TABLE 5. Numerical solutions for $g_1 = 1, g_2 = 0$ and various orders $\alpha = [0.8, 0.6, 0.4, 0.2]$

t	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.2	0.2852557272	0.3873060853	0.4862405550	0.4752544756
0.4	0.4635021734	0.5335392216	0.5655817469	0.2761290230
0.6	0.5926911879	0.6234995920	0.5402102933	-0.0494288250
0.8	0.6901524767	0.6632402623	0.3773264547	-0.4699133430
1.0	0.7595688173	0.6152887298	0.0290085130	-0.9695453240

TABLE 6. Numerical solutions for $g_1 = 2t, g_2 = \cos(2)$ and various orders $\alpha = [0.8,0.6,0.4,0.2]$

t	$\alpha = 0.8$	$\alpha = 0.6$	$\alpha = 0.4$	$\alpha = 0.2$
0	-0.4161468365	-0.4161468365	-0.41614683651	-0.4161468365
0.2	-0.4004301738	-0.3799252056	-0.34048260913	-0.2696723982
0.4	-0.2635554143	-0.1798924137	-0.06296290470	0.08188764502
0.6	-0.0227805005	0.1164338082	0.263437812421	0.36851009539
0.8	0.2924526950	0.4303288541	0.493836744867	0.38953237286
1.0	0.5984456613	0.6098738570	0.409926508089	0.09926055952

5.1 2D GRAPHS REPRESENTATION

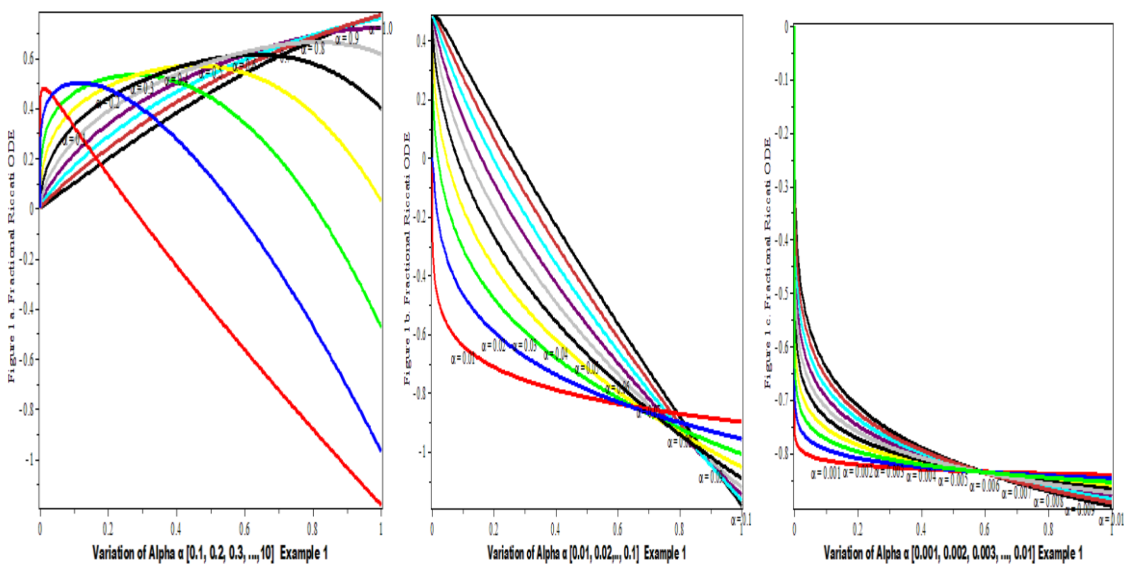

FIGURE 1a. Depicts a 2D plot solution trajectory that describes the system's dynamic response over time for the fractional derivative $0 < \alpha \leq 1$. FIGURE 1b. Depicts a 2D plot solution trajectory that describes the system's dynamic response over time for the fractional derivative $0 < \alpha \leq 0.1$. FIGURE 1c. Depicts a 2D plot solution trajectory that describes the system's dynamic response over time for the fractional derivative $0 < \alpha \leq 0.01$

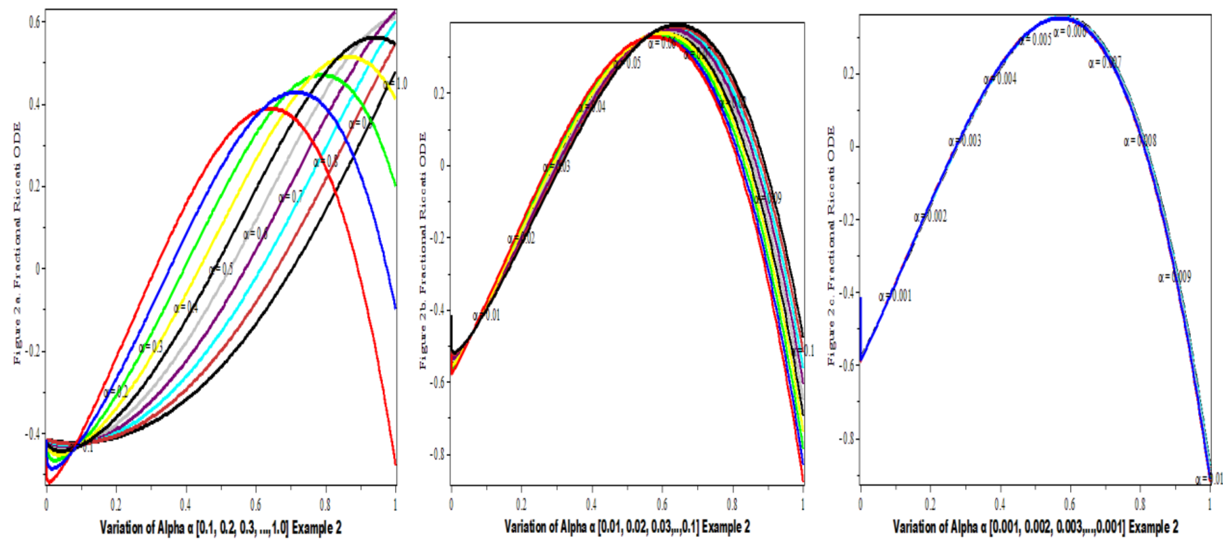

FIGURE 2a. Depicts a 2D plot solution trajectory that describes the system's dynamic response over time for the fractional derivative $0 < \alpha \leq 1$. FIGURE 2b. Depicts a 2D plot solution trajectory that describes the system's dynamic response over time for the fractional derivative $0 < \alpha \leq 0.1$. FIGURE 2c. Depicts a 2D plot solution trajectory that describes the system's dynamic response over time for the fractional derivative $0 < \alpha \leq 0.01$.

6. DISCUSSION AND CONCLUSION

6.1 DISCUSSION

We implemented two numerical techniques, MNIM and HPM computational algorithm based to solve the nonlinear Riccati equation arising in modeling nonlinear phenomena with memory effects across control theory, viscoelasticity, and anomalous transport. The two illustrated examples are considered, and the results are presented in TABLE 3, TABLE 4, TABLE 5, TABLE 6, and 2D plots illustrated in FIGURES 1a–1c and 2a-2c. Below is a concise discussion as follows

FIGURE 1a. Demonstrates that as the fractional order α increases toward 1, the solution gradually converges to classical integer-order behavior with reduced memory effects. This indicates stabilization of the dynamics and diminished influence of anomalous transport as α approaches unity.
FIGURE 1b. Depicted that for very small fractional orders, the solution decays rapidly with time due to dominant memory effects. This behavior highlights the strong influence of fractional dynamics on early-time responses in systems such as control and viscoelastic models.
FIGURE 1c. Indicated that even extremely small variations in α produce noticeable changes in the decay rate of the solution. This emphasizes the high sensitivity of fractional Riccati models and the importance of precise tuning of α for accurately capturing memory-dependent nonlinear dynamics.
FIGURE 2a. Demonstrated that increasing α from 0.1 to 1.0 drives the solution toward classical Riccati behavior with smoother and more distinct oscillations. This reflects the gradual weakening of memory effects as the fractional order approaches the integer case.

FIGURE 2b. Indicated that for α in the range 0.01-0.1, the solutions experience stronger damping and delayed peak responses. This behavior underscores the enhanced role of memory-dependent mechanisms that slow the system dynamics at lower fractional orders.
FIGURE 2c. Illustrate that for very small fractional orders (0.001-0.01), the solutions nearly overlap and display minimal oscillations. This confirms the dominance of long-term memory effects, leading to highly smoothed and significantly retarded system responses.
TABLE 3. The absolute errors show that MNIM consistency achieves higher accuracy than HPM over $t \in [0.1]$, with MNIM errors remaining at 10^{-8} where as HPM errors are typically one order of magnitude larger 10^{-7}, indicating faster convergence and better numerical stability of MNIM for this case.
TABLE 4. A similar trend is observed for MNIM errors, which are uniformly distributed around 3.4×10^{-8}, while HPM errors range from 10^{-7}. The absolute errors confirm MNIM`s superior accuracy across the interval, even as the solution transitions from negative to positive values.
The negative values observed (initially) arise from the chosen coefficients and initial conditions of the Riccati equation, which naturally yield solutions below zero before transitioning upward. These negative results are physically and mathematically consistent with the exact solution and do not indicate instability, as both MNIM and HPM closely track the exact behavior with very small absolute errors.
FIGURE 5. The variation of the fractional order α significantly influences the solution behavior. Higher α values yield smoother, monotonic growth resembling the classical solution, whereas decreasing α introduces stronger memory effects, leading to reduced magnitudes, oscillatory tendencies, and eventual sign changes, highlighting the sensitivity of the system dynamics to fractional order.
FIGURE 6. The fractional order α strongly affects solution evolution. Larger α values produce smoother transitions and delayed sign changes, while smaller α values accelerate growth toward positive values and amplify nonlinear effects. This demonstrates how reducing α enhances memory influence, significantly altering the rate, magnitude, and trend of the numerical solutions.

6.2 CONCLUSION

This study confirms that the modified new iterative method (MNIM) and homotopy perturbation method (HPM) are efficient and reliable techniques for solving fractional-order Riccati differential equations describing nonlinear systems with memory. The results for different fractional orders α illustrate how fractional derivatives govern the transition between oscillatory and diffusive behaviors. MNIM offers rapid convergence and analytical insight, which provides high accuracy and numerical stability. Numerical and graphical comparisons show excellent agreement with exact solutions, validating both approaches. As α decreases, enhanced memory and damping effects emerge, consistent with fractional dynamics. Consequently, MNIM and HPM serve as robust semi-analytical tools for applications in control theory, viscoelasticity, and anomalous transport.

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Authors Contributions.
Falade K.I carried out the methodology, validation, and wrote the original draft. Adeyemo K.A, Muazu N, and Ayodele V. I did the analysis and proofread the article. Raifu S.A,

Babatunde O. V, and Abdullahi B.O corrected and typed the final draft of the article. All the authors read and approved the final manuscript

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REFERENCES

[1] Z. Odibat, S. Momani and H. Xu, A reliable algorithm of homotopy analysis method for solving nonlinear fractional differential equations, *Appl. Math. Model.* 34, (2010), 593-600.
[2] B. Xie and Y. Gao, Numerical analysis for fractional Riccati differential equations, *J. Nonlinear Model. Anal.* 7, (2025), 189-208.
[3] K.I. Falade and S.A. Raifu, Solving classical nonlinear Riccati differential equations using differential transformation method, *Int. J. Math. Appl.* 3(3), (2015), 71–78.
[4] M. Aydemir and M. Merdan, Elzaki-Adomian decomposition method for local fractional Riccati equations with applications in nonlinear ODEs, *AIMS Math.* 10(4), (2025), 9122–9149.
[5] A.F. Fareed, M.S. Semary and H.N. Hassan, Approximate solution of fractional-order Riccati equations based on controlled Picard's method with Atangana-Baleanu derivative, *Alex. Eng. J.* 61(5), (2022), 3673–3678.
[6] E. Abuteen, Solving fractional Riccati differential equations with Caputo–Fabrizio fractional derivative, *Eur. J. Pure Appl. Math.* 17(1), (2024), 372-384.
[7] Numerical studies for solving fractional Riccati differential equations, *Appl. Appl. Math.* 7(2), (2012), 595-608.
[8] K.S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York, (1993).
[9] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, (1999).
[10] V.K. Srivastava, M.K. Awasthi and S. Kumar, Analytical approximations of two- and three-dimensional time-fractional telegraphic equations by reduced differential transform method, *Egypt. J. Basic Appl. Sci.*, (2014), 1-7.
[11] A.K. Farhood and O.H. Mohammed, Homotopy perturbation method for solving time-fractional nonlinear variable-order delay partial differential equations, *Partial Differ. Equ. Appl. Math.* 7, (2023), 1-14.
[12] L. Wang and Y. Wang, A modified iterative method for solving the non-symmetric coupled algebraic Riccati equation, *Taiwanese J. Math.* 28(2), (2024), 377-396.
[13] N. Yadav, A. Das, M. Singh, S. Singh and J. Kumar, Homotopy perturbation method and convergence analysis for nonlinear collisional fragmentation equations, *Proc. R. Soc. A*, (2023), 1–15.
[14] Z. Odibat and S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equations of fractional order, *Chaos Solitons Fractals* 36(1), (2008), 167-174.
[15] Ş. Yüzbaşı, Numerical solutions of fractional Riccati type differential equations by Bernstein polynomials, *Appl. Math. Comput.* 219(11), (2013), 6328–6343.

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