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NUCLEAR IDENTIFICATION OF EXTRA LOOP IDENTITIES OF SECOND BOL-MOUFANG TYPE WITH APPLICATIONS TO SECURE INFORMATION ENCODING

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Abstract. Drpal and Jedlika identified several loop identities, including both BolMoufang and non-BolMoufang varieties through their nuclei. Among these are the extra identities. Subsequently, George and Jaiyeola developed a generalized nuclear identification scheme for identities of the second BolMoufang type. While they discovered twelve new loop identities, their approach did not establish a nuclear identification for the extra identities. This left open the question of whether extra-type identities admit nuclear identification in the second BolMoufang setting. In this paper, we introduced three new loop identities (SBME, SBRE, SBLE) of second Bol-Moufang type. In particular, we show that an extra loop of second BolMoufang type is nuclear-identifiable if it can be expressed as autotopisms $\alpha^{\epsilon} * \eta(x)\alpha^{\omega} * \xi(x)\alpha^{\kappa} * \chi(x)\alpha^{\psi} * \zeta(x)$ and satisfies an identity $(\eta,\ \xi,\ \chi,\ \zeta,\ \epsilon,\ \omega,\ \kappa,\ \psi)$ such that $\eta = \chi \neq \xi \neq \zeta$. Furthermore, we show that the newly introduced extra identities of second BolMoufang type are equivalent to the extra identities of first BolMoufang type. Beyond the algebraic characterization, we illustrate how nuclear identification codes can inspire secure information encoding schemes, where loop identities serve as human-recognizable keys for controlled access in sensitive communication environments.

1. Introduction

The nuclear identification technique introduced by Drpal and Jedlika [2] has proven to be a powerful algebraic tool for detecting and classifying loop identities. Their analysis focused on some BolMoufang identities of the first type, originally introduced by Fenyves in [4], [5] and later completed by Phillips and Vojtẽchovský́ in [23], [24] and some non Bol- Moufang type identities such as the LCC, RCC and the Buchsteiner loops. They successfully identified, among others, the extra loop identities through nuclear characterization.

Recently, George and Jaiyeola [7] extended this framework to develop a generalized nuclear identification model, leading to the discovery of twelve new identities of second BolMoufang type. Among these identities, eight were shown to be genuinely new and nonequivalent to all previously known loop identities, including pairs that axiomatize WIPPACC loops, and identities characterizing Moufang and Buchsteiner loops. However, while the classical nuclear identification scheme captures the extra identities, the generalized model did not recover these extra laws. This raises the question of whether extra-type identities of second second BolMoufang can be recovered via nuclear identification.

The aim of this work is to answer this question: we demonstrate that the extra law does indeed admit a nuclear characterization in the second BolMoufang context, thereby completing the analogy with the first BolMoufang framework and filling the gap left in previous investigations.

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2. Preliminaries

A loop $(Q, \cdot)$ is a set with a binary operation $(\cdot)$ such that for each $a \in Q$, the translation maps $L(a) : Q \to Q : x \to ax$ and $R(a) : Q \to Q : x \to xa$ are bijections and $1x = x1 = x$ for all $x \in Q$. Since the translation maps are bijections, then the inverse maps $L^{-1}(a)$ and $R^{-1}(a)$ exist and are defined by $bL^{-1}(a) = a \setminus b$ and $bR^{-1}(a) = b/a$. The reader is referred to [16, 20, ?, 25] for a general overview on quasigroups and loops.
For $x \in Q$, we denote the left inverse of x by x^λ and the right inverse of x by x^ρ such that $x^\lambda x = 1$ and $xx^\rho = 1$. A loop in which $x^\rho = x^\lambda$ for all elements x is said to have 2-sided inverse.
The left nucleus N_λ, the middle nucleus N_μ and the right nucleus N_ρ of a loop Q are defined by

$$N_\lambda(Q) = \{a \in Q : \ a \cdot xy = ax \cdot y \ \ \forall \ x, \ y \in Q\},$$
$$N_\mu(Q) = \{a \in Q : \ xa \cdot y = x \cdot ay \ \ \forall \ x, \ y \in Q\},$$
$$N_\rho(Q) = \{a \in Q : \ xy \cdot a = x \cdot ya \ \ \forall \ x, \ y \in Q\}.$$

The intersection

$$N(Q) = N_\rho(Q) \cap N_\lambda(Q) \cap N_\mu(Q)$$

is called the nucleus of Q.

A triple of bijections (U, V, W) is called an autotopism of a loop Q provided that

$$xU \cdot yV = (xy)W. \tag{2.1}$$

for all $x, y \in Q$. The set of such triples forms a group $Atp(Q)$ called the autotopism group of Q.
It is easy to see that

$$a \in N_\lambda(Q) \Leftrightarrow (L(a), \ I, \ L(a)) \in Atp(Q) \tag{2.2}$$
$$a \in N_\mu(Q) \Leftrightarrow (R^{-1}(a), \ L(a), \ I) \in Atp(Q) \tag{2.3}$$
$$a \in N_\rho(Q) \Leftrightarrow (I, \ R(a), \ R(a)) \in Atp(Q) \tag{2.4}$$

Code	Identity	Label	Equivalent Form(s) ($\Leftrightarrow$)
$(\mu, \mu, \lambda, \mu; +, +, -, -)$	$x(yx \cdot xz) = ((xy \cdot x)x)z$	Q_1	LB + $P_\lambda(x, y)$
$(\mu, \mu, \rho, \mu; -, -, -, +)$	$(yx \cdot xz)x = y(x(x \cdot zx))$	Q_2	RB + $P_\rho(x, y)$
$(\mu, \mu, \lambda, \mu; +, +, +, -)$	$(xy \cdot x) \cdot xz = x((yx \cdot x)z)$	Q_3	LWPC=LCC + $P_\lambda(x, y)$
$(\mu, \mu, \rho, \mu; -, -, +, +)$	$yx \cdot (x \cdot zx) = (y(x \cdot xz))x$	Q_4	RWPC=RCC + $P_\rho(x, y)$
$(\rho, \rho, \mu, \rho; -, -, -, +)$	$(yx \cdot zx)x = y((xz \cdot x)x)$	Q_5	RM
$(\rho, \rho, \lambda, \rho; +, +, +, -)$	$(xy \cdot zx)x = x((yz \cdot x)x)$	Q_6	MM1 or MM2
$(\rho, \rho, \mu, \rho; -, -, +, +)$	$(y(xz \cdot x))x = yx \cdot (zx \cdot x)$	Q_7	RCC + $P_\lambda(x, y)$
$(\rho, \rho, \lambda, \rho; +, +, -, -)$	$x((y \cdot zx)x) = ((xy \cdot z)x)x$	Q_8	BUCH + $P_\lambda(x, y)$
$(\lambda, \lambda, \rho, \lambda; +, +, +, -)$	$x(xy \cdot zx) = (x(x \cdot yz))x$	Q_9	MM1 or MM2
$(\lambda, \lambda, \mu, \lambda; -, -, +, +)$	$x(xy \cdot xz) = (x(x \cdot yx))z$	Q_{10}	LM
$(\lambda, \lambda, \rho, \lambda; +, +, -, -)$	$(x(xy \cdot z))x = x(x(y \cdot zx))$	Q_{11}	BUCH + $P_\rho(x, y)$
$(\lambda, \lambda, \mu, \lambda; -, -, -, +)$	$x((x \cdot yx)z) = (x \cdot xy) \cdot xz$	Q_{12}	LCC + $P_\rho(x, y)$

Table 1. Summary of new loop identities induced by nuclear identifications and their equivalent forms [7]

A loop identity $\alpha = \beta$ is said to be of first Bol-Moufang type (Bol-Moufang type) if it has 3 distinct variables with two appearing once on both sides, the third variable appears 2 times, the variables appear in the same order on both sides and the only binary operation used is multiplication.
A loop identity of length five is said to be of second Bol-Moufang type if it has 3 distinct variables with two appearing once on both sides, the third variable appears 3 times, the variables appear in the same order on both sides and the only binary operation used is multiplication, see Table 1.

A loop is flexible if it satisfies the identity

$$x \cdot yx = xy \cdot x \tag{FLEX}$$

A loop Q satisfies the left inverse property if

$$x^{\lambda} \cdot xy = y \tag{LIP}$$

and the right inverse property if

$$xy \cdot y^{\rho} = x. \tag{RIP}$$

An inverse property loop is a loop that satisfies both the (LIP) and the (RIP). A loop is a weak inverse property loop if it satisfies any one of the following identities:

$$x(yx)^{\rho} = y^{\rho} \quad or \quad (xy)^{\lambda}x = y^{\lambda}. \tag{WIP}$$

Lemma 2.1 ([22]). *Let Q be a WIPL. If $(U, V, W) \in Atp(Q)$: label=0. $(V,\ \lambda W\rho,\ \lambda U\rho) \in Atp(Q)$.*
lbbel=0. $(\rho W\lambda,\ U,\ \rho V\lambda) \in Atp(Q)$.

Theorem 2.2. [20] *Let $A = (U,\ V,\ W)$ be the autotopism of a loop Q.*
(1) If Q is a left inverse property loop, then $A_\lambda = (IUI,\ W,\ V)$ is also an autotopism of Q.
(2) If Q is a right inverse property loop, then $A_\lambda = (W,\ IVI,\ U)$ is also an autotopism of Q.

A loop $(Q, \cdot)$ is called an extra loop if Q satisfies one (equivalently all) of the following identities:

$$(x \cdot yz)y = xy \cdot zy. \tag{RE}$$
$$yz \cdot yx = y(zy \cdot x). \tag{LE}$$
$$(xy \cdot z)x = x(y \cdot zx). \tag{ME}$$

3. Result

The three newly introduced identities are defined as follows:

Definition 3.1. A loop satisfying the identity

$$x(y(x \cdot zx)) = (xy \cdot xz)x. \tag{SBME}$$

is called second Bol-Moufang middle extra loop.
A loop satisfying the identity

$$(xy \cdot xz)x = (x \cdot yx) \cdot zx. \tag{SBRE}$$

is called second Bol-Moufang right extra loop.
A loop satisfying the identity

$$x((yx \cdot z)x) = (xy \cdot xz)x. \tag{SBLE}$$

is called second Bol-Moufang left extra loop.

Definition 3.2. A loop identity of second Bol-Moufang type is said to be nuclear identifiable if it can be expressed as autotopisms $\alpha_\eta^\epsilon(x)\alpha_\xi^\omega(x)\alpha_\chi^\kappa(x)\alpha_\zeta^\psi(x)$, where $\epsilon,\ \omega,\ \kappa,\ \psi \in \{-1,\ 1\}$, $\eta,\ \xi,\ \chi,\ \zeta \in \{\lambda,\ \rho,\ \mu\}$ and $\alpha_\lambda(x) = (L(x),\ I,\ L(x))$, $\alpha_\rho(x) = (I,\ R(x),\ R(x))$ and $\alpha_\mu(x) = (R^{-1}(x),\ L(x),\ I)$.

We shall code such identity $Id = Id_\alpha$ as $(\eta,\ \xi,\ \chi,\ \zeta\ ;\epsilon,\ \omega,\ \kappa,\ \psi)$ and replace 1 and -1 by $+$ and $-$ in concrete instances

Lemma 3.3. *Let Q be a loop satisfying an identity $(\eta,\ \xi,\ \chi,\ \zeta;\ \epsilon,\ \omega,\ \kappa,\ \psi)$ such that $\eta = \chi \neq \xi \neq \zeta$. Then, $N_\eta = N_\xi = N_\zeta$.*

Proof. Let Q be a loop satisfying an identity $(\eta,\ \xi,\ \chi,\ \zeta,\ \epsilon,\ \omega,\ \kappa,\ \psi)$ which is equivalently expressible by the autotopism α. Then, $\alpha = \alpha_\eta^\epsilon(x)\alpha_\xi^\omega(x)\alpha_\chi^\kappa(x)\alpha_\zeta^\psi(x)$, where $\epsilon, \omega, \kappa, \psi \in \{-1, 1\}$ and $\eta,\ \xi,\ \chi,\ \zeta \in \{\lambda,\ \rho,\ \mu\}$. With the hypothesis $\eta = \chi \neq \xi \neq \zeta$ and $N_\eta = N_\xi = N_\zeta$. □

Remark 3.4. Lemma 3.3 establishes that, for the class of loops considered in this work, the three nuclei coincide; that is, $N_\lambda = N_\rho = N_\mu$.

Theorem 3.5. *Let Q be a loop satisfying the SBME identity. Then $(L^{-1}(x),\ L^{-1}(x)R(x)(L(x),\ R(x)L^{-1}(x))$ is an autotopism of Q and Q can be coded as $(\mu,\ \rho,\ \mu,\ \lambda,\ -,\ +,\ +,\ -)$.*

Proof. Suppose Q satisfies SBME,

$$x(y(x\cdot zx)) = (xy\cdot xz)x \Leftrightarrow x(x\setminus y\ \cdot\ x((x\setminus z)x)) = yz\cdot x \Leftrightarrow (x\setminus y)\ \cdot\ x((x\setminus z)x) = x\setminus(yz\cdot x).$$

Thus $(L^{-1}(x),\ L^{-1}(x)R(x)(L(x),\ R(x)L^{-1}(x))$ is an autotopism of Q. It is then easy to see that Q can be coded as $(\mu,\ \rho,\ \mu,\ \lambda,\ -,\ +,\ +,\ -)$. □

Proposition 3.6. *Let Q be a loop satisfying the SBME identity. Then:*

(1) Q has two sided inverse.
(2) Q is flexible.
(3) Q has WIP.
(4) Q has LIP.

Proof.

(1) Put $y = 1$ and $z = x^\lambda$ in the SBME identity to obtain

$$xx = (x\cdot xx^\lambda)x \Leftrightarrow x = x\cdot xx^\lambda \Leftrightarrow e = xx^\lambda \Leftrightarrow x^\rho = x^\lambda.$$

(2) Replace z with x^λ in the SBME identity,

$$x\cdot yx = (xy\cdot xx^\lambda)x = (xy\cdot xx^\rho)x = xy\cdot x.$$

(3) The SBME identity is equivalent to

$$x((x\setminus y)(x\cdot zx)) = (y\cdot xz)x.$$

Put $y = (xz)^\lambda$ to get

$$x\setminus(xz)^\lambda\ \cdot\ (x\cdot zx) = 1.$$

Therefore,

$$x\setminus(xz)^\lambda = (x\cdot zx)^\lambda \Leftrightarrow (xz)^\lambda = x(x\cdot zx)^\lambda = x(xz\cdot x)^\lambda \Leftrightarrow z^\lambda = x(zx)^\lambda$$

and by (1)

$$z^\rho = x(zx)^\rho.$$

(4) Writing the SBME identity as $x\backslash y\ \cdot x((x\backslash z)x) = x\backslash(yz\cdot x)$, put $y = 1$ to get

$$x^\rho\cdot x((x\backslash z)x) = x\backslash(zx) = (x\backslash z)x,$$

replace z with $(xz)/x$, we obtain

$$x^\rho\cdot xz = z$$

□

Theorem 3.7. *Let Q be a loop satisfying the SBRE identity. Then $(L^{-1}(x)R(x)(L(x),\ L^{-1}(x)R(x),\ R(x))$ is an autotopism of Q and Q can be coded as $(\lambda,\ \mu,\ \lambda,\ \rho,\ -,\ -,\ +,\ +)$.*

Proof. we can write the SBRE identity also as

$$x(x\setminus y)x)\ \cdot(x\setminus z)x = yz\cdot x.$$

Thus,

$$(L^{-1}(x)R(x)(L(x),\ L^{-1}(x)R(x),\ R(x)) \in Atp(Q)$$

and it is easy to see that Q can be coded as $(\lambda,\ \mu,\ \lambda,\ \rho,\ -,\ -,\ +,\ +)$. □

Proposition 3.8. *Let Q be a loop satisfying the SBRE identity. Then:*

(1) Q is flexible.
(2) Q has two sided inverse.
(3) Q has WIP.
(4) Q has LIP.

Proof.

(1) Set $z = 1$ in the SBRE identity to get

$$(xy \cdot x)x = (x \cdot yx)x \Leftrightarrow xy \cdot x = x \cdot yx.$$

(2) Since Q is flexible, i.e., $xy \cdot x = x \cdot yx$, we are done if we put $y = x^\rho$ or $y = x^\lambda$.

(3) The SBRE identity is equivalent to

$$(y \cdot xz)x = (x \cdot (x \setminus y)x) \cdot zx, \tag{3.1}$$

put $y = (xz)^\lambda$ in equation (3.1) and use flexibility law,

$$x = (xz)^\lambda x \cdot zx$$

or

$$x/(zx) = (xz)^\lambda x. \tag{3.2}$$

Again from equation (3.1), put $y = x^\lambda$ and use flexibility law

$$(x^\lambda \cdot xz)x = (x \cdot (x \setminus x^\lambda)x \ \cdot zx \Rightarrow$$
$$(x^\lambda \cdot xz)x = (x(x \setminus x^\lambda)x \ \cdot zx \Rightarrow$$
$$(x^\lambda \cdot xz)x = x^\lambda x \ \cdot zx \Rightarrow$$
$$(x^\lambda \cdot xz)x = zx \Rightarrow$$
$$x^\lambda \cdot xz = z \Rightarrow \tag{3.3}$$
$$x^\lambda = z/(xz). \tag{3.4}$$

Do $x \Leftrightarrow z$ in equation (3.4) and substitute in (3.2), we obtain $(xz)^\lambda x = z^\lambda$.

(4) Equation (3.3) is the left inverse property.

□

Theorem 3.9. *Let Q be a loop satisfying the SBLE identity. Then $(L^{-1}(x)R(x),\ L^{-1}(x),\ R(x)L^{-1}(x)R^{-1}(x))$ is an autotopism of Q and Q can be coded as $(\rho,\ \lambda,\ \rho,\ \mu,\ +,\ -,\ -,\ -)$.*

Proof. Let Q be a SBLE loop, the SBLE identity is equivalent to

$$(x \setminus y)x \cdot x \setminus z = (x \setminus (yz \cdot x))/x,$$

so $(L^{-1}(x)R(x),\ L^{-1}(x),\ R(x)L^{-1}(x)R^{-1}(x))$ is an autotopism of Q and therefore Q can be coded as $(\rho,\ \lambda,\ \rho,\ \mu,\ +,\ -,\ -,\ -)$. □

Proposition 3.10. *Let Q be a loop satisfying the SBLE identity. Then:*

(1) Q is flexible.
(2) Q has two sided inverse.
(3) Q has RIP.
(4) Q has WIP.
(5) Q has LIP.

Proof.

(1) Put $y = 1$ in SBLE identity.

(2) use (1) above.

(3) Replace z with x^ρ in the SBLE identity,

$$x((yx \cdot x^\rho)x) = xy \cdot x = x \cdot yx \Leftrightarrow yx^\rho \cdot x = y.$$

(4) Replace z with $(yx)^\rho$ in the SBLE identity,

$$xx = (xy \cdot x(yx)^\rho)x \Leftrightarrow x = xy \cdot x(yx)^\rho \Leftrightarrow (xy) \setminus x = x(yx)^\rho,$$

by Proposition 3.10 (3), $(xy) \setminus x = y^\rho$, we therefore have

$$x(yx)^\rho = y^\rho.$$

(5) This is from Proposition 3.10 (3) and (4).

□

Code	Autotopism	Identity	Label
$(\mu,\rho,\mu,\lambda;-,+,+,-)$	$(L^{-1}(x), L^{-1}(x)R(x)L(x), R(x)L^{-1}(x))$	$x(y(x\cdot zx)) = ((xy\cdot xz)x$	SBME
$(\lambda,\mu,\lambda,\rho;-,-,+,+)$	$(L^{-1}(x)R(x)L(x), L^{-1}(x)R(x), R(x))$	$(xy\cdot xz)x = (x\cdot yx)\cdot zx$	SBRE
$(\rho,\lambda,\rho,\mu;+,-,-,-)$	$(L^{-1}(x)R(x), L^{-1}(x), R(x)L^{-1}(x)R^{-1}(x))$	$x((yx\cdot z)x) = (xy\cdot xz)x$	SBLE

TABLE 2. Nuclear identification of Extra Loops of Second Bol-Moufang Type

Theorem 3.11. *Let Q be a loop. Then the following are equivalent:*

(1) Q is SBME loop. *(2) Q is SBRE loop.* *(3) Q is SBLE loop.*

Proof. (1) $\Leftrightarrow$ (2)
Let Q be SBME loop. Then $(L^{-1}(x),\ L^{-1}(x)R(x)(L(x),\ R(x)L^{-1}(x))$ is an autotopism of Q and by Proposition 3.6 (3), Q has WIP, using Theorem 2.1 (a) on the autotopism of Q, we obtain

$$(L^{-1}(x)R(x)(L(x),\ L^{-1}(x)R(x),\ )R(x)) \in Atp(Q).$$

Thus Q is SBRE.
The converse follows a similar argument.
(1) $\Rightarrow$ (3), again start with the autotopism $(L^{-1}(x),\ L^{-1}(x)R(x)(L(x),\ R(x)L^{-1}(x))$ and use Theorem 2.1 (b) on the autotopism of Q to obtain $(L^{-1}(x)R(x),\ L^{-1}(x),\ R(x)L^{-1}(x)R^{-1}(x))$, thus showing that a SBME loop implies SBLE.
(3) $\Rightarrow$ (2), let Q be a SBLE loop, then $(L^{-1}(x)R(x),\ L^{-1}(x),\ R(x)L^{-1}(x)R^{-1}(x))$ is an autotopism of Q and by Proposition 3.10 (3), (5), Q has RIP and LIP, then using Theorem 2.2 (2) and (1) on the autotopism of Q, we obtain SBRE. □

Definition 3.12. A loop $(Q,\cdot)$ is called an extra loop of second Bol-Moufang (SE) loop if Q satisfies one (equivalently all) of SBME, SBRE, SBLE. identities:

Example 3.13. An exhaustive search using the `GAP--LOOPS` package [18, 19] shows that there are no non-associative loops of order $n < 16$ satisfying any of the identities *(SBME)*, *(SBRE)*, or *(SBLE)*. At order 16, there are exactly five non-associative loops satisfying these identities, identified in the library as `SmallLoop(16,`m`)` for $m \in \{114, 115, 116, 117, 118\}$. Consequently, these loops represent the minimal non-associative examples satisfying the second Bol–Moufang extra type identities.

Remark 3.14. The examples obtained in our search coincide with the constructions of non-associative Moufang loops of order 16 presented in the work of Chein [1].

Theorem 3.15. *Let Q be a loop. Then:*

(1) Q is SBME loop if and only if Q is ME.
(2) Q is SBRE loop if and only if Q is RE.
(3) Q is SBLE loop if and only if Q is LE.

Proof. (1) Suppose Q is SBME,

$$x(y(x \cdot zx)) = (xy \cdot xz)x,$$
$$x(y(xz \cdot x)) = (xy \cdot xz)x,$$
$$x(y \cdot zx) = (xy \cdot z)x.$$

Thus Q is ME.
Conversely, suppose Q is ME loop, replace z with xz in the middle extra identity to get

$$x(y(xz \cdot x)) = (xy \cdot xz)x,$$
$$x(y(x \cdot zx)) = (xy \cdot xz)x.$$

Thus ME implies SBME.
(2) let Q be SBRE, i.e.,

$$(xy \cdot xz)x = (x \cdot yx) \cdot zx,$$
$$(xy \cdot xz)x = (xy \cdot x) \cdot zx,$$
$$(y \cdot xz)x = yx \cdot zx.$$

Thus Q is right extra loop.
(3) Suppose Q is SBLE, note that Q is flexible, then

$$x((yx \cdot z)x) = (xy \cdot xz)x \Leftrightarrow (x(yx \cdot z))x = (xy \cdot xz)x \Leftrightarrow x(yx \cdot z) = xy \cdot xz.$$

Thus Q is LE loop. The converse holds by a similar argument, reversing the steps of the forward direction.

□

Corollary 3.16. *Let Q be a loop. Then the following are equivalent:*

(1) Q is SBME loop.
(2) Q is SBRE loop.
(3) Q is SBLE loop.
(4) Q is ME loop.
(5) Q is RE loop.
(6) Q is LE loop.

In [7], George and Jaiyeola introduced twelve identities of second Bol-Moufang type, tagged $\{Q_i\}_{i=1}^{12}$, Table 1.
We establish a relationship between this set of loops and the extra loop of second Bol-Moufang type introduced in this work.

Lemma 3.17. [7]

(1) In a loop, each of the following identities Q_1, Q_3, Q_7, Q_8 implies $P_\lambda(x, y)$.
(2) In a loop, each of the following identities Q_2, Q_4, Q_{11}, Q_{12} implies $P_\rho(x, y)$.
(3) A loop in which any of the following is obeyed Q_5, Q_6, Q_9, Q_{10} is a flexible loop.
(4) A flexible loop obeys $P_\lambda(x, y)$ and $P_\rho(x, y)$.

Lemma 3.18. [11] *Let Q be a loop. Then Q is Q_8 if and only if Q is Q_{11} if and only Q is Q_3 and Q_4 if and only Q is Q_7 and Q_{12}.*

Theorem 3.19. *Any extra loop of second Bol-Moufang type obeys any of the identities in the set $\{Q_i\}_{i=1}^{12}$ of the second Bol-Moufang identities introduced in* [7].

Proof. Let Q be an extra loop of second Bol -Mofang type, then Q satisfies SBME and SBRE, thus Q also satisfies

$$(x \cdot yx) \cdot zx = x(y(x \cdot zx)).$$

Replace y with xy and z with z/x and using flexibility property, we get

$$(x(x \cdot yx))z = x(xy \cdot xz).$$

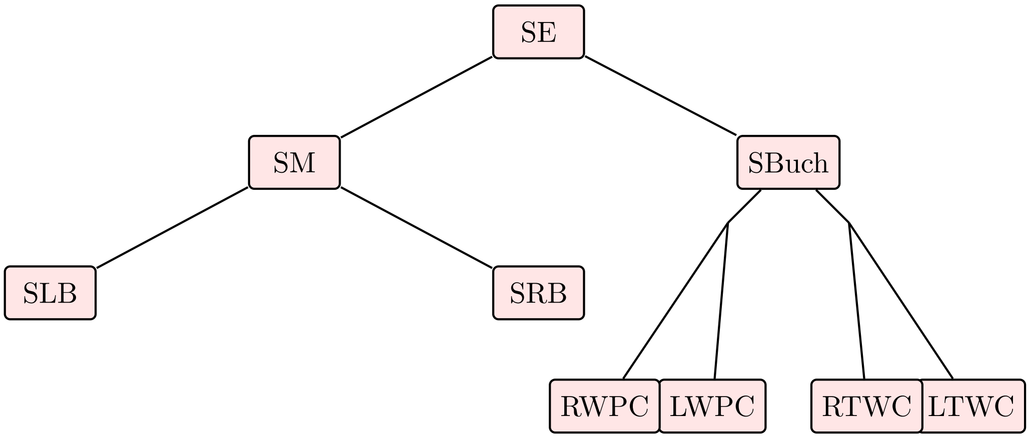

FIGURE 1. Hasse diagram of Second Bol-Moufang Type Loops

Thus Q is Q_{10} and since by [7], $Q_5 \Leftrightarrow Q_6 \Leftrightarrow Q_9 \Leftrightarrow Q_{10}$. Consequently, Q is Q_5, Q_6 and Q_9. Note also that SE satisfies Q_1 and Q_2 since Q_1 and Q_2 are left and right Bol loop satisfying $\underbrace{(xy\cdot x)x = x(yx\cdot x)}_{P_\lambda(x,y)}$ and $\underbrace{x(x\cdot yx) = (x\cdot xy)x}_{P_\rho(x,y)}$ respectively.

Let Q be an extra loop of second Bol-Moufang type, then Q is SBME and Q satisfies

$$x\backslash y \ \cdot x(x\backslash(zx)) = (x\backslash(yz))x,$$

since Q is flexible, the last equation becomes

$$x\backslash y \ \cdot zx = x\backslash(yz\cdot x). \tag{3.5}$$

Again,

$$x\cdot yx = (xy)\cdot x \Leftrightarrow (x\backslash y)x = x\backslash(yx) \Leftrightarrow (x\backslash(yx))x = x\backslash(yx\cdot x) \Leftrightarrow x\backslash(yx) = (x\backslash(yx\cdot x))/x.$$

It is easy to see that Q is Q_8 if we use the last equation on the R.H.S of equation (3.5) and the rest follows from Lemma 3.18. □

The Q_3, Q_4, Q_7 and Q_{12} have been denoted respectively by LWPC, RWPC, LTWC and RTWC, see [6, 9, 10, 14, 15, 8, 13, ?, 21]. Let SM denotes Moufang loops of second Bol-Moufang type (Q_5, Q_6, Q_9, Q_{10}), SLB be Q_1, SRB be Q_2 and SBuch be Buchsteiner loop of second Bol-Moufang type (Q_8, Q_{11}), we therefore have the hasse diagram (Figure 1) connecting all the second Bol-Moufang type in [7] and extra loop of second Bol-Moufang type.

Application: coded access via the identity table. Beyond its algebraic relevance, the nuclear identification framework suggests possible parallels in controlled information access, where code identity pairings govern authorization. In contexts such as military or protective communication systems where multiple personnel may share system access but only designated recipients should read specific messages, the nuclear identification table can serve as a human readable code-book for secure message access.
Concept. Drpal and Jedlika nuclearly identified several loop varieties, both of BolMoufang and nonBolMoufang type. George and Jaiyeola later expanded this approach and George in this present work, introducing a generalized nuclear identification framework and a corresponding Tables (1), (2) of identity codes. Each short `code` in this table uniquely maps to a loop identity name. This correspondence can be viewed as a lightweight two-factor access mechanism in which message decryption depends jointly on a transmitted code and its associated identity name.

Illustrative protocol. The sender selects a code and its corresponding identity name, and derives an encryption key using a standard key-derivation function (e.g., PBKDF2 [17]) combined with an authenticated encryption algorithm such as AES-GCM [3]. Upon receipt, the intended user consults Table 1 to retrieve the identity name linked to the received code, recomputes the KDF, and performs authenticated decryption. A simplified message header could therefore contain the pair `[code, ciphertext]`, while the semantic second key resides in the recipients knowledge of the table.

An interceptor who merely observes the transmitted code cannot determine the correct identity name without access to the table, whereas a user who knows the identity name but not the code likewise fails to obtain the decryption key. The algebraic table of identities therefore functions as a compact, human-usable second factor layered on top of ordinary cryptographic encryption. In this sense, the nuclear identification paradigm demonstrates how algebraic structure can underpin controlled information access in a conceptual and illustrative manner.

Security considerations. Although this design is only conceptual, it illustrates how algebraic nuclear identification can inspire structured access control. Standard precautions should apply: the use of authenticated encryption, slow salted KDFs, and nonces or timestamps to prevent replay attacks. The table should be protected and periodically rotated. We emphasize that this example serves to demonstrate structural analogy rather than to propose a deployable cryptographic system.

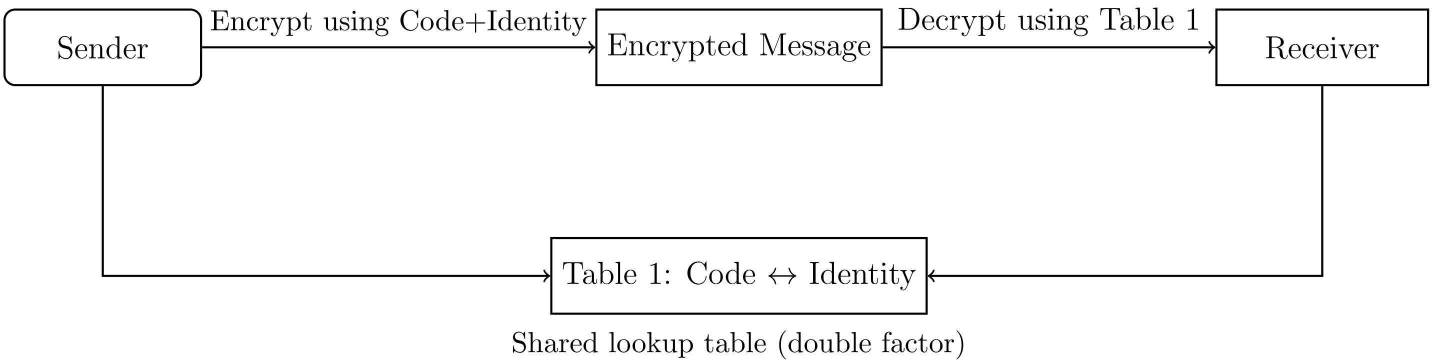

FIGURE 2. Message flow using codeidentity double encryption.

4. Conclusion

In this paper we resolved the open problem of whether extra identities of second Bol–Moufang type admit nuclear identification. We proved that an extra loop of second Bol–Moufang type is nuclear-identifiable precisely when it can be expressed in the form $\alpha_\eta^\epsilon(x)\,\alpha_\xi^\omega(x)\,\alpha_\chi^\kappa(x)\,\alpha_\zeta^\psi(x)$, satisfying an identity $(\eta,\xi,\chi,\zeta;\epsilon,\omega,\kappa,\psi)$ with $\eta=\chi\neq\xi\neq\zeta$. Furthermore, we established that the newly obtained extra identities of the second Bol–Moufang type are equivalent to the classical extra identities of the first Bol–Moufang type.

Beyond the purely algebraic classification, we also indicated that nuclear identification naturally inspires conceptual applications in controlled information access. The correspondence between codes and identity names in Table 1 can be interpreted as the basis of a lightweight two-factor access mechanism, thereby providing a bridge between nonassociative loop theory and idealized models of secure communication.

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