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LYAPUNOV FUNCTIONS AND ASYMPTOTIC EVENTUAL STABILITY FOR IMPULSIVE SYSTEMS WITH NONLINEAR TERMS

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ABSTRACT. This paper investigates the asymptotic eventual stability of a class of nonlinear impulsive differential equations with impulses occurring at fixed moments. The analysis is conducted within a Lyapunov framework by extending the classical concept of vector Lyapunov functions to a generalized class of piecewise continuous Lyapunov functions suitable for impulsive systems. This approach effectively captures both the continuous system evolution and the discrete effects introduced by impulses. By employing appropriate comparison principles, the behavior of the impulsive system is related to that of corresponding comparison systems, enabling the derivation of tractable stability criteria. Based on this methodology, sufficient conditions guaranteeing asymptotic eventual stability are established in terms of inequalities involving the proposed Lyapunov functions and system parameters. The results obtained extend and improve several existing stability criteria in the literature. In particular, the proposed conditions are less restrictive and applicable to a wider class of nonlinear impulsive differential systems, thereby enhancing the scope and effectiveness of Lyapunov-based methods for the qualitative analysis of impulsive dynamics.

1. INTRODUCTION

Stability theory is a cornerstone of the qualitative analysis of differential equations, providing a rigorous framework for assessing and comparing the behavior of solutions generated from distinct initial conditions (see [1]). Within this setting, impulsive differential equations (IDEs) constitute a substantially richer and more complex class than classical ordinary differential equations [9]. IDEs arise naturally in the modeling of many real-world systems in which the state variables undergo sudden changes at prescribed instants. These systems are often subject to perturbations of very short duration relative to the overall

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evolution of the process, which can be appropriately idealized as instantaneous impulses.

Impulsive effects are encountered in a broad spectrum of applications, including biological systems with threshold behavior, bursting rhythm models in medicine and biology, optimal control problems in economics, pharmacokinetics, and frequency-modulated systems (see [9] and the references therein). Consequently, the practical implementation of impulsive differential systems requires the establishment of reliable criteria for determining the stability of their solutions [16]. A variety of analytical techniques have been employed for this purpose, including the monotone iteration method, matrix inequality approaches, Razumikhin techniques, Banach fixed point theory, and Lyapunov-based methods. Among these, Lyapunov's method has proven to be particularly effective and versatile in the stability analysis of both continuous and impulsive dynamical systems. Specifically, Lyapunov's second method is based on the construction of an appropriate positive definite function whose derivative along system trajectories satisfies suitable negativity conditions.

The stability of the trivial solution of impulsive differential equations has been extensively investigated in the literature (see, for example, [2], [4]–[9], [13], [16]–[20]). Nevertheless, as pointed out in [18], numerous perturbation and adaptive control problems involve scenarios in which the object of interest is not an equilibrium point, but rather an eventually stable set that becomes asymptotically invariant as time progresses. This observation provides strong motivation for the study of eventual stability as a meaningful extension of classical Lyapunov stability.

In recent decades, considerable research attention has been directed toward the qualitative properties of impulsive differential equations (see [2], [5], [9], [10], [13], [19]–[21]). For perturbed impulsive systems, eventual stability under bounded perturbations was investigated in [17], while sufficient conditions for uniform eventual stability with non-fixed impulse moments and vanishing perturbations were obtained in [18] using piecewise continuous auxiliary functions regarded as generalized Lyapunov functions. Additional results include uniform eventual stability under bounded perturbations established in [6], and eventual stability and boundedness via Razumikhin-type techniques (see [20]). Fundamental results concerning the existence of maximal solutions for certain classes of impulsive systems were established in [23], while uniform eventual stability results for impulsive systems involving ordinary derivatives were presented in [24]. Further qualitative results on eventual stability were obtained in [26]. Building upon the findings in [24] and [26], the authors in [25] extended eventual stability results from ordinary derivative to fractional-order impulsive systems.

Motivated by these developments, the present work examines the asymptotic eventual stability of the zero solution for a class of impulsive differential equations. By employing vector Lyapunov functions generalized through a class of piecewise continuous Lyapunov functions, together with appropriate comparison principles, sufficient conditions for the asymptotic eventual stability of the perturbed system are derived and illustrated with an example.

1.1. **Preliminaries.** Let R^n be the n-dimensional Euclidean space with norm $\|.\|$. Let Ω be a domain in R^n containing the origin; $R_+ = [0,\infty)$, $R = (-\infty,\infty)$, $t_0 \in R_+$, $t > 0$. Let $J \subset R_+$ and define the following class of functions $PC[J,\Omega] = \alpha : J \to \Omega.$, $\alpha(t)$ as a piecewise continuous function with points of discontinuity $t_k \in J$ at which $\alpha(t_k^+)$ exists.

2. METHODOLOGY

Consider the impulsive differential system of the form,

$$\begin{aligned} x' &= f(t,x), t \neq t_k,\ t \geq t_0, k = 1,2,... \\ \Delta x &= I_k(x),\ t = t_k, k \in N \\ x(t_0^+) &= x_0, \end{aligned} \tag{2.1}$$

under the following assumptions:

A_0 (i) $0 < t_1 < t_2 < ... < t_k < ...,$ and $t_k \to \infty$ $k \to \infty$;

(ii) $f : R_+ \times R^n \to R^n$ is continuous in $(t_{k-1,}\ t_k] \times R^N$ and for each

$x \in R^n$, $k = 1,2,...,$ $\lim_{(t,y)\to(t_k^+,x)} f(t,y) = f(t_k^+, x)$ exists;

(iii) $I_k : R^n \to R^n$

In this paper, we assume that the function f is Lipschitz continuous with respect to its second argument, and $f(t,0) \equiv 0$, $I_k(0) \equiv 0$ for all k, so that we have the trivial solution for (2.1), and the points $t_k, k = 1,2,...$ are fixed such that $0 < t_1 < t_2 < ... and \lim_{k\to\infty} t_k = \infty$. The system (2.1) with initial condition $x(t_0) = x_0$ is assumed to have a solution $x(t; t_0, x_0) \in PC([t_0,\infty), R^N)$. Note that some sufficient conditions for the existence and uniqueness of the global solutions to (2.1) are considered in [9, 15, 16, 18, 26].

Remark 2.1. The second equation in (2.1) is called the impulsive condition, and the function $I_k(x(t_k))$ gives the amount of jump of the solution at the point t_k.

Definition 2.1. Let $V : R_+ \times R^N \to R_+^N$ be a continuous mapping of $R_+ \times R^N$ into R_+^N. Then V is said to belong to class L if,

(i) V is continuous in $(t_{k-1,} \, t_k] \times R^N$ and for each $x \in R^N$, $k = 1,2,...,$ and $\lim\limits_{(t,y)\to(t_k^+,x)} V(t,y) = V(t_k^+, x)$ exists;

(ii) V is locally Lipschitz with respect to its second argument x. For $(t_{k-1,} \, t_k] \times R^N$, we define the upper right Dini derivative of V with respect to (2.1) as,

$$D^+V(t,x) = \limsup_{h\to 0^+} \tfrac{1}{h}[V(t+h, x+hf(t,x)) - V(t,x)] \qquad (2.2)$$

Note that in (2.1), $D^+V(t,x)$ is a functional whereas V is a function.

Definition 2.2. A function $g(t,u)$ is said to be quasimonotone nondecreasing in u, if $u \le v$ and $u_i = v_i$ for $1 \le i \le N$ implies $g_i(t,u) \le g_i(t,v)$ for all i.

Definition 2.3. The trivial solution $x = 0$ of (2.1) is said to be,

(ES1) eventually stable if for every $\varepsilon > 0 \; and \; t_0 \in R_+$ there exist $\delta = \delta(\varepsilon, t_0) > 0$ such that for any $x_0 \in R^n$ the inequality $\|x_0\| \le \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon \; for \; t \ge t_0$;

(ES2) uniformly eventually stable if for every $\varepsilon > 0 \; and \; t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$ such that for any $x_0 \in R^n$, the inequality $\|x_0\| \le \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon \; for \; t \ge t_0$;

(ES3) asymptotically eventually stable if it is stable and if for each $\varepsilon > 0 \; and \; t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0, \varepsilon)$ such that for $t \ge t_0 + T$ and $\|x_0\| \le \delta$ we have $\|x(t,t_0,x_0)\| < \varepsilon$.

(ES4) uniformly asymptotically eventually stable if it is uniformly stable and $\delta_0 = \delta_0(\varepsilon)$ and $T = T(\varepsilon)$ such that for $t \ge t_0 + T$, the inequality $\|x_0\| \le \delta$ implies $\|x(t,t_0,x_0)\| < \varepsilon$.

Definition 2.4. A function $a(r)$ is said to belong to the class K if $a \in C[[0,\rho), R_+], a(0) = 0, \; and \; a(r)$ is strictly monotone increasing in r.

In this paper, we define the following sets:

$$\bar{S}_{\psi} = \{x \in R^N : \|x\| \le \psi\}$$

$$S_{\psi} = \{x \in R^N : \|x\| < \psi\}$$

Remark 2.2.

The inequalities between vectors are understood to be component-wise inequalities.

Definition 2.5. A function $b(r)$ is said to belong to a class L if $b \in C[J, R_+],\ b(t)$ is monotone decreasing in t and $b(t) \to 0$ as $t \to \infty$.

Definition 2.6. A function $a(t,r)$ is said to belong to the class KK if $a \in C[[0,\rho), R_+], a \in K$ for each $t \in J$, and a is monotone increasing in t for each $r > 0$ and $a(t,r) \to \infty$ as $t \to \infty$ for each $r > 0$.

Definition 2.7. A function $V(t,x)$ with $V(t,0) = 0$ is said to be positive definite if there exists a function $a \in K$ such that the relation $V(t,x) \ge a(\|x\|)$ is satisfied for $(t,x) \in J \times S_{\rho}$.

Definition 2.8. A function $V(t,x)$ with $V(t,0) = 0$ is said to be negative definite if there exists a function $a \in K$ such that the relation $V(t,x) \le -a(\|x\|)$ is satisfied for $(t,x) \in J \times S_{\rho}$.

Definition 2.9. A function $V(t,x) \ge 0$ is said to be decrescent if there exists a function $a \in K$ such that the relation $V(t,x) \le a(\|x\|)$ is satisfied for $(t,x) \in J \times S_{\rho}$.

Parallel to (2.1), we shall consider a comparison system of the form,

$$\begin{aligned} u' &= g(t,u), t \ne t_k,\ t \ge t_0, k = 1,2,... \\ \Delta u &= \psi_k(u(t_k)),\ t = t_k, \\ u(t_0^+) &= u_0 \end{aligned} \tag{2.3}$$

existing for $t \ge t_0$, where $u \in R^n$, elation $V(t,x) \le a(\|x\|)$ is satisfied for $(t,x) \in J \times S_{\rho}$ existing for $t \ge t_0$, $u \in R^n$, $R_+ = [t_0, \infty)$, $g: R_+ \times R_+^n \to R^n$, $g(t,0) \equiv 0$, where g is the continuous mapping of $R_+ \times R_+^n$ into R^n. The function $g \in PC[R_+ \times R_+^n, R^n]$ is such that for any initial data $(t_0, u_0) \in R_+ \times R^n$, the system (2.3) with initial condition

$u(t_0) = u_0$ is assumed to have a solution $u(t,t_0,u_0) \in PC([t_0,\infty),R^n)$. Note that some sufficient conditions for the existence of solution of (2.3) has been examined in [9,18, 24, 26].

Lemma 2.1. Assume that the hypotheses $A_0(i), (ii), (iii)$ hold, and that $f(t,0) \equiv 0$ and that $I_k(0) \equiv 0$. Then the interval J can be extended to the maximal interval of existence $[t_0,\infty)$.

Proof.

Since the conditions $A_0(i), (ii), (iii)$ hold, and that $f(t,0) \equiv 0$ and that $I_k(0) \equiv 0$, then from the existence theorem for the equation $x' = f(t,x(t))$ [18] with impulses, it follows that the solution $x(t) = x(t,t_0,x_0)$ of the IVP (2.1) is defined on each of the intervals $(t_{k-1,}\ t_k],\ k = 1,2,....$ Again, since $0 < t_1 < t_2 < ...and \lim_{k\to\infty} t_k = \infty$, then we conclude that the interval J can be continued to $[t_0,\infty)$ for t_0.

3. MAIN RESULTS

In this section we begin by proving the comparison results, then proceed to establish the necessary conditions for the eventual stability of the set $x = 0$ of the impulsive differential systems with fixed moments of impulse.

Using (2.3), Definition 2.2 can be analogously defined as follows:

Definition 3.1.

The trivial solution $u = 0$ of (2.3) is said to be,

(ES_1^*) eventually stable if for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon,t_0) > 0$ such that for any $x_0 \in R^n$ the inequality $\|u_0\| < \delta$ implies $\|u(t,t_0,u_0)\| < \varepsilon\ for\ t \ge t_0$;

(ES_2^*) uniformly eventually stable if the δ in (S_1^*) above is independent of t_0 i.e. for every $\varepsilon > 0\ and\ t_0 \in R_+$ there exist $\delta = \delta(\varepsilon) > 0$ such that the inequality $\|u_0\| < \delta$ implies $\|u(t,t_0,u_0)\| < \varepsilon\ for\ t \ge t_0$;

(ES_3^*) asymptotically eventually stable if S_1^* is satisfied and given $\varepsilon > 0\ and\ t_0 \in R_+$ there exist positive numbers $\delta_0 = \delta_0(t_0) > 0$ and $T = T(t_0,\varepsilon) > 0$ such that for $t \ge t_0 + T$ and $\|u_0\| \le \delta$ we have $\|u(t,t_0,u_0)\| < \varepsilon,\ t \ge t_0 + T$.

(ES_4^*) uniformly asymptotically eventually stable if (ES_2^*) is satisfied (ES_3^*) is independent of t_0.

THEOREM 3.1. (Comparison results) Assume that:

(i) $g \in C[J \times R_+^n, R^n]$, $g(t,0) \equiv 0$, *and* $g(t,u)$ is quasimonotone nondecreasing in u for each $u \in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u) = g(t_k^+,u)$ exists;

(ii) $r(t) = r(t,t_0,u_0) \in PC([t_0,T),R^n)$ is the maximal solution of (2.3) existing for $t \geq t_0$.

(iii) $V \in C[J \times S_\psi, R_+^N]$, $V \in L$ such that for $(t,x) \in J \times S_\psi$

$D^+V(t,x) \leq g(t,V(t,x)),\ t \neq t_k$,

and

$V(t, x + I_k(x(t_k)) \leq \rho_k(V(t,x)),\ t = t_k,\ x \in S_\psi$ and the function

$\rho_k : R_+^N \to R_+^N$ is nondecreasing for $k = 1,2,...$

(iv) $x(t) = x(t,t_0,x_0) \in PC([t_0,T],R^N)$ is a solution of (2.1) such that

$$V(t_0,x_0) \leq u_0 \tag{3.1}$$

existing for $t \geq t_0$. Then

$$V(t,x(t)) \leq r(t) \tag{3.2}$$

Proof.

Let $x(t) = x(t,t_0,x_0)$ be any solution of (2.1) existing for $t \geq t_0$, such that $V(t_0,x_0) \leq u_0$.

Set $m(t) = V(t,x(t))$ for $t \neq t_k$ so that for small $h > 0$, using (2.2) we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t,x(t)) + V(t+h, x(t) + hf(t,x(t)) - V(t,x)$$

Since $V(t,x)$ is locally Lipschitzian in x for $t \in [t_{0,}\ \infty)$, we have

$$m(t+h) - m(t) \leq k\|x(t+h) - (x(t) + hf(t,x(t)))\| + V(t+h, x(t) + hf(t,x(t)) - V(t,x)$$

Dividing through by $h > 0$, and taking the $\limsup$ as $h \to 0^+$ we have

$$\limsup_{h\to 0^+}\tfrac{1}{h}[m(t+h)-m(t)]\le \limsup_{h\to 0^+}\tfrac{i}{h}\Big[k\|x(t+h)-x(t)-hf(t,x(t)\|e\Big]+\limsup_{h\to 0^+}\tfrac{1}{h}[V(t+h,x(t)+ hf(t,x(t))-V(t,x)]$$

where k is the local Lipschitz constant and $e=(1,1,...,1)^T$

It follows that,

$$D^+m(t)=D^+V(t,x(t))\le g(t,m(t)),$$

and using condition (ii) of Theorem 3.1 we arrive at

$$V(t,x(t))\le r(t),$$

provided

$$V(t_0,x_0)\le u_0$$

Also,

$$m(t_k^+)=V(t_k^+,x(t_k)+I_k(x(t_k^+))\le \psi_k(m(t_k^+))$$

Hence, by Cor. 1.7.1 in [13], we obtain the desired estimate of (3.1).

Corollary 3.2. Assume that:

(i) $g\in C[J\times R_+^n,R^n]$, $g(t,0)\equiv 0$, *and* $g(t,u)$ is quasimonotone nondecreasing in u for each $u\in R^n$ and $\lim_{(t,y)\to(t_k^+,u)} g(t,u)=g(t_k^+,u)$ exists;

(ii) $p(t)=p(t,t_0,u_0)\in PC([t_0,T),R^n)$ is the minimal solution of (2.3) existing for $t\ge t_0$.

(iii) $V\in C[J\times S_\psi,R_+^N]$, $V\in L$ such that for $(t,x)\in J\times S_\psi$

$$D^+V(t,x)\ge g(t,V(t,x)),\ t\ne t_k,$$

and

$V(t,x+I_k(x(t_k))\ge \rho_k(V(t,x)),\ t=t_k,\ x\in S_\psi$ and the function $\rho_k: R_+^N\to R_+^N$ is nondecreasing for $k=1,2,...$

(iv) $x(t)=x(t,t_0,x_0)\in PC([t_0,T],R^N)$ is a solution of (2.1) such that

$$V(t_0,x_0)\ge u_0 \tag{3.3}$$

existing for $t\ge t_0$. Then

$$V(t,x(t))\ge p(t) \tag{3.4}$$

Proof.

Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1) existing for $t \geq t_0$, such that $V(t_0, x_0) \geq u_0$.

Set $m(t) = V(t, x(t))$ for $t \neq t_k$ so that for small $h > 0$, using (2.2) we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t)) + V(t+h, x(t) + hf(t, x(t)) - V(t, x)$$

Since $V(t, x)$ is locally Lipschitzian in x for $t \in [t_{0,} \infty)$, we have

$$m(t+h) - m(t) \geq k\|x(t+h) - (x(t) + hf(t, x(t)))\| + V(t+h, x(t) + hf(t, x(t)) - V(t, x)$$

Dividing through by $h > 0$, and taking the $\limsup$ as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \tfrac{1}{h}[m(t+h) - m(t)] \geq \limsup_{h\to 0^+} \tfrac{i}{h}\left[k\|x(t+h) - x(t) - hf(t, x(t)\|\right]e + \limsup_{h\to 0^+} \tfrac{1}{h}[V(t+h, x(t) + hf(t, x(t)) - V(t, x)]$$

where k is the local Lipschitz constant and $e = (1,1,...,1)^T$

It follows from condition (ii) of Cor 3.2. we arrive at the estimate

$$D^+ m(t) = D^+ V(t, x(t)) \geq g(t, m(t)), t \neq t_k,\ m(t_0^+) \geq u_0 \qquad (3.5)$$

Also,

$$m(t_k^+) = V(t_k^+, x(t_k) + I_k(x(t_k^+)) \geq \psi_k(m(t_k^+)) \qquad (3.6)$$

Hence, by Cor. 1.7.1 in [13], we obtain the desired estimate of (3.5).

In what follows, we shall obtain sufficient conditions for the eventual stability of the system (2.3).

THEOREM 3.2. Assume that:

(i) $g \in C[J \times R_+^n, R^n],\ g(t, 0) \equiv 0,\ and\ g(t, u)$ is quasimonotone nondecreasing in u for fixed $t \in J$.

(ii) $V \in C[J \times S_\rho, R_+^n],\ V(t, x)$ is locally Lipschitzian in x and

$$\sum_{i=1}^{N} V_i(t, x) \to 0 \ as \ \|x\| \to 0 \text{ for each } t \text{ and } (t, x) \in J \times S_\rho,$$

$$D^+ V(t, x) \leq g(t, V(t, x)) \qquad (3.7)$$

(iii) $for\ (t, x) \in J \times S_\rho,$

$$b(\|x(t)\|) \le \sum_{i=1}^{N} V_i(t,x) \tag{3.8}$$

where $b \in K$, whence $b \in C[J \times S_\rho, R_+]$.

Then the eventual stability of the set of trivial solution $u = 0$ of the system (2.3) implies the eventual stability of the set of trivial solution $x = 0$ of the system (2.1).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the solution (2.3) is eventually stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each ε such that

$$\sum_{i=1}^{N} u_{i0} \le \delta \quad \Rightarrow \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon), t \ge t_0. \tag{3.9}$$

Since $V(t,x)$ is continuous and mildly unbounded i.e. $V(t,0) \to 0 \; as \; \|x\| \to 0,$ then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ that is continuous in t_0 for each ε such that the inequalities

$$\|x_0\| < \delta_1 \;\; and \;\; V(t_0, x_0) < \delta \tag{3.10}$$

are satisfied simultaneously.

We claim that if $\|x_0\| < \delta_1$ then $\|x(t, t_0, x_0)\| < \varepsilon$ by the stability of $x(t)$.

Now suppose this claim is false, the there would exists a point $t_1 \in [t_0, t)$ and the solution $x(t, t_0, x_0)$ with $\|x_0\| < \delta_1$ such that

$$\|x(t_1)\| = \varepsilon \;\; and \;\; \|x(t)\| < \varepsilon \;\; for \;\; t \in [t_0, t_1). \tag{3.11}$$

So that using equation (3.11); (3.8) reduces to the form

$$b(\|x(t_1)\|) \le \sum_{i=1}^{N} V_i(t_1, x(t_1))$$

implying

$$b(\varepsilon) \le \sum_{i=1}^{N} V_i(t_1, x(t_1)) \tag{3.12}$$

This implies that $x(t) \in S_\psi$ for $t \in [t_0, t_1)$ and from Theorem 3.1,

$$V(t,x(t)) \le r(t,t_0,u_0), \tag{3.13}$$

where $r(t,t_0,u_0)$ is the maximal solution of (2.3).

Then using equations (3.8), (3.9), (3.12) and (3.13) we arrive at the estimate

$$b(\varepsilon) \le V_0(t_1,x(t_1)) \le \sum_{i=1}^{N} r_i(t,t_0,u_0) < b(\varepsilon)$$

which leads to a contradiction.

Hence, the eventual stability of the set of trivial solution $u=0$ of (2.3) implies the eventual stability of the set of trivial solution $x=0$ of (2.1).

In what follows, we shall establish sufficient conditions for the asymptotic eventual stability of the main system (2.1).

THEOREM 3.3. Assume that:

(i) $g \in C[J \times R_+^n, R^n],\ g(t,0) \equiv 0,\ and\ g(t,u)$ is quasimonotone nondecreasing in u for fixed $t \in J$.

(ii) $V \in C[J \times S_\rho, R_+^n],\ V(t,x)$ is locally Lipschitzian in x and

$\sum_{i=1}^{N} V_i(t,x) \to 0\ as\ \|x\| \to 0$ for each t and $(t,x) \in J \times S_\rho$,

$$D^+V(t,x) \le g(t,V(t,x)) \tag{3.7}$$

(iii) $for\ (t,x) \in J \times S_\rho$,

$$b(\|x(t)\|) \le \sum_{i=1}^{N} V_i(t,x) \le a(t,\|x\|) \tag{3.8}$$

where $b \in K,\ a(t,.) \in K$ whence $a \in C[J \times S_\rho, R_+]$

Then the asymptotic eventual stability of the set of trivial solution $u=0$ of the system (2.3) implies the asymptotic eventual stability of the set of trivial solution $x=0$ of the system (2.1).

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in R_+$ be given.

Assume that the solution (2.3) is eventually stable. Then given $b(\varepsilon) > 0$ and $t_0 \in R_+$, there exists a positive function $\delta = \delta(t_o, \varepsilon) > 0$ which is continuous in t_0 for each ε such that

$$\sum_{i=1}^{N} u_{i0} \leq \delta \Rightarrow \sum_{i=1}^{N} u_i(t, t_0, u_0) < b(\varepsilon), t \geq t_0 \tag{3.9}$$

Since $V(t, x)$ is continuous and $V(t,0) \to 0 \; as \; \|x\| \to 0,$ then by the property of continuity, given $\varepsilon > 0$ there exists a positive function $\delta_1 = \delta_1(t_o, \varepsilon) > 0$ that is continuous in t_o for each ε such that the inequalities

$$\|x_0\| < \delta_1 \;\; and \;\; V(t_0, x_0) < \delta \tag{3.10}$$

are satisfied simultaneously.

We claim that if $\|x_0\| < \delta_1$ then $\|x(t, t_0, x_0)\| < \varepsilon.$

Now suppose this claim is false, the there would exists a point $t_1 \in [t_0, t)$ and the solution $x(t, t_0, x_0)$ with $\|x_0\| < \delta_1$ such that

$$\|x(t_1)\| = \varepsilon \;\; and \;\; \|x(t)\| < \varepsilon \;\; for \; t \in [t_0, t_1) \tag{3.11}$$

So that using equation (3.11); (3.8) reduces to the form,

$$b(\|x(t_1)\|) \leq \sum_{i=1}^{N} V_i(t_1, x(t_1)),$$

implying

$$b(\varepsilon) \leq \sum_{i=1}^{N} V_i(t_1, x(t_1)). \tag{3.12}$$

This implies that $x(t) \in S_\psi$ for $t \in [t_0, t_1)$ and from Theorem 3.1,

$$V(t, x(t)) \leq r(t, t_0, u_0), \tag{3.13}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.3).

Then using (3.8), (3.9), (3.12) and (3.13) we arrive at the estimate

$$b(\varepsilon) \leq V_0(t_1, x(t_1) \leq \sum_{i=1}^{N} r_i(t, t_0, u_0) < b(\varepsilon)$$

which leads to an absurdity.

Hence, the asymptotic eventual stability of the set of trivial solution $u = 0$ of (2.3) implies the asymptotic eventual stability of the set of trivial solution $x = 0$ of (2.1).

4. APPLICATION

Consider the system of differential equations

$$\begin{aligned} x_1'(t) &= 3x_1 + x_2 \cos x_1 + x_2(\sin t + 1) - 5x_1 \operatorname{cosec} t,\ t \neq t_k \\ x_2'(t) &= -5x_2 \sec t + 2x_1 \sin t - x_2 + x_1 \cos t,\ \ t \neq t_k \\ \Delta x_1 &= \varpi_k(x(t_k)),\ t = t_k \\ \Delta x_2 &= \zeta_k(x(t_k)),\ t = t_k\ ,\ k = 1,2,... \end{aligned} \tag{4.1}$$

for $t \geq t_0$, with initial conditions,

$$x_1(t_0^+) = \gamma_{10}\ \ and\ x_2(t_0^+) = \gamma_{20}\ ,$$

where $x_1, x_2 \in R^N$ are arbitrary functions.

Equation (4.1) is equivalent to (2.3) and $f = (f_1 f_2)$, where $f_1(t,x_1) = -15x_1 - x_2 \cos x_1 + x_1 \sin x_2 + x_2 \sec x_1$ and

$f_2(t,x_2) = 5x_1 - 4x_2 \sin x_1 - x_2 \sec x_2 - x_1 \cos x_2.$

Consider a vector Lyapunov function of the form $V = (V_1, V_2)^T$, where $V_1(t, x_1, x_2) = |x_1|$ and $V_2(t, x_1, x_2) = |x_2|$. So that $V = (V_1, V_2)^T$ with $x = (x_1, x_2) \in R^2$, so its associated norm defined by $\|x\| = |x_1| + |x_2|$. Now,

$$\sum_{i=1}^{2} V_i(t, x_1, x_2) = |x_1| + |x_2|$$

So that, the assumption,

$$b(\|x\|) \leq \sum_{i=1}^{n} V_i(x, y) \leq a(t, \|x\|) \text{ reduces to,}$$

$$\sqrt{x_1^{\ 2} + x_2^{\ 2}} \leq x_1^{\ 2} + x_2^{\ 2} \leq 2\left(\sqrt{x_1^{\ 2} + x_2^{\ 2}}\ \right)^2$$

with the proviso that $b(r) = r, and\ a(r) = 2r^2$.

Furthermore, we deduce that using equation (3.4) and $V_1(t, x_1, x_2) = |x_1|$

$$D^{+}V(t,x)=\limsup_{h\to 0^{+}}\tfrac{1}{h}\{V(t+h,x+hf(t,x)-V(t,x)\},t\geq t_0$$

$$D^{+}V_1(t,x_1)=\limsup_{h\to 0^{+}}\tfrac{1}{h}\{|x_1+hf_1(t,x_1)|- \quad |x_1|\}$$

$$=\limsup_{h\to 0^{+}}\tfrac{1}{h}\{hf_1(t,x_1)\}$$

$$\leq f_1(t,x_1) \tag{4.2}$$

$$D^{+}V_1(t,x_1)=3x_1+x_2\cos x_1+x_2(\sin t+1)-5x_1\cos ect$$

$$D^{+}V_1(t,x_1)=x_1(3\cos t-5\cos ect)+x_2\sin t+x_2$$

$$\leq |x_1|(3|\cos t|-\frac{5}{|\sin t|})+|x_2|(|\sin t|+1)$$

$$\leq |x_1|(3-5)+|x_2|(1+1)$$

$$\therefore D^{+}V_1(t,x_1)\leq -2V_1+2V_2 \tag{4.3}$$

Also for $x_0\in S_\psi$, $for\ t=t_k, k=1,2,...$ we have

$$V(t,x(t)+\varpi_k)=|\varpi_k+x(t)|\leq V(t,x(t))$$

Again for $V_2(t,x_1,x_2)=|x_2|$ and deducing from (4.2) we have

$$D^{+}V_2(t,x_1)\leq f_2(t,x_2)$$

$$D^{+}V_2(t,x_2)=-5x_2\sec t+2x_1\sin t-x_2+x_1\cos t$$

$$\leq |x_2|(-5|\sec t|-1)+|x_1|(2|\sin t|+|\cos t|)$$

$$\leq |x_2|(-5-1)+|x_1|(2+1)$$

$$\therefore D^{+}V_2(t,x_1)\leq 3V_1-6V_2 \tag{4.4}$$

Also for $x_0\in S_\psi$, $for\ t=t_k, k=1,2,...$ we have,

$$V(t,x(t)+\zeta_k)=|\zeta_k+x(t)|\leq V(t,x(t))$$

Combining (4.3) and (4.4) gives

$$D^+V \leq \begin{pmatrix} -2 & 2 \\ 3 & -6 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = g(t,V) \tag{4.5}$$

$$u' = g(t,u) = Au,$$

where $A = \begin{pmatrix} -2 & 2 \\ 3 & -6 \end{pmatrix}$.

Thus, the vectorial inequality (4.5) and all other conditions of Theorem 3.2 are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (4.1) is asymptotically eventually stable. Therefore, the set $x(t) = 0$ is asymptotically eventually stable.

5. CONCLUSION

In this study, the asymptotic eventual stability of the equilibrium set $x(t) = 0$ associated with a system of impulsive differential equations is analyzed. By adopting vector Lyapunov functions extended via a class of piecewise continuous Lyapunov functions and applying comparison results, sufficient criteria for the asymptotic eventual stability of the perturbed system (2.1) are established, supported by an illustrative example.

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