Unilag Journal of Mathematics and Applications,
Volume 5, Issue 2 (2025), Pages 14–21.
ISSN: 2805 3966. URL: http://lagjma.edu.ng

AN EXPLICIT FORMULA FOR FUZZY SUBGROUPS OF THE ABELIAN GROUP $\mathbb{Z}_{2p^n} \times \mathbb{Z}_2$, $n \geq 1$, $p \geq 3$

MIKE EKPEN OGIUGO*, SUNDAY ADESINA ADEBISI,
OLUSOLA BAMIDELE OGUNFOLU, AND MICHAEL ENIOLUWAFE

Abstract. In this paper, we characterise distinct fuzzy subgroups of the abelian group $\mathbb{Z}_{2p^n} \times \mathbb{Z}_2$, using an enumerative technique derived from the set of representatives of isomorphism classes of subgroups and their sizes. We formulate a linear non-homogeneous recurrence relation of degree one with constant coefficients and apply both the associated linear homogeneous and particular solutions to derive an explicit formula for the number of fuzzy subgroups.

1. Introduction

In 1965, Zadeh [9] introduced the concept of fuzzy subsets, defining a fuzzy subset of a non-empty set as a collection of objects with membership grades ranging between 0 and 1. This foundational idea paved the way for the development of fuzzy set theory, which has found applications in various fields, including decision-making, control theory, and pattern recognition. Building on this concept, Rosenfeld [6] extended the notion to fuzzy subgroups in 1971, marking the beginning of fuzzy abstract algebra. Since then, the classification of fuzzy subgroups in finite groups has been a central problem in fuzzy group theory.

The significance of classifying fuzzy subgroups lies in their potential applications in cryptography, data clustering, and information retrieval, where fuzzy logic provides a more nuanced approach than classical binary logic. Despite numerous studies, including the pioneering works of Das [2], Murali and Makamba [3], and the comprehensive enumeration techniques developed by Adebisi et al. [1], the classification of fuzzy subgroups of finite abelian groups remains an open challenge.

This paper addresses this gap by characterising distinct fuzzy subgroups of the abelian group $\mathbb{Z}_{2p^n} \times \mathbb{Z}_2$ for $n \geq 1$ and prime $p \geq 3$. We employ an enumerative

2010 *Mathematics Subject Classification.* Primary: 20E28, 20K99, 20N25. Secondary: 03E72.

Key words and phrases. Recurrence relation; Isomorphism classes; Enumerative techniques; Fuzzy equivalence relation; Fuzzy subgroups.

Submitted: April 1, 2021. Revised: April 10, 2021. Accepted: April 17, 2021.

* Corresponding author.

technique based on equation (2.1) from Section 2, which relates the total number of fuzzy subgroups to the sizes of isomorphism classes and the number of fuzzy subgroups of representative subgroups. By analysing the structure of $\mathbb{Z}_{2p^n} \times \mathbb{Z}_2$ and formulating a linear non-homogeneous recurrence relation of degree one with constant coefficients, we derive an explicit formula of the form.

$$\delta(\mathbb{Z}_{2p^n} \times \mathbb{Z}_2) = 2^{n-1}[n + (3n+4)(n+4)].$$

This approach builds on the existing body of knowledge [2, 3, 7, 1] and introduces a novel methodology applicable to the broader classification of fuzzy subgroups of finite abelian groups.

2. Preliminaries

Definition 2.1. Let δ be a fuzzy subset of G. Then δ is said to be a **fuzzy subgroup** of G if

i $\delta(xy) \geq \min\{\delta(x), \delta(y)\}$,
ii $\delta(x^{-1}) \geq \delta(x)$

for all $x, y \in G$.

Considering the basic notions of fuzzy subgroups, it follows easily from the axioms that: $\delta(x) = \delta(x^{-1})$ implies $\delta(x) \leq \delta(e)$ for all $x \in G$. Also, δ satisfies conditions (i) and (ii) of Definition 2.1 if and only if $\min\{\delta(x), \delta(y)\} \leq \delta(xy^{-1})$, for all $x, y \in G$. Therefore, $FL(G)$ forms a lattice with respect to the usual ordering of fuzzy set inclusion, called the fuzzy subgroup lattice of G.

Without any equivalence relation on fuzzy subgroups of a group G, the number of fuzzy subgroups is infinite, even for the trivial group. This concept uses the notion of natural fuzzy equivalence classification of $\delta(G)$. We have

$$\delta(x) > \delta(y) \Leftrightarrow \alpha(x) > \alpha(y), \forall x, y \in G.$$

Suppose $\delta(G) = \{\beta_1, \beta_2, \dots, \beta_r\}$, then $\beta_1 < \beta_2 < \cdots < \beta_r$. Therefore, δ computes the subsequent subgroup chains of G that terminates in G:

$$\delta G_{\beta_1} \subset \delta G_{\beta_2} \subset \cdots \subset \delta G_{\beta_m} = G.$$

Moreover, for any $x \in G$ and $i = \overline{1, r}$, we have

$$\delta(x) = \beta_i \Leftrightarrow i = \max\{j \mid x \in \delta G_{\beta_j}\}.$$

Volf [8] disclosed that the necessary condition for two fuzzy subgroups δ, α of G to be equivalent concerning $\sim$. Then, $\delta \sim \alpha$ if and only if δ and α have the same set of level subgroups, that is, they determine the same chain of subgroups. This finding reveals that there exists a bijection between the equivalence classes of fuzzy subgroups of G and the collection of subgroup chains of G which end in G. This fuzzy natural equivalence relation is applied in Tarnauceanu and Bentea [7].

The problem of computing all distinct fuzzy subgroups of G can be transferred into a combinatorial problem on $L(G)$ of G. Let $\delta(G)$ denote the number of fuzzy subgroups of G, $n(H)$ denote the size of the isomorphism class with representative H, and $\mathrm{Iso}(G)$ denote the set of representatives of isomorphism classes of subgroups of G.

We have

$$\delta(G) = \sum_{\text{distinct } H \in \mathrm{Iso}(G)} \delta(H) \times n(H). \tag{2.1}$$

Let $\delta(H_1) = \delta(H_\alpha) = 1$, where H_1 is the trivial group of G and H_α is the improper subgroup of G [4].

It also follows:

(i) $\delta(\mathbb{Z}_p) = 2$ where p is prime
(ii) $\delta(\mathbb{Z}_{pq}) = 6$ where p and q are distinct primes
(iii) $\delta(\mathbb{Z}_{p^2}) = 4$ where p is any prime
(iv) $\delta(\mathbb{Z}_p \times \mathbb{Z}_p) = 2p + 4$ where p is any prime

The above results are special cases [4].

Definition 2.2 (Rosen [5])**.** A linear non-homogeneous recurrence relation of degree k with constant coefficients is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation in the definition is linear since the right-hand side is a sum of multiples of the previous terms of the sequence and all $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$ are raised to power one and non-homogeneous whenever $F(n)$ is not zero otherwise it is homogeneous. The coefficients of the terms of the sequence are all constants, rather than functions that depend on n.

Theorem 2.3 (Rosen [5])**.** *If $\{a_n^{(p)}\}$ is a particular solution to the linear non-homogeneous recurrence relation with constant coefficients*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

Then every solution is of the form:

$$a_n^{(h)} + a_n^{(p)}$$

where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Theorem 2.4 (Rosen [5])**.** *Assume that $\{a_n\}$ satisfies the linear non-homogeneous recurrence relation with constant coefficients*

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n)$$

with $F(n)$ of the form:

$$F(n) = (b_t n^t + b_{t-1} n^{t-1} + \cdots + b_1 n + b_0) s^n$$

where $b_0, b_1, \ldots, b_t$ and s are real numbers.

Case 1: *If s is not a characteristic root of the associated linear homogeneous recurrence relation with constant coefficients, there is a particular solution of the form*

$$(\alpha_t n^t + \alpha_{t-1} n^{t-1} + \cdots + \alpha_1 n + \alpha_0) s^n.$$

Case 2: *If s is a characteristic root of multiplicity m, there is a particular solution of the form*

$$n^m(\alpha_t n^t + \alpha_{t-1}n^{t-1} + \cdots + \alpha_1 n + \alpha_0)s^n.$$

3. Main Results: Fuzzy Subgroups of $\mathbb{Z}_{2p^n} \times \mathbb{Z}_2$, $n \geq 1$, $p \geq 3$

Proposition 3.1. *Let G be $\mathbb{Z}_{2\cdot 3^1} \times \mathbb{Z}_2$, then $\delta(G) = 36$.*

Proof. Let G be $\mathbb{Z}_{2\cdot 3^1} \times \mathbb{Z}_2$. Its isomorphism classes of subgroup representatives and their sizes are as follows: $[e, 1]$, $[\mathbb{Z}_2, 3]$, $[\mathbb{Z}_3, 1]$, $[\mathbb{Z}_2 \times \mathbb{Z}_2, 1]$, $[\mathbb{Z}_6, 3]$, and $[\mathbb{Z}_{2\cdot 3^1} \times \mathbb{Z}_2, 1]$.

Applying equation (2.1),

$$\delta(G) = \delta(e) + 3\delta(\mathbb{Z}_2) + \delta(\mathbb{Z}_3) + \delta(\mathbb{Z}_2 \times \mathbb{Z}_2) + 3\delta(\mathbb{Z}_6) + 1 = 36.$$

Therefore, $\delta(\mathbb{Z}_{2\cdot 3^1} \times \mathbb{Z}_2) = 36$ is obtained. □

Proposition 3.2. *Let G be $\mathbb{Z}_{2\cdot 5^1} \times \mathbb{Z}_2$, then $\delta(G) = 36$.*

Proof. Let G be $\mathbb{Z}_{2\cdot 5^1} \times \mathbb{Z}_2$. Its isomorphism classes of subgroup representatives and their sizes are as follows: $[e, 1]$, $[\mathbb{Z}_2, 3]$, $[\mathbb{Z}_5, 1]$, $[\mathbb{Z}_2 \times \mathbb{Z}_2, 1]$, $[\mathbb{Z}_{10}, 3]$, and $[\mathbb{Z}_{2\cdot 5^1} \times \mathbb{Z}_2, 1]$.

Applying equation (2.1),

$$\delta(G) = \delta(e) + 3\delta(\mathbb{Z}_2) + \delta(\mathbb{Z}_5) + \delta(\mathbb{Z}_2 \times \mathbb{Z}_2) + 3\delta(\mathbb{Z}_{10}) + 1 = 36.$$

Therefore, $\delta(\mathbb{Z}_{2\cdot 5^1} \times \mathbb{Z}_2) = 36$ is obtained. □

Proposition 3.3. *Let G be $\mathbb{Z}_{2\cdot 3^2} \times \mathbb{Z}_2$, then $\delta(G) = 124$.*

Proof. Let G be $\mathbb{Z}_{2\cdot 3^2} \times \mathbb{Z}_2$. Its isomorphism classes of subgroup representatives and their sizes are as follows: $[e, 1]$, $[\mathbb{Z}_2, 3]$, $[\mathbb{Z}_3, 1]$, $[\mathbb{Z}_2 \times \mathbb{Z}_2, 1]$, $[\mathbb{Z}_6 \times \mathbb{Z}_2, 1]$, $[\mathbb{Z}_6, 3]$, $[\mathbb{Z}_9, 1]$, $[\mathbb{Z}_{18}, 3]$, and $[\mathbb{Z}_{2\cdot 3^2} \times \mathbb{Z}_2, 1]$.

Applying equation (2.1),

$$\begin{aligned}\delta(G) &= \delta(e) + 3\delta(\mathbb{Z}_2) + \delta(\mathbb{Z}_3) + \delta(\mathbb{Z}_2 \times \mathbb{Z}_2) + \delta(\mathbb{Z}_6 \times \mathbb{Z}_2) \\ &\quad + 3\delta(\mathbb{Z}_6) + \delta(\mathbb{Z}_9) + 3\delta(\mathbb{Z}_{18}) + 1 = 124. \end{aligned} \tag{3.1}$$

Therefore, $\delta(\mathbb{Z}_{2\cdot 3^2} \times \mathbb{Z}_2) = 124$ is obtained. □

Proposition 3.4. *Let G be $\mathbb{Z}_{2\cdot 5^2} \times \mathbb{Z}_2$, then $\delta(G) = 124$.*

Proof. Given G to be $\mathbb{Z}_{2\cdot 5^2} \times \mathbb{Z}_2$, its isomorphism classes of subgroups are represented by the following set of representations, along with their corresponding sizes: $[e, 1]$, $[\mathbb{Z}_2, 3]$, $[\mathbb{Z}_5, 1]$, $[\mathbb{Z}_2 \times \mathbb{Z}_2, 1]$, $[\mathbb{Z}_{10} \times \mathbb{Z}_2, 1]$, $[\mathbb{Z}_{10}, 3]$, $[\mathbb{Z}_{25}, 1]$, $[\mathbb{Z}_{50}, 3]$, and $[\mathbb{Z}_{2\cdot 5^2} \times \mathbb{Z}_2, 1]$.

Applying equation (2.1),

$$\begin{aligned}\delta(G) &= \delta(e) + 3\delta(\mathbb{Z}_2) + \delta(\mathbb{Z}_5) + \delta(\mathbb{Z}_2 \times \mathbb{Z}_2) + \delta(\mathbb{Z}_{10} \times \mathbb{Z}_2) \\ &\quad + 3\delta(\mathbb{Z}_{10}) + \delta(\mathbb{Z}_{25}) + 3\delta(\mathbb{Z}_{50}) + 1 = 124. \end{aligned} \tag{3.2}$$

Therefore, $\delta(\mathbb{Z}_{2\cdot 5^2} \times \mathbb{Z}_2) = 124$ is obtained. □

Theorem 3.5. *Let G be $\mathbb{Z}_{2p^n} \times \mathbb{Z}_2$, $n \geq 1$, $p \geq 3$. Then, the explicit formula for $\delta(G)$ is*

$$\delta(\mathbb{Z}_{2p^n} \times \mathbb{Z}_2) = 2^{n-1}[n + (3n+4)(n+4)].$$

Proof. Let $\delta(\mathbb{Z}_{2p^n} \times \mathbb{Z}_2) = b_n$ where $b_1 = 36$, $b_2 = 124$, $b_3 = 376$.

We establish a recurrence relation from the following pattern:

$$b_2 - 2b_1 = (\cdots)2^2, \tag{3.3}$$

$$b_3 - 2b_2 = (\cdots)2^3, \tag{3.4}$$

$$b_n - 2b_{n-1} = (\cdots)2^n. \tag{3.5}$$

The isomorphism classes of subgroup representatives and their sizes are as follows: $[e, 1]$; $[\mathbb{Z}_{p^1}, 1]$, $[\mathbb{Z}_{p^2}, 1]$, $\ldots$, $[\mathbb{Z}_{p^n}, 1]$; $[\mathbb{Z}_{2p^0}, 3]$, $[\mathbb{Z}_{2p^1}, 3]$, $[\mathbb{Z}_{2p^2}, 3]$, $\ldots$, $[\mathbb{Z}_{2p^n}, 3]$; $[\mathbb{Z}_{2p^0} \times \mathbb{Z}_2, 1]$, $[\mathbb{Z}_{2p^1} \times \mathbb{Z}_2, 1]$, $[\mathbb{Z}_{2p^2} \times \mathbb{Z}_2, 1]$, $\ldots$, $[\mathbb{Z}_{2p^{n-1}} \times \mathbb{Z}_2, 1]$; and the improper subgroup $[\mathbb{Z}_{2p^n} \times \mathbb{Z}_2, 1]$.

Applying equation (2.1), we obtain

$$\delta(\mathbb{Z}_{2p^n} \times \mathbb{Z}_2) = \delta(e) + \sum_{j=1}^{n} \delta(\mathbb{Z}_{p^j}) + 3\sum_{j=0}^{n} \delta(\mathbb{Z}_{2p^j}) + \sum_{j=0}^{n-1} \delta(\mathbb{Z}_{2p^j} \times \mathbb{Z}_2) + 1. \tag{3.6}$$

Replacing n by $n-1$ in equation (3.6), we have

$$\delta(\mathbb{Z}_{2p^{n-1}} \times \mathbb{Z}_2) = \delta(e) + \sum_{j=1}^{n-1} \delta(\mathbb{Z}_{p^j}) + 3\sum_{j=0}^{n-1} \delta(\mathbb{Z}_{2p^j}) + \sum_{j=0}^{n-2} \delta(\mathbb{Z}_{2p^j} \times \mathbb{Z}_2) + 1. \tag{3.7}$$

Subtracting equation (3.7) from equation (3.6), noting that the terms $\delta(e)$ and the improper subgroup term cancel, and using the fact that $2\delta(\mathbb{Z}_{2p^{n-1}} \times \mathbb{Z}_2)$ accounts for the factor of 2, we obtain:

$$\delta(\mathbb{Z}_{2p^n} \times \mathbb{Z}_2) - 2\delta(\mathbb{Z}_{2p^{n-1}} \times \mathbb{Z}_2) = \delta(\mathbb{Z}_{p^n}) + 3\delta(\mathbb{Z}_{2p^n}). \tag{3.8}$$

From special cases in Section 2, we know that.

$$\delta(\mathbb{Z}_{p^n}) = 2^n, \quad \text{and} \quad \delta(\mathbb{Z}_{2p^n}) = 2^n(n+2).$$

Thus, equation (3.8) becomes:

$$\delta(\mathbb{Z}_{2p^n} \times \mathbb{Z}_2) - 2\delta(\mathbb{Z}_{2p^{n-1}} \times \mathbb{Z}_2) = 2^n + 3(2^n(n+2)) = (3n+7)2^n. \tag{3.9}$$

However, verifying with initial conditions $b_1 = 36$, $b_2 = 124$:

$$b_2 - 2b_1 = 124 - 72 = 52 = (3 \cdot 2 + 7) \cdot 2^2 = 13 \cdot 4 = 52.$$

This suggests the recurrence relation is:

$$b_n - 2b_{n-1} = (3n+7)2^n. \tag{3.10}$$

Solution of the recurrence relation:

The homogeneous part of equation (3.10) is $b_n = 2b_{n-1}$. The characteristic equation is $r - 2 = 0$, giving the characteristic root $r = 2$.

By Theorem 2.3, the homogeneous solution is:

$$b_n^{(h)} = \alpha_1(2^n),$$

Where α_1 is a constant.

For the particular solution, we have $F(n) = (3n+7)2^n$. By Theorem 2.4, since $s = 2$ is a characteristic root with multiplicity 1, the particular solution has the form:

$$b_n^{(p)} = n(\alpha_2 n + \alpha_3)2^n,$$

where α_2 and α_3 are constants to be determined.

Substituting $b_n^{(p)}$ into equation (3.10):

$$\begin{aligned} n(\alpha_2 n + \alpha_3)2^n - 2[(n-1)\{\alpha_2(n-1) + \alpha_3\}]2^{n-1} &= (3n+7)2^n, \\ n(\alpha_2 n + \alpha_3)2^n - [(n-1)\{\alpha_2(n-1) + \alpha_3\}]2^n &= (3n+7)2^n, \\ (\alpha_2 n^2 + \alpha_3 n) - [\alpha_2(n^2 - 2n + 1) + \alpha_3 n - \alpha_3] &= 3n+7, \\ 2\alpha_2 n - \alpha_2 + \alpha_3 &= 3n + 7. \end{aligned} \tag{3.11}$$

Matching coefficients:

$$\begin{aligned} 2\alpha_2 = 3 \quad &\Rightarrow \quad \alpha_2 = \frac{3}{2}, \\ -\alpha_2 + \alpha_3 = 7 \quad &\Rightarrow \quad \alpha_3 = 7 + \frac{3}{2} = \frac{17}{2}. \end{aligned}$$

Therefore, the particular solution is:

$$b_n^{(p)} = n\left(\frac{3}{2}n + \frac{17}{2}\right)2^n = n(3n+17)2^{n-1}.$$

The general solution is:

$$b_n = \alpha_1 \cdot 2^n + n(3n+17)2^{n-1}.$$

Using the initial condition $n = 1$ and $b_1 = 36$:

$$36 = \alpha_1 \cdot 2 + 1(3+17) \cdot 1 = 2\alpha_1 + 20,$$

which gives $\alpha_1 = 8$.

Therefore, the general solution is:

$$\begin{aligned} b_n &= 8 \cdot 2^n + n(3n+17)2^{n-1} \\ &= 2^n\left(8 + \frac{3n^2}{2} + \frac{17n}{2}\right) \\ &= 2^{n-1}(16 + 3n^2 + 17n) \\ &= 2^{n-1}(3n^2 + 17n + 16) \\ &= 2^{n-1}[n + (3n+4)(n+4)]. \end{aligned} \tag{3.12}$$

Therefore, $\delta(\mathbb{Z}_{2p^n} \times \mathbb{Z}_2) = 2^{n-1}[n + (3n+4)(n+4)]$ for $n \geq 1$, $p \geq 3$. $\square$

3.1. **Verification of the Formula.** To verify the derived formula, we check it against the known values:

- For $n = 1$: $\delta(\mathbb{Z}_{2p^1} \times \mathbb{Z}_2) = 2^0[1 + (3+4)(1+4)] = 1[1 + 7 \cdot 5] = 36$
- For $n = 2$: $\delta(\mathbb{Z}_{2p^2} \times \mathbb{Z}_2) = 2^1[2 + (6+4)(2+4)] = 2[2 + 10 \cdot 6] = 2 \cdot 62 = 124$
- For $n = 3$: $\delta(\mathbb{Z}_{2p^3} \times \mathbb{Z}_2) = 2^2[3 + (9+4)(3+4)] = 4[3 + 13 \cdot 7] = 4 \cdot 94 = 376$

4. Conclusion

The main contribution of this paper is the explicit formula

$$\delta(\mathbb{Z}_{2p^n} \times \mathbb{Z}_2) = 2^{n-1}[n + (3n+4)(n+4)],$$

This provides a direct computational method for determining the number of fuzzy subgroups of this abelian structure without requiring exhaustive enumeration.

This formula demonstrates exponential growth in the number of fuzzy subgroups as n increases, which has implications for the complexity of fuzzy classification problems in these groups. The methodology employed, combining enumerative techniques with recurrence relations, can be extended to other classes of finite abelian groups, particularly those of the form $\mathbb{Z}_{mp^n} \times \mathbb{Z}_m$ where m and p are coprime.

Future research directions include:

- Extending this approach to groups of the form $\mathbb{Z}_{2p^n} \times \mathbb{Z}_{2q^m}$ for distinct primes p and q;
- Investigating the asymptotic behavior of $\delta(G)$ for more general abelian group structures;
- Applying these results to practical problems in cryptography and data clustering, where fuzzy group structures arise naturally.

Acknowledgment. The authors acknowledge the support provided by their respective institutions in conducting this research.

Authors Contributions. All authors contributed to the writing and revision of the manuscript.

Authors' Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

Funding Statement. No specific funding was received for this research work.

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Mike Ekpen Ogiugo*
Department of Mathematics, Yaba College of Technology, Lagos, Nigeria.
E-mail address: ekpenogiugo@gmail.com

Sunday Adesina Adebisi
Department of Mathematics, University of Lagos, Lagos, Nigeria.
E-mail address: adesinasunday@yahoo.com

Olusola Bamidele Ogunfolu
Department of Mathematics, University of Lagos, Lagos, Nigeria.
E-mail address: teminiyidele@gmail.com

Michael Enioluwafe
Department of Mathematics, University of Ibadan, Ibadan, Nigeria.
E-mail address: michael.enioluwafe@gmail.com