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VISCOSITY-BASED GRADIENT TECHNIQUES FOR SPLIT FEASIBILITY PROBLEMS

LAWAN BULAMA MOHAMMED* AND ADEM KILICMAN

Abstract. The Split Feasibility Problem (SFP) is a key optimization model with diverse applications in fields such as inverse problems, signal processing, and medical imaging. This paper presents novel viscosity algorithms for solving the SFP in infinite-dimensional Hilbert spaces. Building on existing methods, we introduce new inertial techniques to improve convergence properties. The proposed algorithms feature adaptive step size strategies, which eliminate the need for prior knowledge of operator norms, ensuring greater computational efficiency. Strong convergence results are demonstrated under mild assumptions. These results generalize many existing findings in the literature.

1. Introduction

Let G_1 and G_2 be Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and corresponding norms $\| \cdot \|$. Consider the nonempty, closed, and convex subsets $D \subseteq G_1$ and $R \subseteq G_2$, and let $B \colon G_1 \to G_2$ be a bounded linear operator. The *split feasibility problem* (SFP) is formulated as finding a point

$$y^* \in D \quad \text{such that} \quad By^* \in R. \tag{1.1}$$

The SFP was introduced by Censor and Elfying in 1994 [6] and has become a cornerstone for various applications, especially in inverse problems such as phase retrieval and medical imaging [4]. It also finds widespread use in areas like signal processing, optimization, and intensity-modulated radiation therapy [5].

Several iterative methods have been proposed to solve the SFP. Among these, Byrne's algorithm [3] is notable for its simplicity. Starting from an initial point $y_0 \in G_1$, the sequence $\{y_k\}$ is iteratively defined as follows:

$$y_{k+1} = P_D\big(y_k - \beta B^*(I - P_R)By_k\big), \quad k \in \mathbb{N}, \tag{1.2}$$

where P_D and P_R denote the metric projections onto D and R, respectively, B^* is the adjoint of B, I is the identity operator, and β is a step size chosen such

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* Correspondence .

that $0 < \beta < \frac{2}{L}$, with $L = \|B^*B\|$. Under appropriate conditions, the sequence $\{y_k\}$ converges weakly to the SFP (1.1).

Byrne's algorithm (1.2) is often regarded as a special case of the gradient projection method, as the SFP can be reformulated as the constrained convex minimization problem:

$$\min_{y\in D} g(y), \quad \text{where} \quad g(y) = \frac{1}{2}\|(I - P_R)By\|^2. \tag{1.3}$$

The dependence of β on $\|B^*B\|$ in algorithm (1.2), which is known to be difficult to compute, has motivated numerous researchers to devise adaptive step-size strategies. Notable examples include:

Dynamic Step Size by Lpez et al. [9].

$$\beta_k := \frac{\sigma_k\|(I - P_R)By_k\|^2}{\|B^*(I - P_R)By_k\|^2}, \quad \sigma_k \in (0, 4). \tag{1.4}$$

Adaptive Step Size by Anh et al. [1].

$$\beta_k := \frac{\sigma_k}{\max\{1, \|B^*(I - P_R)By_k\|\}}, \tag{1.5}$$

where the sequence $\{\sigma_k\}$ satisfies $\lim_{k\to\infty} \sigma_k = 0$ and $\sum_{k\geq 0} \sigma_k = \infty$.

Alternatively, the *Armijo Line Search Rule* selects $\beta_k = \mu\nu^{\tau_k}$, with $\mu, \nu \in (0, 1)$ and τ_k being the smallest nonnegative integer satisfying:

$$\mu\nu^{\tau_k}\|B^*(I - P_R)By_k - B^*(I - P_R)Bz_k\| \leq \lambda\|y_k - z_k\|, \tag{1.6}$$

for $z_k = P_D(y_k - \beta_k B^*(I - P_R)By_k)$ and $\lambda \in (0, 1)$.

Polyak [14] first proposed an inertial-type algorithm designed to accelerate the solution of smooth convex optimization problems. This approach involves a two-step iterative scheme, where each new iteration is determined by the previous two iterates. Building on this idea, Nesterov [13] developed an enhanced heavy-ball method aimed at further improving the convergence rate. The method is described as follows:

$$\begin{cases} y_{k+1} = w_k - \rho_k \nabla g(w_k), & \forall k \geq 1, \\ w_k = y_k + \theta_k(y_k - y_{k-1}), \end{cases} \tag{1.7}$$

where $\theta_k \in [0, 1)$, and $\{\rho_k\}$ is a sequence of positive numbers. Here, the inertia is represented by the term $\theta_k(y_k - y_{k-1})$.

The integration of inertial terms into iterative algorithms is well-known for its effectiveness in accelerating the convergence of generated sequences. As a result, substantial research has been devoted to developing and analyzing inertial-based algorithms for the SFP, as detailed in [7, 8, 12, 10, 15, 16, 17].

On the other hand, Yao et al., [20] developed an improved self-adaptive viscosity methods for solving the SFP as follows:

$$y_{k+1} = P_D\left(\alpha_k\pi(y_k) + (1 - \alpha_k)\big(y_k - \beta_k\nabla g(y_k)\big)\right), k \geq 0, \tag{1.8}$$

where $\{\alpha_k\} \subset (0,1)$, $\{\rho_k\} \subset (0,2)$, and the step size

$$\beta_k := \frac{\rho_k g(y_k)}{\|\nabla g(y_k)\|^2},$$

after some conditions imposed on the operators and parameters involved, they established the convergent of the proposed algorithm to the solution of variational inequality problem:

$$y \in D \text{ such that } \langle\, y - \pi(y), y - z\rangle \leq 0 \text{ for all } z \in D. \tag{1.9}$$

In particular, Vinh et al., [12] proposed an inertial approach combined with Polyak's step size to design an algorithm specifically tailored for solving the SFP. This method incorporates the projection of the inertial term onto the feasible set, under standard assumptions on the parameters and operators, they established the weak convergence of the proposed scheme, described as follows:

$$\begin{cases} x_{k+1} = \beta_k \varphi(w_k) + (1-\beta_k)\big(w_k - \lambda_k \nabla f(w_k)\big), \quad \forall k \geq 1, \\ w_k = P_D(y_k + \alpha_k(y_k - x_{k-1})), \end{cases} \tag{1.10}$$

where the step size is given by

$$\beta_k = \rho_k \frac{g(w_k)}{\|\nabla g(w_k)\|^2}.$$

Building on these results, this study seeks to:

- Integrate algorithms (1.7) and (1.8) with Polyak's step size to establish a new viscosity-based inertial self-adaptive gradient algorithm for solving the SFP. The convergence of the proposed methods will be extensively analyzed.
- The step size do not require prior knowledge of the bounded linear operator norm or additional projection steps.
- Convex inertial terms are incorporated to boost the convergence rate of the proposed algorithms.
- Our iterative algorithms demonstrate strong convergence to a solution of the SFP, a critical consideration in infinite-dimensional spaces.

The remainder of this paper is organized as follows: Section 2 presents the preliminary concepts and foundational results essential for establishing the main result, Section 3 discusses the main result of the study and finally, Section 4 presents the conclusion.

2. Materials and Methods

In this section, we present preliminary concepts, including key definitions and lemmas, which will be instrumental in the subsequent discussions.

For any element $x \in G_1$, there exists a unique point in D, denoted by $P_D x$, called the metric projection of x onto D. This projection satisfies the property:

$$\|x - P_D x\| = \inf_{y \in D} \|x - y\|,$$

where $P_D x$ is the nearest point to x within the set D.

Remark 2.1. *The metric projection P_D from G_1 onto D has the following properties:*

a. $\langle x - P_D x, y - P_D x\rangle \leq 0$ $\forall x \in G_1$ *and* $y \in D$*;*
b. $\|x - P_D x\|^2 \leq \langle x - P_D x, x - y\rangle$, $\forall x \in G_1$ *and* $y \in D$*;*
c. $\|P_D x - y\|^2 \leq \|x - y\|^2 - \|(I - P_D)x\|^2$, $\forall x \in G_1$ *and* $y \in D$.

Lemma 2.2 (Aubin [2])**.** *Let $g : G_1 \to \mathbb{R}$ be a function defined by*

$$g(x) := \frac{1}{2}\|Bx - P_D Bx\|^2, \forall x \in G_1, \tag{2.1}$$

then

a. *g is convex and differentiable;*
b. *g is weakly lower semicontinuous (w-lsc) on G_1;*
c. $\nabla g(x) = B^*(I - P_D)Bx$ $\forall x \in G_1$*;*
d. *∇g is $\frac{1}{\|B\|^2}$-inverse strongly monotone, that is,*

$$\langle \nabla g(x) - \nabla g(y), x - y\rangle \geq \frac{1}{\|B\|^2}\|\nabla g(x) - \nabla g(y)\|^2, \quad \forall x, y \in G_1.$$

Lemma 2.3 (Xu [18])**.** *Consider the SFP (1.1), and let g be defined as in equation (2.1). Then the following statements are equivalent:*

a. *The point x solves the SFP;*
b. *The point x solves*

$$x = P_D(x - \beta\nabla g(x)) = P_D(x - \beta B^*(I - P_R)Bx), \beta > 0;$$

c. *The point x solves the following variational inequality problem:*

$$\langle \nabla g(x), y - x\rangle \geq 0, \quad \forall x, y \in D.$$

Lemma 2.4 (Xu [19])**.** *Assume that $\{\beta_k\}$ is a sequence of nonnegative real numbers such that*

$$\beta_{k+1} \leq (1 - \eta_k)\beta_k + \zeta_k + \epsilon_k,$$

where $\{\eta_k\}$ is a sequence of real numbers in $(0, 1)$, and $\{\zeta_k\}$ and $\{\epsilon_k\}$ are sequences of nonnegative real numbers such that:

a. $\sum_{k=1}^{\infty} \eta_k = \infty$*;*
b. $\limsup_{k\to\infty} \frac{\zeta_k}{\eta_k} \leq 0$ *or* $\sum_{k=1}^{\infty} \zeta_k < \infty$*;*
c. $\sum_{k=1}^{\infty} \epsilon_k < \infty$.

Then, $\lim_{k\to\infty} \beta_k = 0$.

Lemma 2.5 (Maing [11])**.** *Let $\{t_k\}$ be a sequence of real numbers that does not decrease at infinity, meaning there exists a subsequence $\{t_{m_i}\}$ of $\{t_k\}$ such that $t_{m_i} \leq t_{m_{i+1}}$ for all $i \geq 0$. For every $m \geq m_0$, define an integer sequence $\{\sigma(m)\}$ as*

$$\sigma(m) = \max\{k \leq m : t_{m_k} < t_{m_{k+1}}\}.$$

Then, $\sigma(m) \to \infty$ as $m \to \infty$, and for all $m \geq m_0$, $\max\{t_{\sigma(m)}, t_k\} \leq t_{\sigma(m)+1}$.

3. Results

In this section, we present the main findings of our study. In what follows, the solution set of the SFP (1.1) will be denoted by $\mathcal{S}$, that is

$$\mathcal{S} = \{x \in \mathcal{D} \text{ such that } \mathcal{B}x \in \mathcal{R}\}. \tag{3.1}$$

Suppose the following presumptions hold:

(C1) A mapping $\mathcal{F} : G_1 \to G_1$ is a contraction with contraction's constant $\eta \in (0,1]$, $P_{\mathcal{D}} : \mathcal{D} \to G_1$ and $P_{\mathcal{R}} : \mathcal{R} \to G_2$ are metric projections onto $\mathcal{D}$ and $\mathcal{R}$, respectively, and $B : G_1 \to G_1$ is linear and bounded operator with its adjoint B^*;
(C2) $g(u_k) := \frac{1}{2}\|Bu_k - P_{\mathcal{R}}Bu_k\|^2$ and $\nabla g(u_k) := B^*(I - P_{\mathcal{R}})Bu_k$;
(C3) Algorithm: Let $\{y_k\}$ be a sequence define by

$$\begin{cases} \gamma_k = \alpha_k u_k + (1-\alpha_k)\big(u_k - \lambda_k \nabla g(u_k)\big), \\ u_k = (1-\tau_k)y_k + \tau_k(y_{k-1} - y_k), \\ y_{k+1} = P_{\mathcal{D}}\left(\beta_k \mathcal{F}(\gamma_k) + (1-\beta_k)\gamma_k\right), \forall k \geq 1. \end{cases} \tag{3.2}$$

Let $(x_0, x_1) \in \mathcal{S}$ be chosen arbitrarily, $\rho_k \in (0,4)$ such that $\inf\limits_{k\geq 1} \rho_k(4-\rho_k) > 0$, and let the step size be defined by $\lambda_k := \frac{\rho_k g(u_k)}{\|\nabla g(u_k)\|^2}$, $\beta_k \in (0,1)$ with $\sum_{k=0}^{\infty} \beta_k = \infty$ and $\lim\limits_{k\to\infty} \beta_k = 0$. Additionally, suppose $(1-\beta_k(1-\eta))(4-\rho_k)\rho_k \geq \sigma$, and $\epsilon_k \in (0,1)$ satisfies $\sum_{k=0}^{\infty} \epsilon_k^2 < \infty$ and $\lim\limits_{k\to\infty} \epsilon_k^2 = 0$, and let $\|y_k - y_{k-1}\| \geq b$, where $b > 0$ and τ_k be a sequence define by

$$\tau_k := \begin{cases} \min\left\{1, \frac{\epsilon_k^2}{2\|y_k - y_{k-1}\|^2}\right\}, & \text{if } y_k \neq y_{k-1}; \\ 0, & \text{if } y_k = y_{k-1}. \end{cases}$$

Lemma 3.1. *Assume that conditions (C1)-(C3) are satisfied, and that $\mathcal{S} \neq \emptyset$. Then $\{\|y_{k+1} - q\|\}$ is bounded, and*

$$\begin{aligned} \|y_{k+1} - q\|^2 \leq & (1-\beta_k(1-\eta))\|y_k - q\|^2 + \left(1 + \frac{\|q\|^2}{b^2}\right)\epsilon_k^2 \\ & - (1-\beta_k(1-\eta))(4-\rho_k)\lambda_k g(u_k) + 2\beta_k\langle \mathcal{F}(q) - q, y_{k+1} - q\rangle. \end{aligned}$$

Proof. Let $q \in \mathcal{S}$, then

$$
\begin{aligned}
\|y_{k+1}-q\|^2 =& \|P_{\mathcal{D}}\left(\beta_k\mathcal{F}(\gamma_k)+(1-\beta_k)\gamma_k\right)-q\|^2 \\
\leq& \|\beta_k\mathcal{F}(\gamma_k)+(1-\beta_k)\gamma_k-q\|^2 \\
& -\|\beta_k\mathcal{F}(\gamma_k)+(1-\beta_k)\gamma_k-P_D\left(\beta_k\mathcal{F}(\gamma_k)+(1-\beta_k)\gamma_k\right)\|^2 \\
\leq& \|\beta_k(\mathcal{F}(\gamma_k)-q)+(1-\beta_k)(\gamma_k-q)\|^2 \\
=& \beta_k\|\mathcal{F}(\gamma_k)-q\|^2+(1-\beta_k)\|\gamma_k-q\|^2 \\
& -\beta_k(1-\beta_k)\|\mathcal{F}(\gamma_k)-\gamma_k\|^2 \\
\leq& 2\eta^2\beta_k\|\gamma_k-q\|^2+2\beta_k\|\mathcal{F}(q)-q\|^2+(1-\beta_k)\|\gamma_k-q\|^2 \\
=& \big(1-\beta_k(1-2\eta^2)\big)\|\alpha_k(u_k-q)+(1-\alpha_k)\big(u_k-\lambda_k\nabla g(u_k)-q\big)\|^2 \\
& +2\beta_k\|\mathcal{F}(q)-q\|^2 \\
=& \alpha_k\big(1-\beta_k(1-2\eta^2)\big)\|u_k-q\|^2 \\
& +(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)\|u_k-\lambda_k\nabla g(u_k)-q\|^2 \\
& -\alpha_k(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)\|\lambda_k\nabla g(u_k)\|^2+2\beta_k\|\mathcal{F}(q)-q\|^2 \\
=& \alpha_k\big(1-\beta_k(1-2\eta^2)\big)\|u_k-q\|^2+(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big) \\
& \times\left[\|u_k-q\|^2-2\lambda_k\langle u_k-q,\nabla g(u_k)\rangle+\|\lambda_k\nabla g(u_k)\|^2\right] \\
& -\alpha_k(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)\|\lambda_k\nabla g(u_k)\|^2+2\beta_k\|\mathcal{F}(q)-q\|^2 \\
\leq& \big(1-\beta_k(1-\eta^2)\big)\|u_k-q\|^2 \\
& -2(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)\lambda_k\langle u_k-q,\nabla g(u_k)\rangle \\
& +(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)\|\lambda_k\nabla g(u_k)\|^2+2\beta_k\|\mathcal{F}(q)-q\|^2 \\
\leq& \big(1-\beta_k(1-\eta^2)\big)\|u_k-q\|^2-4(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)\lambda_k g(u_k) \\
& +(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)\|\lambda_k\nabla g(u_k)\|^2+2\beta_k\|\mathcal{F}(q)-q\|^2 \\
=& \big(1-\beta_k(1-\eta^2)\big)\|u_k-q\|^2-(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)(4-\rho_k)\lambda_k g(u_k) \\
& +2\beta_k\|\mathcal{F}(q)-q\|^2. \qquad (3.3)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\|u_k-q\|^2 =& \|(1-\tau_k)(y_k-q)+\tau_k(y_{k-1}-y_k-q)\|^2 \\
\leq& (1-\tau_k)\|y_k-q\|^2+\tau_k\|y_{k-1}-y_k-q\|^2 \\
\leq& (1-\tau_k)\|y_k-q\|^2+2\tau_k\|y_{k-1}-y_k\|^2+2\tau_k\|q\|^2 \\
\leq& (1-\tau_k)\|y_k-q\|^2+\epsilon_k^2+\|q\|^2\frac{\epsilon_k^2}{b^2}. \qquad (3.4)
\end{aligned}
$$

Thus, by equation (3.3) and (3.4) , we have that

$$
\begin{aligned}
\|y_{k+1}-q\|^2 \leq & \big(1-\beta_k(1-2\eta^2)\big)\left((1-\tau_k)\|y_k-q\|^2+\left(1+\frac{\|q\|^2}{b^2}\right)\epsilon_k^2\right) \\
& -(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)(4-\rho_k)\lambda_k g(u_k)+2\beta_k\|\mathcal{F}(q)-q\|^2 \\
\leq & \big(1-\beta_k(1-2\eta^2)\big)\|y_k-q\|^2+\big(1-\beta_k(1-2\eta^2)\big)\left(1+\frac{\|q\|^2}{b^2}\right)\epsilon_k^2 \\
& -(1-\alpha_k)\big(1-\beta_k(1-2\eta^2)\big)(4-\rho_k)\lambda_k g(u_k)+2\beta_k\|\mathcal{F}(q)-q\|^2 \\
\leq & \big(1-\beta_k(1-2\eta^2)\big)\|y_k-q\|^2+\epsilon_k^2+2\beta_k\|\mathcal{F}(q)-q\|^2 \\
\leq & \max\left\{\|y_k-q\|^2,\frac{2\|\mathcal{F}(q)-q\|^2}{1-2\eta^2}\right\}+\left(1+\frac{\|q\|^2}{b^2}\right)\sum_{k=1}^{\infty}\epsilon_k^2 \\
& . \\
& . \\
& . \\
\leq & \max\left\{\|y_1-q\|^2,\frac{2\|\mathcal{F}(q)-q\|^2}{1-2\eta^2}\right\}+\left(1+\frac{\|q\|^2}{b^2}\right)\sum_{k=1}^{\infty}\epsilon_k^2<\infty. \quad (3.5)
\end{aligned}
$$

This implies that $\{\|y_{k+1}-q\|\}$ is bounded.
On the other hand, by using Remark 2.1 (c), we have

$$
\begin{aligned}
\|y_{k+1}-q\|^2 = & \|P_D\left[\beta_k\mathcal{F}(\gamma_k)+(1-\beta_k)\gamma_k\right]-q\|^2 \\
\leq & \langle\beta_k(\mathcal{F}(\gamma_k)-q)+(1-\beta_k)(\gamma_k-q),y_{k+1}-q\rangle \\
= & \beta_k\langle\mathcal{F}(\gamma_k)-q,y_{k+1}-q\rangle+(1-\beta_k)\langle\gamma_k-q,y_{k+1}-q\rangle \\
\leq & \beta_k\eta\|\gamma_k-q\|\|y_{k+1}-q\|+\beta_k\langle\mathcal{F}(q)-q,x_{k+1}-q\rangle \\
& +(1-\beta_k)\|\gamma_k-q\|\|y_{k+1}-q\| \\
= & (1-\beta_k(1-\eta))\|\gamma_k-q\|\|x_{k+1}-q\|+\beta_k\langle\mathcal{F}(q)-q,x_{k+1}-q\rangle \\
= & (1-\beta_k(1-\eta))\|\alpha_k(u_k-q)+(1-\alpha_k)\big(u_k-\lambda_k\nabla g(u_k)-q\big)\|\|y_{k+1}-q\| \\
& +\beta_k\langle\mathcal{F}(q)-q,y_{k+1}-q\rangle \\
\leq & (1-\beta_k(1-\eta))\Big(\alpha_k\|u_k-q\|+(1-\alpha_k)\|u_k-\lambda_k\nabla g(u_k)-q\|\Big)\|y_{k+1}-q\| \\
& +\beta_k\langle\mathcal{F}(q)-q,y_{k+1}-q\rangle \\
\leq & \frac{(1-\beta_k(1-\eta))}{2}\Big(\alpha_k\|u_k-q\|+(1-\alpha_k)\|u_k-\lambda_k\nabla g(u_k)-q\|\Big)^2 \\
& +\frac{(1-\beta_k(1-\eta))}{2}\|y_{k+1}-q\|^2+\beta_k\langle\mathcal{F}(q)-q,y_{k+1}-q\rangle. \quad (3.6)
\end{aligned}
$$

This turns to implies that

$$
\begin{aligned}
\|y_{k+1}-q\|^2 \leq &(1-\beta_k(1-\eta))\Big(\alpha_k\|u_k-q\|+(1-\alpha_k)\|u_k-\lambda_k\nabla g(u_k)-q\|\Big)^2 \\
&+2\beta_k\langle \mathcal{F}(q)-q, y_{k+1}-q\rangle \\
\leq &(1-\beta_k(1-\eta))\Big(\alpha_k\|u_k-q\|^2+(1-\alpha_k)\|u_k-\lambda_k\nabla g(u_k)-q\|^2\Big) \\
&+2\beta_k\langle \mathcal{F}(q)-q, y_{k+1}-q\rangle \\
\leq &(1-\beta_k(1-\eta))\Big(\|u_k-q\|^2-(4-\rho_k)\lambda_k g(u_k)\Big) \\
&+2\beta_k\langle \mathcal{F}(q)-q, y_{k+1}-q\rangle. \qquad (3.7)
\end{aligned}
$$

By equation (3.4) and (3.7), we have

$$
\begin{aligned}
\|y_{k+1}-q\|^2 \leq &(1-\beta_k(1-\eta))\Big((1-\tau_k)\|y_k-q\|^2+\epsilon_k^2-(4-\rho_k)\lambda_k g(u_k)\Big) \\
&+2\beta_k\langle \mathcal{F}(q)-q, y_{k+1}-q\rangle \\
\leq &(1-\beta_k(1-\eta))\|y_k-q\|^2+\left(1+\frac{\|q\|^2}{b^2}\right)\epsilon_k^2 \\
&-(1-\beta_k(1-\eta))(4-\rho_k)\lambda_k g(u_k)+2\beta_k\langle \mathcal{F}(q)-q, y_{k+1}-q\rangle.
\end{aligned}
$$

Thus, we deduce that

$$
\begin{aligned}
\|y_{k+1}-q\|^2 \leq &(1-\beta_k(1-\eta))\|y_k-q\|^2+\left(1+\frac{\|q\|^2}{b^2}\right)\epsilon_k^2 \\
&-(1-\beta_k(1-\eta))(4-\rho_k)\lambda_k g(u_k)+2\beta_k\langle \mathcal{F}(q)-q, y_{k+1}-q\rangle. \qquad (3.8)
\end{aligned}
$$

□

Theorem 3.2. *Assume that conditions (C1)-(C3) are satisfied, and that $\mathcal{S} \neq \emptyset$. Then the sequence $\{y_k\}$ defined by algorithm (3.2) converges strongly to $y \in \mathcal{S}$, which also solve the following variational inequality:*

$$y \in \mathcal{S} \text{ such that } \langle \mathcal{F}(y)-y, z-y\rangle \leq 0 \text{ for all } z \in \mathcal{S}.$$

Proof. $q \in \mathcal{S}$, then by Lemma 3.1, we see that $\{\|y_k-q\|\}$ is bounded, and

$$
\begin{aligned}
(1-\beta_k(1-\eta))(4-\rho_k)\rho_k\frac{g^2(u_k)}{\|\nabla g(u_k)\|^2} \leq &s_k-s_{k+1}+\left(1+\frac{\|q\|^2}{b^2}\right)\epsilon_k^2 \\
&+2\beta_k\|\mathcal{F}(q)-q\|\|y_{n+1}-q\| \\
\leq &s_k-s_{k+1}+\left(1+\frac{\|q\|^2}{b^2}\right)\epsilon_k^2+\beta_k N. \qquad (3.9)
\end{aligned}
$$

where $s_k=\|y_k-q\|^2$, and $N>0$ is a constant number chosen arbitrarily such that $\sup_k\{2\|\mathcal{F}(q)-q\|\|y_{k+1}-q\|\} \leq N$.

We are now in the position to prove that $y_k \to y$, as $n \to \infty$. To do this, we consider the following cases:

Case 1: Suppose that $\{s_k\}$ is a decreasing sequence, by equation (3.9), we deduce that

$$\lim_{n\to\infty} \frac{g^2(u_k)}{\|\nabla g(u_k)\|^2} = 0, \tag{3.10}$$

this further implies that

$$\lim_{n\to\infty} g(u_k) = 0.$$

Thus $\{u_k\}$ is bounded, therefore, there exists a subsequence $\{u_{k_n}\}$ of $\{u_k\}$ converging weakly to $q \in \mathcal{D}$, and this turn to implies that $\{y_k\}$ converges weakly to $q \in \mathcal{D}$.

From the weak lower semicontinuity of g, we have

$$0 \le g(q) \le \liminf_{n\to\infty} g(u_{k_n}) = \lim_{k\to\infty} g(u_k) = 0.$$

Hence, $g(q) = 0$, i.e., $Aq \in \mathcal{R}$. This indicates that

$$\omega_w(y_k) \subset \mathcal{R}.$$

Furthermore, due to the metric projection property, we have that

$$\limsup_{n\to\infty} \langle \mathcal{F}(q) - q, y_{k+1} - q\rangle = \max_{\omega\in\omega_w(y_k)} \langle \mathcal{F}(q) - q, \omega - q\rangle \le 0.$$

By equation (3.8), we deduce that

$$s_{k+1} \le (1 - \beta_k(1-\eta))s_k + \left(1 + \frac{\|q\|^2}{b^2}\right)\epsilon_k^2 + 2\beta_k\langle \mathcal{F}(q) - q, y_{k+1} - q\rangle. \tag{3.11}$$

Thus, we deduce that

(i) $\sum_{k\ge 0} \beta_k(1-\eta) = \infty$,

(ii) $\limsup_{k\to\infty} \frac{2\langle \mathcal{F}(q)-q, y_{k+1}-q\rangle}{(1-\eta)} = 0$,

(iii) $\sum_{k\ge 0} \left(1 + \frac{\|q\|^2}{b^2}\right)\epsilon_k^2 < \infty$.

Therefore, applying Lemma 2.4 to equation (3.11), we have $y_k \to q$. Since $q \in \mathcal{S}$ was chosen arbitrarily, we therefore conclude that $y_k \to y$ as required.

Case 2: Assume $\{\|y_k - q\|\}$ is an increasing sequence. That is,

$$\|y_k - q\| \le \|y_{k+1} - q\|.$$

Thus, we can define an integer sequence $\{\xi(k)\}$ for all $k \ge n_0$ as follows:

$$\xi(k) = \max\{m \in \mathbb{N} \mid n_0 \le m \le k, \|y_k - q\| \le \|y_{k+1} - q\|\}.$$

Clearly, $\xi(k)$ is an increasing sequence such that $\xi(k) \to \infty$ as $k \to \infty$ and

$$\|y_{\xi(k)} - q\| \le \|y_{\xi(k)+1} - q\|, \forall k \ge n_0.$$

On the other hand, by equation (3.9), we deduce that

$$\frac{g^2(u_{\xi(k)})}{\|\nabla g(u_{\xi(k)})\|^2} \le \frac{1}{\sigma}\Big(\|y_{\xi(k)} - q\|^2 - \|y_{\xi(k)+1} - q\|^2 + \left(1 + \frac{\|q\|^2}{b^2}\right)\epsilon_{\xi(k)}^2 + \beta_{\xi(k)}N\Big), \tag{3.12}$$

where $\sigma > 0$ was chosen such that $(1-\beta_k(1-\eta))(4-\rho_k)\rho_k \geq \sigma$. This turns to implies that

$$\lim_{n\to\infty} g(u_{\xi(k)}) = 0.$$

Following the same argument as in Case 1, we see that $\{y_{\xi(k)}\}$ is in the solution set $\mathcal{S}$; i.e.,

$$\omega_w(y_{\xi(k)}) \subset \mathcal{S}.$$

Furthermore,

$$\limsup_{n\to\infty} \langle \mathcal{F}(z) - z, y_{\xi(k)+1} - z\rangle = 0.$$

The fact that $\{\|y_{\xi(k)} - q\|\}$ is monotone increasing, and couple with equation (3.11), we have

$$\begin{aligned}\beta_{\xi(k)}(1-\eta)\|y_{\xi(k)} - q\|^2 \leq & \|y_{\xi(k)} - q\|^2 - \|y_{\xi(k)+1} - q\|^2 + \left(1 + \frac{\|q\|^2}{b^2}\right)\epsilon^2_{\xi(k)} \\ & + 2\beta_{\xi(k)}\langle \mathcal{F}(q) - q, y_{\xi(k)+1} - q\rangle \\ & \leq \epsilon^2_{\xi(k)} + 2\beta_{\xi(k)}\langle \mathcal{F}(q) - q, y_{\xi(k)+1} - q\rangle. \end{aligned} \tag{3.13}$$

Since $\beta_{\xi(k)}$ is bounded, there exist M such that $\beta_{\xi(k)} > M$, this and couple with equation (3.13), we have

$$\|y_{\xi(k)} - q\|^2 \leq \frac{\left(1 + \frac{\|q\|^2}{b^2}\right)\epsilon^2_{\xi(k)} + 2\beta_{\xi(k)}\langle \mathcal{F}(q) - q, y_{\xi(k)+1} - q\rangle}{M(1-\eta)}. \tag{3.14}$$

This turn to implies that

$$\lim_{n\to\infty} \|y_{\xi(k)} - q\|^2 = 0,$$

which further implies that

$$\lim_{n\to\infty} \|y_{\xi(k)+1} - q\|^2 = 0.$$

Thus, by Lemma 2.5, we deduce that

$$\begin{aligned} 0 \leq \|y_k - q\|^2 &\leq \max\{\|y_{\xi(k)} - q\|^2, \|y_k - q\|^2\} \\ &\leq \|y_{\xi(k)+1} - q\|^2 \to 0. \end{aligned}$$

Therefore, $\|y_k - q\|^2 \to 0$. That is, $y_k \to q$. Since $q \in \mathcal{S}$ was chosen arbitrarily, we conclude that $y_k \to y$ as required. This completes the proof. □

The following results can be easily deduced from Theorem 3.2.

Corollary 3.3. *Assume that conditions (C1) and (C2) are satisfied, and let* $\{y_k\}$ *be a sequence defined by*

$$\begin{cases} \gamma_k = u_k - \lambda_k \nabla g(u_k), \\ u_k = (1-\tau_k)y_k + \tau_k(y_{k-1} - y_k), \\ y_{k+1} = P_{\mathcal{D}}\left(\beta_k \mathcal{F}(\gamma_k) + (1-\beta_k)\gamma_k\right), \forall k \geq 1, \end{cases} \tag{3.15}$$

let $(x_0, x_1) \in \mathcal{S}$ be chosen arbitrarily, with $\rho_k \in (0,4)$ such that $\inf_{k\geq 1} \rho_k(4-\rho_k) > 0$. The step size is defined as $\lambda_k := \frac{\rho_k g(u_k)}{\|\nabla g(u_k)\|^2}$, and we have $\beta_k \in (0,1)$ with $\sum_{k=0}^{\infty} \beta_k = \infty$ and $\lim_{k\to\infty} \beta_k = 0$. Additionally, $(1-\beta_k(1-\eta))(4-\rho_k)\rho_k \geq \sigma$, and let $\epsilon_k \in (0,1)$ satisfies $\sum_{k=0}^{\infty} \epsilon_k^2 < \infty$ and $\lim_{k\to\infty} \epsilon_k^2 = 0$, and let $\|y_k - y_{k-1}\| \geq b$, where $b > 0$ and the sequence τ_k is defined as:

$$\tau_k := \begin{cases} \min\left\{1, \frac{\epsilon_k^2}{\|y_k - y_{k-1}\|^2}\right\} & \text{if } y_k \neq y_{k-1}; \\ 0, \text{if } y_k = y_{k-1}. \end{cases}$$

Then $y_k \to y \in \mathcal{S}$.

Corollary 3.4. *Suppose that condition C1 and C2 are satisfied, and let $\{y_k\}$ be a sequence define by*

$$\begin{cases} \gamma_k = y_k - \lambda_k \nabla g(y_k), \\ y_{k+1} = P_{\mathcal{D}}\left(\beta_k \mathcal{F}(\gamma_k) + (1-\beta_k)\gamma_k\right), \forall k \geq 1, \end{cases} \tag{3.16}$$

let $(x_0, x_1) \in \mathcal{S}$ be chosen arbitrarily, with $\rho_k \in (0,4)$ such that $\inf_{k\geq 1} \rho_k(4-\rho_k) > 0$, and let $\beta_k \in (0,1)$ with $\sum_{k=0}^{\infty} \beta_k = \infty$ and $\lim_{k\to\infty} \beta_k = 0$. Additionally, $(1-\beta_k(1-\eta))(4-\rho_k)\rho_k \geq \sigma$. The step size is defined as $\lambda_k := \frac{\rho_k g(u_k)}{\|\nabla g(u_k)\|^2}$. Then $y_k \to y \in \mathcal{S}$.

Corollary 3.5. *Suppose that condition C1 and C2 are satisfied, and let $\{y_k\}$ be a sequence define by*

$$\left\{ y_{k+1} = P_{\mathcal{D}}\left(\beta_k \mathcal{F}(y_k) + (1-\beta_k)\Big(y_k - \lambda_k \nabla g(y_k)\Big)\right), \forall k \geq 1, \right. \tag{3.17}$$

let $(x_0, x_1) \in \mathcal{S}$ be chosen arbitrarily, with $\rho_k \in (0,4)$ such that $\inf_{k\geq 1} \rho_k(4-\rho_k) > 0$, and let $\beta_k \in (0,1)$ with $\sum_{k=0}^{\infty} \beta_k = \infty$ and $\lim_{k\to\infty} \beta_k = 0$. Additionally, $(1-\beta_k(1-\eta))(4-\rho_k)\rho_k \geq \sigma$. The step size is defined as $\lambda_k := \frac{\rho_k g(y_k)}{\|\nabla g(y_k)\|^2}$. Then $y_k \to y \in \mathcal{S}$.

Proof. Algorithm 3.17 was proposed by Yoa et al., [20], and the proof follows directly from Corollary 3.5 by taking $\gamma_k = y_k$.

□

Conclusion

This study successfully introduces and analyzes novel viscosity-based inertial gradient algorithms for solving the Split Feasibility Problem (SFP) in infinite-dimensional Hilbert spaces. Our primary contributions are fourfold.

First, we developed a new iterative scheme (Algorithm 3.2) that effectively integrates new inertial techniques—inspired by Polyak's heavy-ball method — with the strong convergence guarantees of viscosity methods.

Second, we incorporated a self-adaptive step size defined by

$$\lambda_k := \frac{\rho_k g(u_k)}{\|\nabla g(u_k)\|^2}.$$

This design entirely eliminates the dependency on the operator norm $\|B^*B\|$, which is often computationally challenging to obtain in practice.

Third, we have rigorously established the strong convergence of the generated sequence $\{y_k\}$ to a solution of the SFP under mild and standard assumptions.

Finally, our results serve as a significant extension of numerous existing algorithms in the literature. For instance, our method generalizes the work of Yao et al. [20] (Corollary 3.5). Furthermore, by setting $\tau_k = \alpha_k = \beta_k = 0$, our algorithm reduces to those studied by Anh et al. [1] and Byrne [3].

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Lawan Bulama Mohammed*
Department of Mathematics, Federal University Dutse, PMB 7156, Dutse, Jigawa State, Nigeria.
E-mail address: lawanbulama@gmail.com

Adem Kilicman
College of Computing, Informatics and Mathematics, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia.
E-mail address: kilicman@uitm.edu.my.