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ALGEBRAIC PROPERTIES OF MONOGENIC SOFT QUASIGROUPS

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Abstract. This paper explores the properties of soft monogenic quasigroups, a class of algebraic structures that combine soft set theory and quasigroups. We define soft monogenic quasigroups and investigate their generation criteria, structural properties, and invariance under automorphism. Our results contribute to the advancement of soft algebraic structures, with potential applications in fields like cryptography and coding theory. We establish key characteristics of soft monogenic quasigroups, including their generation by a single element and structural invariance. This work lays the foundation for further research in soft quasigroups and their applications.

1. Introduction

Soft set theory, introduced by Molodtsov, provides a robust framework for handling uncertainty and vagueness in mathematical modeling. Building on this foundation, recent research has explored algebraic structures like quasigroups within the soft set context. Quasigroups, known for their non-associative properties, have applications in cryptography, coding theory, and combinatorics. This paper focuses on soft monogenic quasigroups, a specific class of soft quasigroups generated by a single element.

Marty [19] introduced algebraic hyperstructures in 1934, demonstrating their generalization of algebraic structures like quasigroups and others. This concept has since flourished in various mathematical disciplines, including Automata, Combinatorics, Cryptography, Geometry, Fuzzy sets, Rough sets and Artificial intelligence.

A Quasigroup is an hyper algebraic framework which is not group but generalises group because it dispenses with the non - associative property. Quasigroups

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are considered as generalisation of groups and many authors like Ajala et al. [1] Brucks [8], Chein et al. [11] Jonathan [16] , Moufang [22] Pflugfelder [28] have done great works on quasigroups and loops.

In 2022, Falcon [13] did analysis and recognised the fractal image pattern derived from Cayley tables, which has played an important role in distributing unique types of quasigroup algebraic structures into different isomorphic classes.

While Mckay et al. [20] introduced the order of automorphism of quasigroup, were they proved the quadratic upper bounds on the order of any autotopism of a quasigroup and their order, and that quasigroup of prime order can possess autotopism that consist of three permutations with different cycle structures.

Due to uncertainties associated with the set theory, Molodtsov [21] introduced soft set theory to generalise set theory problems that are not clearly defined. Soft set is considered a better mathematical method for unravelling mathematical questions associated with uncertainty and not well defined objects, because of its freedom from parametrization inadequacies.

After Molodtsov's pioneer work [21], different authors studied the theoretical aspects of soft sets which have become very active recently. Maji et al. [18] gave elaborate theoretical account of soft set by analysing several algebraic operations of soft sets.

Expanding on Maji et al.'s research work [18], Ali et al. [5] developed innovative operations for soft sets. Aktas and Ozlu [3] built upon this by defining soft groups and addressing inconsistencies in earlier work by Aktas and Cagman [2], while also exploring normalistic soft groups and their homomorphisms.

Recent studies have applied algebraic concepts to soft quasigroup and its parastrophes, such as soft quasigroups to egalitarianism. Oyem et al. [23] explored how these mathematical structures can model and analyze egalitarian relationships, establishing algebraic connections that promote fairness and equality.

Oyem et al. [24] studied different algebraic properties of soft quasigroups and prove that some of the operations do not distribute over each other. Oyem et al. [25] studied the parastrophes and cosets of soft quasigroups and furthermore introduced soft Neutrosophic quasigroup Oyem et al. [26].

Our goal in this paper is to investigate monogenic quasigroups fundamental characteristics, generation criteria and structural invariance under automorphism, contributing to the advancement of soft algebraic structures and their potential applications.

1.1. **Preliminaries.** We review some definitions and results on quasigroups. See [1, 4, 7, 8, 11, 16, 22].

1.2. **Quasigroups.**

Definition 1.1. (Groupoid, Quasigroup)

Supposing that Q_λ is a non trivial set under a binary mapping $(\cdot)$ on on $Q_\lambda \times Q_\lambda \to Q_\lambda$, the pair $(Q_\lambda, \cdot)$ will be referred as a *groupoid* or *Magma*, if there exists $\tau \cdot \beta \in Q_\lambda$ and for any $d, h \in Q_\lambda$, Hence the equations:

$$d \cdot \tau = h \qquad \text{and} \qquad \beta \cdot d = h$$

will be solve uniquely in Q_λ for τ and β correspondingly, then $(Q_\lambda, \cdot)$ forms a *quasigroup*.

If z is any fixed element in the groupoid $(Q_\lambda, \cdot)$, and there is a left $(\backslash)$ and the right $(/)$ mapping function of Q_λ, illustrated by R_z and L_z defined as;

$$yR_z = y \cdot z \qquad \text{and} \qquad yL_z = z \cdot y.$$

known as translation maps and if they are bijective, and the inverse mappings L_z^{-1} and R_z^{-1} exists, such that;

$$z\backslash y = yL_z^{-1} \qquad \text{and} \qquad z/y = zR_y^{-1}$$

Then, $(Q_{\lambda,\cdot,/,\backslash})$ will be regarded as a quasigroup.

Definition 1.2. Loops

$(Q_\lambda, \cdot)$ forms a loop with identity e if $e \cdot p = p \cdot e = p$ holds uniquely for all $p \in Q_\lambda$.

Definition 1.3. Subquasigroups

If $(Q_{\lambda,\cdot,/,\backslash})$ is regarded as a quasigroup and it has a subset H represented as $H \subset Q_\lambda$. So H will be referred as a subquasigroup of $(Q_{\lambda,\cdot,/,\backslash})$ provided it satisfies quasigroup properties in its own right. We write $H \subset Q_\lambda$.

Definition 1.4. Power Associativity [11]

A power associative quasigroup is one for which the subquasigroup generated by each single element $< a >$ is associative. This is equivalent to saying that all powers are unambiguous and the index rule holds.

Remark 1.1. If in a quasigroup Q_λ the (right multiplied) power rule holds for some element $x \in Q_\lambda$, that is, $x^m \cdot x^n = x^{m+n}$ and $m, n \in N$, then $< x >$ is power associative.

Definition 1.5. Idempotent Quasigroups [11]

We call a finite quasigroup Q_λ an idempotent quasigroup provided $\forall x \in Q_\lambda$, $x^2 = x$ where $x^2 = x \cdot x$; x is call idempotent element.

Definition 1.6. Cyclic presentation [28]

Let G_F be a free group, and if $\{k_1, k_2, \cdots, k_n\}$ are generators on (G_F), then if β is a reduced word in some of the $k_i s$ and α is an automorphism of G_F caused by $k_1 \to k_2 \to \cdots \to k_n \to k_1$, and if $\beta\alpha^j$ represents the word formed by applying the jth power of a transformation to each $k_i's$ in β , then the resulting group presentation is;

$$< k_1, k_2, \cdots, k_n | \beta, \beta\alpha, \beta\alpha^2, \cdots, \beta\alpha^{n-1} >$$

is called a cyclic presentation.

Definition 1.7. Quasigroup Presentation [11]

By quasigroup presentation defined as

$$< x_1, x_2, \cdots, x_t; r_1 = s_1, r_2 = s_2, \cdots r_t = s_t >,$$

we mean the quotient F/θ of the free quasigroup on $x_1, x_2 \cdots x_n$ by the congruence θ generated by the pairs $(r_i, s_i), i = 1, 2, 3 \cdots t$.

Definition 1.8. Finitely presented Quasigroup [17, 11, 28]

Supposing a quasigroup Q_λ is generated by the elements $x_1, x_2, \cdots, x_t$, and consider a set of relations defined by $h \star p = z$, where $h, p, z \in \{x_1, x_2, \cdots, x_t\}$. These relations satisfy the following rules:

i For arbitrary $h, p, z \in \{x_1, x_2, \cdots, x_n\}$, such that any of $h \cdot p = z, h \backslash z = p, z/p = p$ is a defining relation, so also are the others.

ii There can be no two defining relations in form of $h \circ p = z_1, h \circ p = z_2 (z_1 \neq z_2)$ with $(\circ)$ being the same operation.

1.3. **Soft sets.** Now, we consider the notion of soft sets. We referred to [2, 3, 5, 6, 9, 10, 12, 18, 21, 27] for this purpose. If Q_λ represents an original universe, and E also represents the defined variables, while $A \subseteq E$.

Definition 1.9. Soft Sets [2, 3, 5]

Let Q^* is a universal set under consideration and there exists E, as a set of defined parameters over Q^* and $K \subset E$, we called the pair (G, K) a Soft set over Q^*, whenever $G(k) \subset Q^* \ \forall \ k \in K$; and F is a function that mappings K to all the non-empty subsets of Q^*, i.e $G : K \longrightarrow 2^{Q^*} \backslash \{\emptyset\}$. A soft set (G, K) over a set Q^* is described as a set of ordered pairs: $(G, K) = \{(k, G(k)) : k \in K \text{ and } G(k) \in 2^{Q^*}\}$.

Given soft sets (Z, K) and (V, D) over Q^*, (V, D) is considered a soft subset of (Z, K) if.

i $D \subseteq K$

ii (V, D) is a soft superset of (Z, K), denoted $(V, D) \supseteq (Z, K)$, if $K \subseteq D$ and $V(o) \supseteq Z(o)$ for all $o \in K$. This implies (Z, K) is a soft subset of (V, D).

1.4. **Soft Quasigroups.**

Definition 1.10. Soft quasigroups [23, 24]

A soft quasigroup over a quasigroup Q_λ is defined as $(F_\alpha, A^\star)$ where F_α maps $A^\star \subseteq K^\star$, to subsets of Q_λ, such that $F(o)$ is a subquasigroup of Q_λ for all $o \in A^\star$, where $K^\star$ is the set of defined parameters.

Let take Table 1 to be a table of a quasigroup $Q_\lambda = \{1, 2, 3, 4, 5, 6, 7, 8\}$ on 8 symbols.

Then let $A^\star = \{\gamma_1, \gamma_2, \gamma_3\} \subset K^\star$ is any defined set of variables. Let F_α be a function mapping $A^\star$ into all the subsets of Q_λ, such that $F_\alpha : A^\star \longrightarrow 2^{Q_\lambda} \backslash \emptyset$ is defined by,

$$F_\alpha(\gamma_1) \equiv \{1, 2\}.$$
$$F_\alpha(\gamma_2) \equiv \{1, 2, 3, 4\}.$$
$$F_\alpha(\gamma_3) \equiv \{1, 2, 7, 8\}.$$

Therefore, $(F_\alpha, A^\star)$ is a soft quasigroup over Q_λ, with $F_\alpha(\gamma_i)$ expressible as Latin squares for $i = 1, 2, 3$.
If $B^\star = \{\gamma_1, \gamma_2, \gamma_3\}$ is another set of parameters over Q_λ in Table 1, and let Σ be a function that maps $B^\star$ to all the subset of Q_λ, such that $\Sigma : B^\star \to 2^{Q_\lambda} \backslash \emptyset$, where, $\Sigma(\gamma_1) \equiv \{1\}$, $\Sigma(\gamma_2) \equiv \{1, 2\}$, $\Sigma(\gamma_3) \equiv \{1, 2, 5, 6\}$, $\Sigma(\gamma_4) = Q_\lambda$. Accordingly, $(\Sigma, B^\star)$ is a soft quasigroup over Q_λ, with each $\Sigma(\gamma_i)$ manifested as a Latin square table.

Thus, if $(F_\alpha, A^\star)$ forms a soft quasigroup over Q_λ, then $(F_\alpha, A^\star) = \{F_\alpha(\gamma_1), F_\alpha(\gamma_2), F_\alpha(\gamma_3), \cdots F_\alpha(\gamma_n)$ and it has $\sum\limits_{i=1}^{n} F_\alpha(\gamma_i)$ soft subquasigroups.

2. Result: Soft Monogenic Quasigroups

Definition 2.1. Soft Monogenic Quasigroups

A finite quasigroup Q_λ is called a monogenic quasigroup, if Q_λ is singly generated by any of its elements. It is denoted as $Q_\lambda =< o >$ for $o \in Q_\lambda$ and Q_λ is non - associative.
Therefore, an element generates a monogenic quasigroup if and only if it is not an element of any proper subquasigroup.

Definition 2.2.

Given soft quasigroup $(\mathcal{F}_\alpha, A^\star)$ over Q_λ and $J_\alpha(o) \subseteq Q_\lambda$. Define $K = \{(o, \langle\kappa(o)\rangle) : \kappa(o) \in J_\alpha(o), o \in A^\star, F_\alpha(o) = \langle\kappa(o)\rangle\}$ is regarded as a soft subquasigroup of $(\mathcal{F}_\alpha, A^\star)$ is based on the set $F_\alpha(o)$ and represented as $< f(o) >$. Hence, if $(\mathcal{F}_\alpha, A^\star) =< \kappa(o) >$, then the soft quasigroup $(\mathcal{F}_\alpha, A^\star)$ is regarded as soft monogenic quasigroup.

Q, ·	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	6	5
4	4	3	2	1	8	7	5	6
5	6	5	8	7	2	1	4	3
6	5	6	7	8	1	2	3	4
7	8	7	5	6	3	4	1	2
8	7	8	6	5	4	3	2	1

Table 1. Quasigroup $(Q_\lambda, \cdot)$ of order 8

If $(\mathcal{F}_\alpha, A^\star)$ is a soft monogenic quasigroup over Q_λ, it can be represented as $(\mathcal{F}_\alpha, A^\star) = \{J(o) = \langle J_\alpha(o) \rangle : o \in A^\star, \kappa(o) \in J_\alpha(o) \subseteq 2^{Q_\lambda}\}$, where each element is generated by some $o \in A^\star$ and $J_\alpha(o) \in 2^{Q_\lambda}$, then $(\mathcal{F}_\alpha, A^\star)$ is a soft monogenic quasigroup over Q_λ since all subquasigroups of monogenic quasigroup are monogenic.

We state some definitions of monogenic quasigroup presentation as defined in cyclic group presentation in [28].

Consider a free soft quasigroup $F_{(\mathcal{F}_\kappa, A^\star)}$ with p generators $\{\kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_p)\}$, where $\zeta_1, \zeta_2, \cdots, \zeta_p$ are reduced words. An automorphism α cycles these generators: $\kappa(\zeta_1) \to \kappa(\zeta_2) \to \cdots \to \kappa(\zeta_p) \to \kappa(\zeta_1)$. Define $L = \{\zeta_i \alpha^j : 1 \leq i \leq p, 1 \leq j \leq p\}$. Then we give the following definitions;

Definition 2.3. The presentation $< F_{(\mathcal{F}_\alpha, A^\star)} | L >$ is called a soft monogenic quasigroup presentation, given $(\mathcal{F}_\alpha, A^\star)$ and P as defined above.

Definition 2.4. A soft quasigroup is soft monogenically generated when defined by a soft monogenic quasigroup presentation.

Definition 2.5. The generating set from soft monogenic quasigroup presentation is regarded as a soft monogenic generating set.

Definition 2.6. If α is automorphism of a soft quasigroup (F, A) and if $\kappa(\alpha) \in (F, A)$ and $\{\kappa(\zeta), \kappa(\zeta)\alpha, \kappa(\zeta)\alpha^2, \cdots\}$ is a monogenic generating set of Q_λ. Then α is regarded as a monogenic automorphism

Theorem 2.1. A soft quasigroup $(\mathcal{F}_\alpha, A^\star)$ is monogenically generated exactly when it has a soft monogenic quasigroup presentation

Proof. Let a soft quasigroup (F, A) be monogenically generated such that $(\mathcal{F}_\alpha, A^\star) = \{\kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_p)\}$. Then let α be an automorphism of $(\mathcal{F}_\alpha, A^\star)$ so that $\kappa(\zeta_1) \longrightarrow \kappa(\zeta_2) \longrightarrow \cdots \longrightarrow \kappa(\zeta_p) \longrightarrow \kappa(\zeta_1)$, and if $Q^\star_\lambda =< (\mathcal{F}_\alpha, A^\star)|L >$ is a presentation for $(\mathcal{F}_\alpha, A^\star)$ on the generating set of $(\mathcal{F}_\alpha, A^\star)$, then any relator $l \in L$ in $\kappa(a)_i s$, is a word, i.e. $l = l(\kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_{(p-1)}), \kappa(\zeta_p))$. Since, α is an automorphism then by applying l and α yields

$$l\alpha = (\kappa(\zeta_1)\alpha, \ldots, \kappa(\zeta_p)\alpha) = l(\kappa(\zeta_2), \ldots, \kappa(\zeta_1)) = 1.$$

So the relation $l\alpha = 1$ comes from the relators L. Therefore, by summing the relators $l\alpha$ and the presentation $Q^\star_\lambda$ will not change the quasigroup defined by $Q^\star_\lambda$ and let $L = \{l_1, l_2, \cdots, l_k\}$, for each l_i and α^j , so that $1 \leq j \leq p-1$, by adding $l_i\alpha^j$ to the relators in $Q^\star_\lambda$ will not alter the quasigroup defined by the resultant presentation. So (F, A) will now be defined by monogenic presentation;

$$< F(a)|l_1, l_1\alpha, l_1\alpha^2, \cdots, l_1\alpha^{p-1}, l_2, l_2\alpha, l_2\alpha^2, \cdots, l_2\alpha^{p-1}, \cdots l_k, l_k\alpha, l_k\alpha^2, \cdots, l_k\alpha p - 1 > .$$

Conversely, if we define $(\mathcal{F}_\alpha, A^\star)$ by the monogenic presentation
$< \kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_p)|l_{11}, l_{12}, \cdots, l_{1\ p1}, l_{1\ p}, l_{2\ 1}, l_{2\ 2}, \cdots, l_{2\ p-1}, l_{2\ p}, ...l_{p-1\ 1}, l_{p-1\ 2}, \cdots, l_{p-1\ n-1}, l_{p-1\ p}, l_{n\ 1}, l_{p\ 2}, \cdots, l_{n\ p-1}, l_{p\ pi}. >$

where $l_{i\ 1} = l_i(\kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_{(p-1)}, \kappa(\zeta_p)$ and $l_{i\ j}$ is the result of replacing $\kappa(\zeta_1)$ with $\kappa(\zeta_j)$ *and* $\kappa(\zeta_2)$ with $\kappa(\zeta_{(j+1)})$, e.t.c, in $l_{i\ 1}$, i.e.
$l_{i\ j} = l_i(\kappa(\zeta_j), \kappa(\zeta_{(j+1)}), \cdots \kappa(\zeta_{(p-1)}), \kappa(\zeta_p), \kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_{(j-1)})$.
Let α be the map defined by

$$\kappa(\zeta_1) \longrightarrow \kappa(\zeta_2)\kappa(\zeta_2) \longrightarrow \kappa(\zeta_3)...\kappa(\zeta_{(p-1)}) \longrightarrow \kappa(\zeta_p)\kappa(\zeta_p) \longrightarrow \kappa(\zeta_1).$$

With α being onto function, we aim to show that it is a homomorphism. This involves proving that replacing $\kappa(\zeta)i$ with $\kappa(\zeta)_i\alpha$ in the presentation's relators gives a word equal to 1 in $(\mathcal{F}_\alpha, A^\star)$, specifically for $l_{i\ j}$.
$l_i(\kappa(\zeta_j)\alpha, \kappa(\zeta_{(j+1)})\alpha, \cdots, \kappa(\zeta_{(p-1)})\alpha, \kappa(\zeta_n)\alpha, \kappa(\zeta_1)\alpha, \cdots, \kappa(\zeta_{(j-1)})\alpha)$
$= l_i(\kappa(\zeta_{(j+1)}), \kappa(\zeta_{(j+2)}), \cdots, \kappa(\zeta_p), \kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_j) = l_{i\ j+1} = 1.$
Hence, α is an automorphism of $(\mathcal{F}_\alpha, A^\star)$ that permutes the generators $\{\kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_p)\}$, implying $(\mathcal{F}_\alpha, A^\star)$ is monogenically generated. □

Theorem 2.2. Every soft monogenic quasigroup is monogenically generated by a single generator

Proof. Given $(\mathcal{F}_\alpha, A^\star) = J(\zeta_p) \times J(b_p)$, the presentation $< \kappa(\zeta), \kappa(b) | \kappa(\zeta)^n, \kappa(b)^p >$ is monogenic for $(\mathcal{F}, \mathcal{A})$. If $(\mathcal{F}\alpha, A^\star)$ consists of q replicas of a monogenic quasigroup of order p, then:

$$\langle \kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_q) \mid \kappa(\zeta_i)^p = 1, [\kappa(\zeta_i), \kappa(\zeta_j)] = 1 \rangle$$

where $i = 1, 2, \cdots, q$ and $1 \leq i < j \leq q$. This presentation clearly induces an automorphism via the cyclic mapping:

$$\kappa(\zeta_1) \to \kappa(\zeta_2) \to \cdots \to \kappa(\zeta_q) \to \kappa(\zeta_1)$$

□

Theorem 2.3. Every finite abelian soft quasigroup is monogenically generated.

Proof. We can write any finite abelian soft quasigroup $(\mathcal{F}_\alpha, A^\star)$ in the form $(\mathcal{F}_\alpha, A^\star) = \kappa(\zeta_{(q_1)}) \times \kappa(\zeta_{(q_2)}) \times \cdots \times \kappa(\zeta_{(qp)})$ where $mn|q\ p-1| \cdots q_2|q_1$. Assuming $\kappa(\zeta_i)$ is a generator of the ith monogenic factor $\kappa(\zeta_{(qi)})$, then
$(\mathcal{F}_\alpha, A^\star) = \kappa(\zeta_{(q_1)}) \times \kappa(\zeta_{(q_2)} \times \cdots \times \kappa(\zeta_{qp}) =< \kappa(\zeta_1), \kappa(\zeta_2), \cdots, \kappa(\zeta_p) | \kappa(\zeta)^{qi}$,
$(i = 1, 2, \cdots, p), [\kappa(\zeta_i), \kappa(\zeta_j)], (1 \leq i < j \leq p) >$.
Defining an onto map;

$$\kappa(\zeta_i) \to \kappa(\zeta_i)\kappa(\zeta_{i+1}) \text{ for } i = 1, 2, \cdots, p-1$$
$$\kappa(\zeta_p) \to \kappa(\zeta_p)$$

Therefore, we can extend this map to a homomorphism θ in $(\mathcal{F}_\alpha, A^\star)$. If θ is maps onto $(\mathcal{F}_\alpha, A^\star)$, then θ is regarded as an automorphism. So, if consecutive powers of θ is applied to $\eta(\zeta_1)$, we have the set;

$$\kappa(\zeta_1) \longrightarrow \kappa(\zeta_1)\kappa(\zeta_2) \longrightarrow \kappa(\zeta_1)\kappa(\zeta_2)^2\kappa(\zeta_3) \longrightarrow \cdots \longrightarrow \kappa(\zeta_1) \prod \kappa(\zeta_p) \longrightarrow \cdots \kappa(\zeta_1)$$

where $\prod$ is a product of powers of $\kappa(\zeta_2), \kappa(\zeta_3), \cdots, \kappa(\zeta_{(p-1)})$. The set $\{\kappa(\zeta_1), \kappa(\zeta_1)\theta, \kappa(\zeta_1)\theta^2, \cdots\}$ clearly generates $(\mathcal{F}_\alpha, A^\star)$, and hence $(\mathcal{F}_\alpha, A^\star)$ is monogenically generated. □

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Theorem 2.4. Every nonabelian soft quasigroup is monogenically generated by two elements

Proof. Assuming $(\mathcal{F}\star_\alpha, A^\star)$ is a nonabelian soft quasigroup. Suppose $\kappa(t)$ has order 2 and $\kappa(b)$ is a distinct element of $(\mathcal{F}\star_\alpha, A^\star)$, then $\kappa(\zeta)$ and $\kappa(b)$ generates $(\mathcal{F}\star_\alpha, A^\star)$.
Therefore, $\kappa(\zeta)$ and $\kappa(b)$ exists in $(\mathcal{F}\star_\alpha, A^\star)$ since every nonabelian quasigroup can be generated by an element of order 2.
Let α be conjugated by $\kappa(\zeta)$ and if we let the subquasigroup $J(\zeta)$ be generated by $\kappa(\zeta)$ and $\kappa(b)\alpha$, then if $J(\zeta) \neq (\mathcal{F}\star_\alpha, A^\star)$; then $\kappa(\zeta)$ cannot be in $J(\zeta)$, so $\kappa(\zeta)J(\zeta) \neq J(\zeta)$, but $J(\zeta)$ has index 2 in $(\mathcal{F}\star_\alpha, A^\star)$ because $\kappa(\zeta)J(\zeta) \cup J(\zeta) = (\mathcal{F}\star_\alpha, A^\star)$. Thus, $J(\zeta)$ being normal in $(\mathcal{F}\star\alpha, A^\star)$ leads to a contradiction, implying $(\mathcal{F}\star\alpha, A^\star)$ is generated by two elements. □

Theorem 2.5. Let $(\mathcal{F}_\alpha, A^\star)$ and $(\mathcal{G}_\alpha, B^\star)$ be soft monogenic quasigroups, if $(\mathcal{F}_\alpha, A^\star)$ is generated by elements of $|J(\zeta)| = m$ and $(\mathcal{G}_\alpha, B^\star)$ is generated by element of $|G(b)| = n$. Therefore, the direct product of $J(\zeta) \times G(b)$ will be monogenically generated if m and n are coprime.

Proof. Let $J(\zeta) \in (\mathcal{F}_\alpha, A^\star)$ and $\alpha \in AutJ(\zeta)$ monogenically generate $(\mathcal{F}_\alpha, A^\star)$, and $G(b) \in (\mathcal{G}_\alpha, B^\star)$ and $\phi \in AutG(b)$ monogenically generate $(\mathcal{G}_\alpha, B^\star)$. Consider the elements of $(\mathcal{F}_\alpha, A^\star) \times (\mathcal{G}_\alpha, B^\star)$ as pairs $(J(\zeta), G(b))$ and define a map $\gamma : J(\zeta) \times G(b) \longrightarrow J(\zeta) \times G(b)$ by,

$$(J(\zeta), G(b))\gamma = (J(\zeta)\alpha, G(b)\phi).$$

With α and ϕ defined as automorphisms, so γ is also an automorphism. Applying γ's powers to generating set $(J(\zeta), G(b))$ gives;

$$(J(\zeta), G(b)) \longrightarrow (J(\zeta)\alpha, G(b)\phi) \longrightarrow (J(\zeta)\alpha^2, G(b)\phi^2) \longrightarrow ... \longrightarrow (J(\zeta), G(b)).$$

Let $sm + tn = 1$, where s and t are integers and there exists m and n as coprime, such that for any j; it implies that,

$$((F(a), G(b)\gamma^j)^{tn} = (J(\zeta)\alpha^j, G(b)\phi^j)^{tn} = ((F(a)\alpha^j)^{tn}, (G(b)\phi^j)^{tn}) = ((J(\zeta)\alpha^j)^{1-sm}, 1) = (J(\zeta)\alpha^j, 1)$$

and similarly,

$$((J(\zeta), G(b)\gamma^j)^{sm} = (1, G(b)\gamma^j).$$

As the $J(\zeta)\gamma^j s$ generate $(\mathcal{F}_\alpha, A^\star)$ and the $G(b)\phi^j s$ generate $(\mathcal{G}_\alpha, B^\star)$, the $(J(\zeta), G(b))\gamma^j s$ must generate $(\mathcal{F}_\alpha, A^\star) \times (\mathcal{G}_\alpha, B^\star)$. □

Definition 2.7. Soft Monogenic Quasigroup Generators

Let $(\mathcal{F}_\alpha, A^\star)$ be a soft quasigroup, then the only soft monogenic quasigroup $F(u)$, $F(u) \in (\mathcal{F}_\alpha, A^\star)$ $\forall u \in A$ of any given order, is the soft cyclic group, and this is always generated by any element which is relatively prime with the order of $(\mathcal{F}_\alpha, A^\star)$.
So except for cyclic groups, monogenic quasigroups are not necessarily power associative. Then a soft quasigroup is not monogenic if every elements is in some proper subquasigroups and every non - generator is sufficiently in some proper subquasigroups.
We want to investigate how many elements of a soft monogenic quasigroup can

be its generators.
$F(u)$ of order 1 and 2 are monogenic quasigroups generated by all of its elements because there are no ways of forming a proper subquasigroup out of it. For $|(\mathcal{F}_\alpha, A^\star)| = 3$ there are soft monogenic quasigroups with 2 and with 3 generators, and there is no order 3 quasigroups with exactly 2 non - generators. For $|(\mathcal{F}_\alpha, A^\star)| = 4$ there are examples for all possible numbers of generators.
There are soft monogenic quasigroups $|(\mathcal{F}_\alpha, A^\star)| = p$, where $|(\mathcal{F}_\alpha, A^\star)| \geq 4$ with exactly d generators for any $d \leq p$, meaning that it is separately generated by each of d of its elements, such that $(\mathcal{F}_\alpha, A^\star) =< u_1 >=< u_2 >= \cdots =< u_d >$.

Proposition 2.1. Given a soft quasigroup $(\mathcal{F}_\alpha, A^*)$ of order p, any soft subquasigroup $J(u)$ has order at most $\frac{p}{2}$.

Proof. Since $J(u)$ is a subquasigroup of $(\mathcal{F}_\alpha, A^\star)$, then if $x \in (\mathcal{F}_\alpha, A^\star) - J(u)$ and $y \in J(u)$, then $xy \in (\mathcal{F}_\alpha, A^\star) - J(u)$. So $xJ(u) \subset (\mathcal{F}_\alpha, A^\star) - J(u)$. But $xJ(u)$ has the order h, since $(\mathcal{F}_\alpha, A^\star) - J(u)$ has order $p - h$; that is
$h \leq p - h = 2h \leq p$.
$\Longrightarrow h \leq \frac{p}{2}$. □

Theorem 2.6. The set of all non - generators form a subquasigroup of $(\mathcal{F}_\alpha, A^\star)$

Proof. Given soft quasigroup $(\mathcal{F}_\alpha, A^\star)$ so that $J(o), J(p) \in (\mathcal{F}_\alpha, A^\star)$, then if $J(o^\star) \in J(o)$ is the set of non - generator of $(\mathcal{F}_\alpha, A^\star)$, and $J(p) \neq \emptyset$ is the set of generators of $(\mathcal{F}_\alpha, A^\star)$

Given $J(o_1), J(o_2) \in J(o)$, if both elements belong to $J(o)$, then $J(o^\star)$ is contained in $J(o)$. For a proper subquasigroup $J(p^\star)$ of $(\mathcal{F}_\alpha, A^*)$, the following holds:

$$\langle J(p^\star), J(o_1) \cdot J(o_2) \rangle \leq \langle J(p^\star), J(o_1) \rangle \cdot \langle J(o_2), J(p^\star) \rangle$$

So, $J(o_1) \cdot J(o_2)$ are also a non - generator of $(\mathcal{F}_\alpha, A^\star)$ and forms a subquasigroup of $(\mathcal{F}_\alpha, A^\star)$, where $(\cdot)$ is any quasigroup operation, since each non-generator element is necessarily in some proper soft subquasigroup. □

Any quasigroup generated by all of its elements is called Monoquasigroup as stated by Izbash [14], who noted that monoquasigroups exist for all order n of a quasigroup. We consider two propositions as follows;

Proposition 2.2. Let l be the number of generators of $(\mathcal{F}_\alpha, A^\star)$, then we consider;

i. $l > |J(u)|$, where $|J(u)| \subseteq (\mathcal{F}_\alpha, A^\star)$
ii. When $l \leq |J(u)|$, where l is the number of generators denoted by $1, 2, \cdots, u$, and non - generators will be denoted by $l+1, l+2, \cdots, u$.

Proof. In each case we consider the structure of $J(u)$ as a subquasigroup of $(\mathcal{F}_\alpha, A^\star)$ and do a construction, followed by some examples as illustrations, since $|J(u)|$ is at most $\frac{p}{2}$, and $|(\mathcal{F}_\alpha, A^\star)| = p$.
Let $|(\mathcal{F}_\alpha, A^\star)| = p$ and $J(u) \subset (\mathcal{F}_\alpha, A^\star)$, where the elements of $J(u) = \{l+1,\ l+2,\ \cdots,\ u\}$ then $|J(u)| = p - l$ and $((\mathcal{F}_\alpha, A^\star)) - J(u) = \{1,\ 2,\ \cdots,\ p\}$.
So, if $|(\mathcal{F}_\alpha, A^\star)| = p$ is even, so $|J(u)| = \frac{p}{2}$, so elements of $J(u) = \{\frac{p}{2}+1,\ \frac{p}{2}+2,\ \cdots,\ p\}$.

If $|(\mathcal{F}_\alpha, A^\star)|$ is odd, and $|J(u)| = \frac{p-1}{2}$, then the largest order of $J(u) = \frac{p-1}{2}$, and the elements of $J(u) = \{1, 2, \cdots, l\}$. □

Theorem 2.7. There exists a soft monogenic quasigroup $(\mathcal{F}_\alpha, A^\star)$ of order p generated by exactly k of its elements, where $u > p$.

Proof. Since $J(u) \subset (\mathcal{F}_\alpha, A^\star)$ then from proposition 2.1 $|J(u)|$ is at most $\frac{p}{2}$ and we set;

$$1 \leq j \leq t \qquad\qquad j^2 = j + 1 \ mod\ k$$

and letting

$$u + 1 \leq j \leq n \qquad\qquad j^2 = j.$$

For each of the elements $j \in \{1, ..., u\}$, successive squaring of j produces t distinct self-products of u. As $u > \frac{p}{2}$, this ensures that $< j >= F(u)$, so j is a generator. The remaining elements are either idempotent or are members of a proper subquasigroup and therefore can not generate $J(u)$ and thus $J(u)$ has exactly u generators. □

Proposition 2.3. There exists a soft quasigroup $(\mathcal{F}_\alpha, A^\star)$ of order p generated by any number of generators u of its elements, where $u \leq p$.

Proof. Let $|J(u)| \subseteq (\mathcal{F}_\alpha, A^\star) = \frac{p}{2}$ and its elements be $\frac{p}{2} + 1, \cdots, n$ form a subquasigroup $J(u)$ of index 2, with u_1, u_2 as generators of $J(u)$ and $f(u_1), f(u_2)$ also in $J(u)$ as idempotent elements., then we have two sets of elements of $J(u)$ i.e, $\frac{p}{2} + 1, ..., p$ and $1, 2, ..., \frac{p}{2}$.
Then letting,

$$1 \leq u \leq p \qquad\qquad u^2 = p,$$

and

$$u + 1 \leq \frac{p}{2} \leq p \qquad\qquad u^2 = f(u)_1$$

and

$$p \cdot f(u_1) = f(u_1) \cdot p = p.$$

Hence if $u \neq 1$, it gives us u generators as elements $1 - k$ each generates $J(u)$. The remaining elements are in some proper subquasigroup. The result follows that to be true for $|J(u)| \geq 4$, for $|J(u)| = 4$ and $|J(u)| = 5$ with $|J(u)| \leq \frac{p}{2}$, and for $|J(u)| = 5$ with explicit examples.
For $|J(u)| = 4$ the first table with 1 generator, ie it is generated by the element (1), and other elements are idempotent elements. In the second table below, it has 2 generators, namely 1 and 2, the remaining two elements 3 and 4 are idempotent elements.

.	1	2	3	4
1	3	4	2	1
2	1	2	4	3
3	4	1	3	2
4	2	3	1	4

.	1	2	3	4
1	2	3	4	1
2	4	1	2	3
3	1	4	3	2
4	3	2	1	4

□

Example 2.1. Below is a soft quasigroup $(\mathcal{F}_\alpha, A^\star)$ of order 5, with 3 generators.

·	1	2	3	4	5
1	2	3	1	5	4
2	4	1	5	3	2
3	1	5	4	2	3
4	5	2	3	4	1
5	3	4	2	1	5

Table 2. Soft Quasigroup of order 5

Here the three generators are 1, 2 and 3, while 4, 5 are idempotent elements.

3. CONCLUSION

This study significantly advances the understanding of soft quasigroups by introducing and characterizing soft monogenic quasigroups. We established key properties, including generation criteria and structural invariance under automorphism. These findings provide a foundation for further research in algebraic structures and potential applications in fields like cryptography and coding theory. Future work may explore extensions to other algebraic systems and practical implementations.

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