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SOME FIXED POINT RESULTS FOR REICH TYPE CONTRACTION MAPPINGS IN BIPOLAR METRIC SPACES WITH APPLICATIONS

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Abstract. In this study, we present the novel idea of contravariant mappings with Reich convex contraction type in bipolar metric spaces, contributing to the understanding of distances between disparate entities. Furthermore, we establish the existence of a singular fixed point for contravariant mappings of Reich convex contraction-type within complete bipolar metric spaces. Our investigation extends to obtaining solutions for integral and fractional differential equations through the application of this operator. To validate our findings, we presented examples to illustrate the implications of the results.

1. Introduction

In 1906, Frechet[3] introduced the concept of a metric space with a domain consisting of the product of an arbitrary non-empty set. Subsequently, many researchers expanded upon this space by either relaxing the axioms or modifying the distance function. Specifically, Mutlu and Gurdal [9] extended the metric space by broadening the domain of a metric to the product of two distinct non-empty sets.

Imaga et al. [6] resolved a fractional-order p-Laplacian boundary value problem involving left Caputo fractional derivatives on the half-line. Gurdal et al. [5] introduced contractive-type covariant and contravariant mappings in bipolar metric spaces, proving fixed point theorems for these operators in complete bipolar metric spaces. Recently, Gaba et al. [4] established the existence of a unique fixed point for Reich contraction mapping in bipolar metric space.

In 1981, Istratescu [7] generalized Banach contraction mappings in [1] by introducing the convexity condition, thereby creating another class of mappings known as convex contraction mappings. Istratescu[7] demonstrated fixed point

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theorems for convex contraction mappings, including the Reich convex contraction mapping, in a metric space. This result was later extended to Hardy and Rogers convex contraction mappings of type 2 by Eke et al. [2]. The authors in [2] verified the existence of unique fixed points for these operators in a complete metric space, applying the results to establish solutions for nonlinear Fredholm integral equations.
Nallaselli et al. [10] proved the existence of a unique fixed point for alpha-f-convex contraction mappings in complete metric spaces, using the result to establish the solution of an integral equation. For the purpose of this research, the outcome of Istratescu [7]'s work is adopted.
It is our aim in this study to introduce the concepts of contravariant mappings with Reich convex contraction and establish the fixed point theorem for this operator in bipolar metric spaces. Removing the contravariant condition from this newly introduced operator yields the result presented by[7] within the framework of metric spaces. Similarly, removing the convexity condition from this newly introduced operator leads to the result obtained by [4].

2. Preliminaries

We shall need the following definitions and theorems by previous authors in proving our main results:

Definition 2.1. [9] Let X and Y be nonempty sets and $d : X \times Y \to R^+$ be a function where R^+ denotes the set of nonnegative real numbers. Consider the following :
(i) $d(x, y) = 0$ if and only if $x = y$ for all $(x, y) \in X \times Y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X \cap Y$;
(iii) $d(x_1, y_2) \leq d(x_1, y_1) + d(x_2, y_1) + d(x_2, y_2)$ for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.
Then, the triple (X, Y, d) is called a bipolar metric space.
The set X and Y are respectively called the left pole and the right pole of (X, Y, d).

Let (X_1, Y_1, d_1) and (X_2, Y_2, d_2) be bipolar metric spaces and $f : X_1 \cup Y_1 \to X_2 \cup Y_2$ be a function. If $f(X_1) \subseteq X_2$ and $f(Y_1) \subseteq Y_2$ then f is called covariant map and it is written as $f : (X_1, Y_1, d_1) \rightrightarrows (X_2, Y_2, d_2)$. If $f : (X_1, Y_1, d_1) \leftrightarrows (X_2, Y_2, d_2)$ is a map, then f is called a contravariant map from (X_1, Y_1, d_1) to (X_2, Y_2, d_2).

Theorem 2.2. [4] *Let (X, Y, d) be a complete bipolar metric space and $F : (X, Y, d) \leftrightarrows (X, Y, d)$ be a contravariant map such that there exist constants $a_1, a_2, a_3 \geq 0$ with $a_1 + a_2 + a_3 < 1$ so that*

$$d(fy, fx) \leq a_1 d(x, y) + a_2 d(x, fx) + a_3 d(fy, y),$$

whenever $(x, y) \in X \times Y$. Then the function $f : X \cup Y \leftrightarrows X \cup Y$ has a unique fixed point.

The following results are found in [7].

Definition 2.3. Let X be a complete metric space and $f : X \to X$ be a continuous self mapping. Then, f is said to be a convex contraction of order 2 if there exist $a, b \in (0, 1)$ such that for all $x, y \in X$,

$$d(f^2x, f^2y) \leq ad(fx, fy) + bd(x, y),$$

where $a + b < 1$.

Definition 2.4. Let X be a complete metric space and $f : X \to X$ be a continuous self mapping. Then, f is called Kannan two- sided convex contraction mappings if there exist positive numbers $a_1, a_2, b_1, b_2 \in (0, 1)$ such that the following inequality holds:

$$d(f^2x, f^2y) \leq a_1d(x, fx) + a_2d(fx, f^2x) + b_1d(y, fy) + b_2d(fy, f^2y),$$

for all $x, y \in X$ and $a_1 + a_2 + b_1 + b_2 < 1$.

Theorem 2.5. *Let f be a continuous self mapping of a complete metric space satisfying the condition:*

$$\begin{aligned} d(f^2x, f^2y) \leq\ & a_1d(x, y) + a_2d(fx, fy) + b_1d(x, fx) + b_2d(fx, f^2x) \\ & +c_1d(y, fy) + c_2d(fy, f^2y), \end{aligned}$$

where $0 \leq a_1 + a_2 + b_1 + b_2 + c_1 + c_2 < 1$ distinct $x, y \in X$. Then, f has a unique fixed point.

3. Results

Within this section, we provide the definition of contravariant mappings with Reich convex contraction in bipolar metric spaces and establish the existence of a unique fixed point for this operator within such spaces. Furthermore, we offer two illustrative examples to validate the obtained results.

Definition 3.1. Let (X, Y, d) be a bipolar metric space, $f : (X, Y, d) \leftrightarrows (X, Y, d)$ be a contravariant map, and $a_1, a_2, b_1, b_2, c_1, c_2 \geq 0$ be constants. f is called Reich convex contraction contravariant mappings if the following conditions hold:

$$\begin{aligned} d(f^2x, f^2y) \leq\ & a_1d(x, y) + a_2d(fx, fy) + b_1d(x, fx) + b_2d(fx, f^2x) \\ & +c_1d(y, fy) + c_2d(fy, f^2y), \end{aligned}$$

where $0 \leq a_1 + a_2 + b_1 + b_2 + c_1 + c_2 < 1$, whenever $(x, y) \in X \times Y$.

Example 3.2. Let $X = \{0, 1, 2, 7\}$, $Y = \{1, \frac{1}{4}, \frac{1}{2}, 3\}$ and $d : X \times Y \to R$ be defined by $d(x, y) = |x - y|$ for all $(x, y) \in X \times Y$. Then (X, Y, d) is a complete bipolar metric space. Define $f : X \cup Y \leftrightarrows X \cup Y$ by $fx = \frac{x^2}{5} + \frac{1}{5}$ for all $x \in X \cup Y$. Then f is called Reich convex contraction contravariant mappings.

Theorem 3.3. *Let (X, Y, d) be a complete bipolar metric space and f be Reich convex contraction contravariant mappings. Then the function $f : X \cup Y \leftrightarrows X \cup Y$ has a unique fixed point.*

Proof. Let $x_0 \in X$ and $y_0 \in Y$ then we define $y_n = f(x_n)$ and $x_{n+1} = fy_n$ for each $n \geq 0$. Thus (x_n, y_n) is a bisequence in (X, Y, d).

$$
\begin{aligned}
d(x_{n+1}, y_{n+1}) &= d(f^2x_n, f^2y_n) \\
&\leq a_1d(x_n, y_n) + a_2d(fx_n, fy_n) + b_1d(x_n, fx_n) + b_2d(fx_n, f^2x_n) \\
&\quad +c_1d(y_n, fy_n) + c_2d(fy_n, f^2y_n) \\
&= a_1d(x_n, y_n) + a_2d(y_n, x_{n+1}) + b_1d(x_n, y_n) + b_2d(y_n, x_{n+1}) \\
&\quad +c_1d(y_n, x_{n+1}) + c_2d(x_{n+1}, y_{n+1}) \\
&= (a_1 + b_1)d(x_n, y_n) + (a_2 + b_2 + c_1)d(y_n, x_{n+1}) + c_2d(x_{n+1}, y_{n+1}) \\
&= (a_1 + b_1 + a_2 + b_2 + c_1)max\{d(x_n, y_n), d(y_n, x_{n+1})\} \\
&\quad +c_2d(x_{n+1}, y_{n+1}) \\
&\leq \lambda\ max\{d(x_n, y_n), d(y_n, x_{n+1})\},
\end{aligned}
$$

where $\lambda = \frac{a_1+b_1+a_2+b_2+c_1}{1-c_2} < 1$. If max $\{d(x_n, y_n), d(y_n, x_{n+1})\} = d(x_n, y_n)$, then we have $d(x_{n+1}, y_{n+1}) \leq \lambda d(x_n, y_n)$.
Consequently,

$$
\begin{aligned}
d(x_{n+1}, y_{n+1}) &\leq \lambda d(x_n, y_n) \\
&\vdots \\
&\leq \lambda^{n+1}d(x_0, y_0). \qquad (3.1)
\end{aligned}
$$

Suppose max $\{d(x_n, y_n), d(y_n, x_{n+1})\} = d(y_n, x_{n+1})$, then we have

$$d(x_{n+1}, y_{n+1}) \leq \lambda d(y_n, x_{n+1})$$

and

$$
\begin{aligned}
d(y_n, x_{n+1}) &= d(f^2y_{n-1}, f^2x_n) \\
&\leq a_1d(y_{n-1}, x_n) + a_2d(fy_{n-1}, fx_n) + b_1d(y_{n-1}, fy_{n-1}) \\
&\quad +b_2d(fy_{n-1}, f^2y_{n-1}) + c_1d(x_n, fx_n) + c_2d(fx_n, f^2y_{n-1}) \\
&= a_1d(y_{n-1}, x_n) + a_2d(x_n, y_n) + b_1d(y_{n-1}, x_n) + b_2d(x_n, y_n) \\
&\quad +c_1d(x_n, y_n) + c_2d(y_n, y_n) \\
&= (a_1 + b_1)d(y_{n-1}, x_n) + (a_2 + b_2 + c_1)d(x_n, y_n) \\
&= (a_1 + b_1 + a_2 + b_2 + c_1)max\{d(y_{n-1}, x_n), d(x_n, y_n)\} \\
&\leq kmax\{d(y_{n-1}, x_n), d(x_n, y_n)\} \\
&\leq kd(x_n, y_n),
\end{aligned}
$$

where $k = a_1 + b_1 + a_2 + b_2 + c_1 < 1$. Consequently,

$$
\begin{aligned}
d(y_n, x_{n+1}) &\leq kd(x_n, y_n) \\
&\vdots \\
&\leq k^nd(x_0, y_0). \qquad (3.2)
\end{aligned}
$$

In view of (3.1), (3.2) and for positive integers m and n, we have

$$\begin{aligned} d(x_n, y_m) &\leq d(x_n, y_n) + d(x_{n+1}, y_n) + d(x_{n+1}, y_m) \\ &\leq (\lambda^n + k^{n+1}) d(x_0, y_0) + d(x_{n+1}, y_m) \\ &\vdots \\ &\leq (\lambda^n + k^{n+1} + \cdots + k^m) d(x_0, y_0). \end{aligned} \tag{3.3}$$

For $m > n$,

$$\begin{aligned} d(x_n, y_m) &\leq d(x_{m+1}, y_m) + d(x_{m+1}, y_{m+1}) + d(x_n, y_{m+1}) \\ &\leq (k^n + \lambda^{n+1}) d(x_0, y_0) + d(x_n, y_{m+1}) \\ &\vdots \\ &\leq (k^n + \lambda^{n+1} + \cdots + k^m) d(x_0, y_0). \end{aligned} \tag{3.4}$$

Since $\lambda < 1$ and $k < 1$, it implies that $d(x_n, y_m)$ can be arbitrarily small by larger m and n, hence (x_n, y_m) is a Cauchy bisequence. Since (X, Y, d) is complete, (x_n, y_m) is convergent and it is biconvergent since it is convergent Cauchy bisequence. Let r be the point to which (x_n, y_m) biconverges. Then $x_n \to r$, $y_n \to r$ and $r \in X \times Y$. Also, $y_n = fx_n \to fr$. Since $\{y_n\}$ has a limit in $X \cap Y$, then the limit is unique. Hence $fr = r$ and so f has a fixed point.
Suppose there is a different fixed point of f say v such that $fv = v$, then it implies that $v \in X \cap Y$. Since $y_n = f^2 y_{n-1} \to f^2 r$ and $y_n = fx_n \to fr$, we have that $fr = f^2 r$.
Hence,

$$\begin{aligned} d(v, r) &= d(f^2 v, f^2 r) \leq a_1 d(v, r) + a_2 d(fv, fr) + b_1 d(v, fv) + b_2 d(fv, f^2 v) \\ &\quad + c_1 d(r, fr) + c_2 d(fr, f^2 r) \\ &= a_1 d(v, r) + a_2 d(v, r) \\ &= (a_1 + a_2) d(v, r). \end{aligned}$$

Since $a_1 + a_2 < 1$, then $d(v, r) = 0$. Hence $r = v$. □

If $c_1 = c_2 = 0$ in Theorem 3.2, then we obtain the following result.

Corollary 3.4. *Let (X, Y, d) be a complete bipolar metric space and f be Kannan convex contraction contravariant mappings. Then the function $f : X \cup Y \leftrightarrows X \cup Y$ has a unique fixed point.*

If $b_1 = b_2 = c_1 = c_2 = 0$ in Theorem 3.3 then, we obtain the following result.

Corollary 3.5. *Let (X, Y, d) be a complete bipolar metric space and f be convex contraction contravariant mappings. Then, the function $f : X \cup Y \leftrightarrows X \cup Y$ has a unique fixed point.*

Remark 3.6. (i) If $a_2 = b_2 = c_2 = 0$ and the order of the mapping is one, then Theorem 3.3 reduces to the result of [4] (Theorem 1).
(ii) If the contrvariant condition is remove from the operator of Theorem 3.3 and $b_1 = b_2 = c_1 = c_2 = 0$ then we obtain the result of [7] for convex contraction

mappings.
(iii) if $a_2 = b_1 = b_2 = c_1 = c_2 = 0$ and the order of the mapping is one then Theorem 3.3 reduces to the result of [9] (Theorem 5.2).

Example 3.7. Let $X = \{5, 6, 11, 17\}$ and $Y = \{2, 4, 11, 20\}$. Define $d : X \times Y \to [0, \infty)$ as the usual metric, $d(x, y) = |x - y|$. Then the triple (X, Y, d) is a bipolar metric space. The contravariant mapping $f : X \cup Y \leftrightarrows X \cup Y$, is defined by

$$fx = \begin{cases} 11, \ if \ x \in X \cup \{20\}, \\ \\ \frac{x^2}{7} + \frac{1}{14}, \ if \ x = 2, \end{cases}$$

whenever $(x, y) \in X \times Y$ we obtain

$|fx - fy| \leq \frac{1}{7}|x^2 - y^2| \leq |x - y|$.

$$\begin{aligned} |f^2x - f^2y| &= |(\frac{x^2}{7} + \frac{1}{14})^2 - (\frac{y^2}{7} + \frac{1}{14})^2| \\ &= |\frac{x^4}{49} + \frac{x^2}{49} + \frac{1}{196} - \frac{y^4}{49} - \frac{y^2}{49} - \frac{1}{196}| \\ &= |\frac{x^4}{49} + \frac{x^2}{49} - \frac{y^4}{49} - \frac{y^2}{49}| \\ &= \frac{1}{49}|x^4 - y^4 + x^2 - y^2| \\ &\leq \frac{1}{7}|x - y| + \frac{1}{14}|fx - fy| = \frac{1}{7}d(x, y) + \frac{1}{14}d(fx, fy) \\ &\leq \frac{1}{7}d(x, y) + \frac{1}{14}d(fx, fy) + \frac{1}{7}d(x, fx) + \frac{1}{28}d(fx, f^2x) \\ &+\frac{1}{14}d(y, fy) + \frac{1}{28}d(fy, f^2y). \end{aligned}$$

Thus, the conditions of Theorem 3.3 are satisfied and $\{11\} \in X \cap Y$ is the unique fixed point of f.

4. APPLICATIONS

This section deals with the utilization of Theorem 3.3 to establish the solutions of integral and fractional differential equations as follows.

4.1. **An Application to Integral Equation.**

In this study, we investigate the existence and uniqueness of solutions for an integral equation.

Theorem 4.1. *Consider the integral equation*

$$f(\gamma(x)) = f(x) + \int_{X \cup Y} k(x, y, \gamma(s))ds, x \in X \cup Y,$$

where $X \cup Y$ is a lebesgue measurable set. Suppose
(i) $K : (X^2 \cup Y^2) \times [0, \infty) \to [0, \infty)$ and $f \in L^\infty(X) \cup L^\infty(Y)$, satisfies

$$\begin{aligned} |K(x, y, f(\gamma(s))) - K(x, y, f(\beta(s)))| &\leq a_1|x - y| + a_2|fx - fy| + b_1|x - fx| \\ &+b_2|fx - f^2x| + c_1|y - fy| + c_2|fy - f^2y|, \end{aligned}$$

for $(x, y) \in (X^2 \cup Y^2)$ *and* $a_1, a_2, b_1, b_2, c_1, c_2 \in [0, 1)$ *with* $\int_{X\cup Y} ds < 1$. *Then, the integral equation has a unique solution in* $L^\infty(X) \cup L^\infty(Y)$.

Proof. Let $A = L^\infty(X)$ and $B = L^\infty(Y)$ be two normed linear spaces, where X and Y are Lebesgue measurable sets and $m(X \cup Y) < \infty$. Consider $d : A \times B \to [0, +\infty)$ to be defined by $d(x, y) = ||x - y||_\infty$ for all $(x, y) \in A \times B$. Then (A, B, d) is a complete bipolar metric space. Define the contravariant mapping $f : A \cup B \leftrightarrows A \cup B$ by
$f(\gamma(x)) = f(x) + \int_{X\cup Y} k(x, y, \gamma(s))ds$, where $x \in X \cup Y$.
Then,
$d(f^2\gamma(x), f^2\beta(x)) = ||f^2\gamma(x) - f^2\beta(x)||$

$= | \int_{X\cup Y} k(x, y, f(\gamma(s)))ds - \int_{X\cup Y} k(x, y, f(\beta(s)))ds|$

$\leq \int_{X\cup Y} |k(x, y, f(\gamma(s))) - k(x, y, f(\beta(s)))|ds$

$\leq \int_{X\cup Y}(a_1|x(s) - y(s)| + a_2|fx(s) - fy(s)| + b_1|x(s) - fx(s)|$

$+b_2|fx(s) - f^2x(s)| + c_1|y(s) - fy(s)| + c_2|fy(s) - f^2y(s)|)ds$

$\leq (a_1 + a_2 + b_1 + b_2 + c_1 + c_2)$

$\int_{X\cup Y} max\{|x(s) - y(s)|, |fx(s) - fy(s)|, |x(s) - fx(s)|,$

$|fx(s) - f^2x(s)|, |y(s) - fy(s)|, |fy(s) - f^2y(s)|\}ds$

$= kmax\{|x(s) - y(s)|, |fx(s) - fy(s)|, |x(s) - fx(s)|,$

$|fx(s) - f^2x(s)|, |y(s) - fy(s)|, |fy(s) - f^2y(s)|\} \int_{X\cup Y} ds$

$\leq kmax\{d(x, y), d(fx, fy), d(x, fx), d(fx, f^2x), d(y, fy), d(fy, f^2y)\}$,
where $k = a_1 + a_2 + b_1 + b_2 + c_1 + c_2 < 1$, for $(x, y) \in (X \times Y)$.
Therefore, the contravariant mapping f has a unique solution in $A \cup B$ according to Theorem 3.3.

4.2. **An Application to Fractional Differential Equation.** Consider the following fractional differential equation:

$$^eD^\mu\rho(\varphi)+f(\varphi, \rho(\varphi)) = 0,\ 1 \leq \varphi \leq 0,\ 2 \leq \mu > 1,\ \rho(0) = \rho(1) = 0, \tag{4.2.1}$$

where f is a continuous function from $[0, 1] \times \mathbb{R} \to \mathbb{R}$ and $^eD^\mu$ represent the Caputo fractional derivative of order μ and it is defined by

$^eD^\mu = \frac{1}{\Gamma(\tau-\mu)} \int_0^1 \frac{\rho^\tau(e)de}{(\mu-e)^{\mu-\tau+1}}$.

Let $A = (C[0, 1], [0, \infty))$ be the set of all continuous functions defined on $[0, 1]$

with values in the interval $[0,\infty)$ and $B = (C[0,1],[0,\infty))$ be the set of all continuous functions defined on $[0,1]$ with values in the interval $[0,\infty)$.
Consider $d : A\times B \to \mathbb{R}^+$ to be defined by $d(\rho,\rho') = sup_{\mu\in[0,1]}|\rho(\mu)-\rho'(\mu)|$, for all $(\rho,\rho')\in A\times B$. Then (A,B,d) is a complete bipolar metric space.

Theorem 4.2. *Consider the fractional differential equation in (4.2.1). Assuming that the following conditions hold:*
(i) there exists $\varphi\in[0,1]$, $\beta\in(0,1)$ and $(\rho,\rho')\in A\times B$ such that

$$\begin{aligned}|f(\varphi,\rho)-f(\varphi,\rho')| \;\leq\;& a_1d(\rho(\varphi),\rho'(\varphi))+a_2d(f\rho(\varphi),f\rho'(\varphi))+b_1d(\rho(\varphi),f\rho(\varphi))\\&+b_2d(f\rho(\varphi),f^2\rho(\varphi))+c_1d(\rho'(\varphi),f\rho'(\varphi))\\&+c_2d(f\rho'(\varphi),f^2\rho'(\varphi)),\end{aligned}$$

(ii) $sup_{\varphi\in[0,1]}\int_0^1|G(\varphi,e)|dq\leq 1$.
Then, the FDE(4.2.1) has a unique solution in $A\cup B$.

Proof. Equation (4.2.1) is equivalent to $\rho(\varphi)=\int_0^1 G(\varphi,e)f(\varphi,\rho(e))de$

where

$$G(\varphi,e)=\begin{cases}\frac{|\varphi(1-e)|^{\mu-1}-(\varphi-e)^{\mu-1}}{\Gamma(\mu)},\ 0\leq e\leq\varphi\leq 1,\\ \\ \frac{|\varphi(1-e)|^{\mu-1}}{\Gamma(\mu)},\ 0\leq\varphi\leq e\leq 1.\end{cases}$$

Define the contravariant mapping $F : A\cup B \leftrightarrows A\cup B$ by
$f^2\rho(\varphi)=\int_0^1 G(\varphi,e)f(\varphi,\rho(e))de.$

If $\rho*$ is the fixed point of f then $\rho*$ is the solution to the problem (1).
Now
$|f^2(\varphi,\rho)-f^2(\varphi,\rho')| = |\int_0^1 G(\varphi,e)f(\varphi,\rho(e))de-\int_0^1 G(\varphi,e)f(\varphi,\rho'(e))de|$

$\leq\int_0^1|G(\varphi,e)|de\times\int_0^1|f(\varphi,\rho(e))-f(\varphi,\rho'(e))|de$

$\leq|f(\varphi,\rho)-f(\varphi,\rho')|$

$=a_1d(\rho(\varphi),\rho'(\varphi))+a_2d(f\rho(\varphi),f\rho'(\varphi))+b_1d(\rho(\varphi),f\rho(\varphi))$

$+b_2d(f\rho(\varphi),f^2\rho(\varphi))+c_1d(\rho'(\varphi),f\rho'(\varphi))+c_2d(f\rho'(\varphi),f^2\rho'(\varphi))$

$\leq(a_1+a_2+b_1+b_2+c_1+c_2)max\{d(\rho(\varphi),\rho'(\varphi)),d(f\rho(\varphi),f\rho'(\varphi)),$

$d(\rho(\varphi),f\rho(\varphi)),d(f\rho(\varphi),f^2\rho(\varphi)),d(\rho'(\varphi),f\rho'(\varphi)),d(f\rho'(\varphi),f^2\rho'(\varphi))\}$

$=\beta max\{d(\rho(\varphi),\rho'(\varphi)),d(f\rho(\varphi),f\rho'(\varphi)),d(\rho(\varphi),f\rho(\varphi)),$

$d(f\rho(\varphi),f^2\rho(\varphi)),d(\rho'(\varphi),f\rho'(\varphi)),d(f\rho'(\varphi),f^2\rho'(\varphi))\}.$

Taking the supremum over φ on both sides gives

$d(f^2\rho, f^2\rho') \leq \beta\ max\{d(\rho,\rho'), d(f\rho, f\rho'), d(\rho, f\rho), d(f\rho, f^2\rho), d(\rho', f\rho'), d(f\rho', f^2\rho')\}$

where $\beta = a_1 + a_2 + b_1 + b_2 + c_1 + c_2 < 1$ for $(\rho, \rho') \in (A \times B)$. Hence the contravariant mapping f has a unique fixed point in $A \cup B$ according to Theorem 3.3. Thus the fractional differential equation (4.2.1) has a unique solution.

5. Conclusion

In conclusion, this study introduces the category of contravariant mappings into the pre-existing realm of Reich convex contraction mappings within bipolar metric spaces. The existence and uniqueness of these contraction mappings are established within the context of bipolar metric spaces. Utilizing the fixed point results of this new operator, solutions for integral and fractional differential equations are derived. Moreover, this outcome has the potential for extension to diverse abstract spaces and application in solving a range of differential equations beyond the scope of this research.

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