Unilag Journal of Mathematics and Applications,
Volume 5, Issue 2 (2025), Pages 65–79.
ISSN: 2805 3966. URL: http://lagjma.edu.ng

BI-G-STARLIKE FUNCTION ASSOCIATED WITH NORMALIZED ONE VARIABLE GEGENBAUER AND BELL-SHEFFER POLYNOMIALS INVOLVING GREGORY COEFFICIENTS

IBRAHIM TUNJI AWOLERE AND ADENIYI MUSIBAU GBOLAGADE*

Abstract. In this enquiry, applying normalized one variable Gegenbauer and Bell-Sheffer Polynomials, the authors introduce a new class of bi-G-starlike functions defined by Gregory and Caratheodory coefficients. Coefficient bounds for the new class were obtained and were consequently used to investigate the concept of Fekete-Szego functional and Toeplitz determinant in this direction.

1. Introduction

The branch of complex analysis that studies the multitude of geometric characteristics of analytic functions is called geometry function theory. It commenced in the mid-1900s, it has persisted as one of the most vibrant aspect of geometry that has been captivating critical development for research focuses. Louis de Branges answered the "Bieberbach conjecture" [16] in 1984, opening up new avenues and methods for the study of geometric function theory [14]. Finding fresh observational and theoretical insights in this subject with a range of applications is crucial for us. One of the most exquisite aspect of geometric function theory is the theory of univalent functions. Numerous fields, including contemporary mathematical physics, fluid dynamics, non-linear integrable system theory of partial differential equations, computer science, electrostatic potential, and heat conditions as well as statistics, are among those that continue to find novel uses for it [13]. Koebe was the first to introduce the notion of univalent function [25]. He started by applying the traditional Reimann mapping theory. Its original source is the Koebe paper.

A simply-connected domain that is analytic function g(z) is said to be univalent if it never takes the same value twice [18,33]. The most intriguing part of geometry is probably how it interacts with analysis. The concept of the theory of univalent

Key words and phrases. Analytic function, Combinaterial Analysis, Coefficient Estimate, FeketeSzego functional, Toeplitz determinant

2010, Mathematics Subject Classification: 30C45, 30C50.

Submitted: June 30, 2025. Revised: September 4, 2025. Accepted: November 18, 2025.

* Correspondence.

functions was introduced in early nineteen century and has found its use in solving a broad range of problems in hydrodynamic, aerodynamics, thermodynamics, electrodynamics, natural sciences and neural network.
Let the functions $g(z)$ be regular in the unit disk $D = \{z \in C : |z| < 1\}$ and the function is considered to have have a Maclaurin series expansion of the form

$$g(z) = b_0 + b_1 z + b_2 z^2 + b_3 z^3 + ... = \sum_{k=0}^{\infty} b_k z^k. \tag{1.1}$$

Normalizing equation (1.1) we have

$$g(z) - b_0 = b_1 z + b_2 z^2 + b_3 z^3 + ... = \sum_{k=1}^{\infty} b_k z^k. \tag{1.2}$$

Dividing eqn (1.2) by b_1, we get

$$\frac{g(z) - b_0}{b_1} = z + \frac{b_2}{b_1} z^2 + \frac{b_3}{b_1} z^3 + \frac{b_4}{b_1} z^4 + ... = \sum_{k=2}^{\infty} b_k z^k + ... = z + \sum_{k=2}^{\infty} \frac{b_k}{b_1} z^k. \tag{1.3}$$

If we let $f(z) = \frac{g(z)-b_0}{b_1}$, then

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in D \tag{1.4}$$

where

$$a_k = \frac{b_k}{b_1}, \quad for \;\; k = 2, 3, 4, , \quad D = \{z \in C : |z| < 1\} \;\; with \;\; f(0) = f'(0) - 1 = 0.$$

The classes of starlike, convex and close-to-convex functions consist of functions $f \in A$ with the geometric representation which are given in eqn (1.5) to (1.7) in the form

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \tag{1.5}$$

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \tag{1.6}$$

and

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0, \tag{1.7}$$

respectively, where g in eqn (1.7)denote the class of starlike function.
Just of recent, echelon of researchers [7, 8, 9, 23, 33] investigated the approach introduced by the author in [11] and this consequently improved the class of A. In particular, the class of $\lambda-$pseudo starlike functions that is denoted by $L_\lambda(\beta)$ has geometric representation of the form

$$Re\left(\frac{zf'(z)^\lambda}{f(z)}\right) > \beta \tag{1.8}$$

for $0 \leq \beta < 1$ and $\lambda \geq 1$ and λ is real.
The classes of $F-$ starlike and $F-$ convex functions represented by FS^* and FC, respectively, which are characterized by

$$Re\left(\frac{F(z)f'(z)}{f(z)}\right) > 0, \tag{1.9}$$

$$Re\left(1 + \frac{F(z)f''(z)}{f'(z)}\right) > 0 \tag{1.10}$$

respectively. The special case for the inequality in eqn (1.10) takes the form $f(0) = 0$ and this suggest that the real is greater than 0. For (9) and (10) see [1, 4, 5, 6, 29, 30].
Bi-univalent functions play crucial role in geometric function theory and these are in literature. On the other hand a function that is both onto(also known as surjective) and injective (sometime known as one-to-one) is called bijective. The discipline of complex analysis is where the idea of a "bi-univalent function" usually appears. A function that is both onto (also known as surjective) and injective (sometimes known as one-to-one) is called bi-univalent. Put more simply, it is a function that maps each input value to a distinct output value, with a corresponding input value for each output value. Mbius transformation is among the most well-known transformation that supports the concept of bi-univalent function. The evolution of geometric characteristic as a whole support the study of the functions of complex variables which invariably are closely related to the concept of bi-univalent functions. The special properties that are observed injectivity and subjectivity by several researchers have huge importance in varieties of mathematical settings such as mapping of a real in the complex plane, conformal mapping and it is application in physics and engineering. Furthermore, univalent function f along with its inverse f^{-1} takes the form

$$f^{-1}(f(z)) = z \quad (z \in D)$$

and

$$f^{-1}(f(w)) = w, \quad (|w| < r(f); r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 + ... \tag{1.11}$$

A function $f \in A$ is said to be bi-univalent in D if both $f(z)$ and $f^{-1}(z)$ are univalent in D and it is represented by $\sum$. The authors in [24] and [15] studied the concept of bi-univalent functions from interesting perspectives and they established useful sharp estimates. The authors in [32] also introduced and investigated from extended from extended perspective two classes of bi-univalent and obtained useful result. Consequently , different authors [6-10, 20, 21, 22, 25] have investigated and examined the various classes of bi-univalent functions.
Furthermore, in the recent past some authors looked into new sets of Sheffer and Brenke polynomials based on higher order Bell numbers with special consideration for exponential and logarithm polynomials for exponential and logarithm polynomials were investigated and special cases were pointed out. For more on

Bell-Sheffer polynomial sets see [17, 26, 27, 28, 31]. Bell polynomials and Sheffer sequences have diverse applications across various branches of mathematics, including combinatorics, probability theory, mathematical physics, and control theory. They provide powerful tools for analyzing and solving problems in these areas

Of particular interest to the present research is the recent work of Natalini and Ricci [28] where the authors considered the Sheffer polynomials defined through their generating functions as follows

$$A(z)E_r(z)+1=e^{E_{r-1}(z)},\quad H(t)=E_{r-1}(z)$$

$$G(z,x)=exp[(1+x)E_{r-1}(z)]=\sum_{k=0}^{\infty}\chi_k^{(r)}(x)\frac{z}{k!} \tag{1.12}$$

where $x\in N\cup\{0\}, r\in N=\{1,2,3,...\}$.

We observed that Bell-Sheffer polynomial sets denoted by $G(z,x)$ does not belong to the family of analytic univalent functions. Thus, we provide normalization using convolution (Hadamard) principle as

$$\Phi(z)=f(z)*G(z,x)=f(z)*exp[(1+x)E_{r-1}(z)]=z+\sum_{k=2}^{\infty}\frac{\chi_{k-1}^{(r)}(x)}{(k-1)!}a_kz^k, \tag{1.13}$$

which means

$$\Phi(z)=z+\sum_{k=2}^{\infty}\frac{\chi_{k-1}^{(r)}(x)}{(k-1)!}a_kz^k.$$

Netalini and Ricci [28] showed the first few values of Bell-Sheffer polynomials $\chi_k^{(1)}(x)$, for $r=1$ is defined by generating function $(7)_2$ as follows

$$\chi_0^{(1)}(x)=1,$$
$$\chi_1^{(1)}(x)=x+1,$$
$$\chi_2^{(1)}(x)=x^2+3x+2$$
$$\chi_3^{(1)}(x)=x^3+6x^2+10x+5$$
$$\chi_4^{(1)}(x)=x^4+10x^3+31x^2+37x+15$$
$$\chi_5^{(1)}(x)=x^5+15x^4+75x^3+160x^2+151x+52$$

and they also show the first few values for Bell-Sheffer polynomials $\chi_k^{(2)}(x)$, defined by the generating function $(7)_2$

$$\chi_0^{(2)}(x)=1,$$
$$\chi_1^{(2)}(x)=x+1,$$
$$\chi_2^{(2)}(x)=x^2+4x+3,$$
$$\chi_3^{(2)}(x)=x^3+9x^2+20x+12,$$
$$\chi_4^{(2)}(x)=x^4+16x^3+74x^2+119x+60,$$
$$\chi_5^{(2)}(x)=x^5+25x^4+200x^3+635x^2+817x+358.$$

Let p represent the class of the Caratheodory functions with the representation

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k, \tag{1.14}$$

with the conditions $Rep(z) > 0$ and $p(0) = 1$.
The functions which has the representation

$$v(z) = \frac{z}{log(1+z)} = \sum_{k=0}^{\infty} \Delta_k z^k = 1 + \frac{1}{2}z - \frac{1}{12}z^2 + \frac{1}{24}z^3 - \frac{109}{720}z^4 + ... \tag{1.15}$$

referred to as Gregory coefficient and for more information, the reader should see [25]. Gregory coefficients was named after James Gregory in 1670 in numerical integration context also refer to reciprocal logarithmic numbers, Bernoulli numbers of second kind , or Cauchy numbers of the first kind are decrease rational numbers $\frac{1}{2},\ -\frac{1}{12},\ \frac{1}{24},\ -\frac{109}{720}, ...$ which occur in expansion of the reciprocal logarithm.
Gegenbauer polynomials or ultraspherical polynomials $C_k^{(\alpha)}(x)$ are orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight function $(1-x^2)^{\alpha-\frac{1}{2}}$. The generalize Legendre polynomials and Chebyshev polynomials, and are special cases of Jacobi polynomials. Amourah [2] applied Gegenbauer polynomial in the direction of geometric property that align with caratheodory functions in order to investigate coefficient bounds. Gegenbauer polynomials have diverse applications across various branches of mathematics and its applications, including approximation theory, mathematical physics, numerical analysis, signal processing, and statistics. They provide valuable tools for solving problems in these areas and are widely used in theoretical and applied research. The authors in [3, 12] investigated Gegenbauer polynomial involving some characterization from interesting directions. Further, a normalized one variable Gegenbauer polynomial [30] is of the form

$$G(z, m) = z + \sum_{k=2}^{\infty} C_{k-1}^{\gamma}(m) z^k, \tag{1.16}$$

where $\gamma > -0.5$. The first k coefficients of the representations are

$$C_0^{\gamma}(m) = 1,$$

$$C_1^{\gamma}(m) = 2\gamma m,$$

$$C_2^{\gamma}(m) = 2\gamma(\gamma+1)m^2 - \gamma, ,$$

$$C_3^{\gamma}(m) = \frac{4\gamma(\gamma+1)(\gamma+2)}{3} - 2\gamma(\gamma+1)m$$

and

$$C_k^{\gamma}(m) = \frac{2m(k+\gamma-1)C_{k-1}^{\gamma}(m) - (k+2\gamma-2)C_{k-2}^{\gamma}(m)}{k}. \tag{1.17}$$

Equation (16) can also be obtained from integral representation of the form

$$\varrho(z) = \int_{-1}^{1} G(z, m) d\mu(m),$$

where

$$G(z,m)=\frac{z}{(1-2mz+z^2)^{\gamma}} \tag{1.18}$$

and here μ is a probability measure on the interval $[-1,1]$. We notice that by $\gamma=1$ in (1.18) we have the popular Chebyshev polynomial while setting $\gamma=0.5$, (1.18) will produce the Legendre polynomial.
Now, by (11) and (16) we have the form

$$g^{-1}(m,\ w)=w-[C_1^{\gamma}(m)]w^2+(2[C_1^{\gamma}(m)]^2-[C_2^{\gamma}(m)])w^3+... \tag{1.19}$$

In the same vain, by (11) and (13) we conclude that

$$g^{-1}(w)=w-[X_1^{(r)}(x)]a_2w^2+\left(2a_2^2-a_3\right)\frac{[X_2^{(r)}(x)]}{2}w^3-\left(5a_2^3-5a_2a_3+a_4\right)\frac{[X_3^{(r)}(x)]}{6}w^4+.... \tag{1.20}$$

Not too long ago, Olatunji et al.[30] investigated G-starlike functions by using the convolution of error and generalized distribution functions defined in terms of subordination associated with Carathodory, Bell numbers, and modified sigmoid functions for a specific class of functions. Their work aimed to find the early coefficient bounds and establish the usual FeketeSzeg inequalities. Therefore, our motivation is derived from their research to explore the possibility of finding the inverse of G-starlike functions defined by the Bell-Sherffer polynomial subordinate to Carathodory and Gregory coefficients.
Assuming $g(z)$ is analytic and univalent in D. If $f(z)$ is analytic in D, $f(0)=g(0)$, and $f(D)\subset g(D)$, then we observe that $f(z)$ is subordinate $g(z)$ which is write as $f(z)\succ g(z)$. For details see [19].
Definition 1.1: A function $f\in A$ is said to be in the class $GS^*_{\sum}(m,\ x,\ r,\ \gamma)$ if the subordination holds:

$$\frac{G(z,m)\Phi'(z)}{\Phi(z)}\prec\sqrt{1+z} \tag{1.21}$$

and

$$\frac{g^{-1}(w,m)(g^{-1}(w))'}{g^{-1}(w)}\prec\sqrt{1+w}. \tag{1.22}$$

Definition 1.2: A function $f\in A$ is said to be in the class $GS^*_{\sum}(m,\ x,\ r,\ \gamma)$ if the subordination holds:

$$\frac{G(z,m)\Phi'(z)}{\Phi(z)}\prec\sqrt{\nu(z)} \tag{1.23}$$

and

$$\frac{g^{-1}(w,m)(g^{-1}(w))'}{g^{-1}(w)}\prec\sqrt{\nu(w)} \tag{1.24}$$

for $m\in[-1,1]$, $x\in NU\{0\},\ r=\{1,2,3,...\},\ \gamma\geq 1$ with condition $G(0)=0$.

2. Result

The main focus of this paper is to investigate the coefficient problems for the class of bi-univalent function $\Theta S^m_{\sum}(r,x,\gamma)$ defined in this study. The coefficient estimates are obtained using the Carathodory function p(z) defined by (1.14), the Gregory coefficient (15) and Bell-Sherffer polynomials generated by the function given in (21) involving functions associated with Gegenbauer polynomials.

Theorem 2.1. Let $f(z)$ be form (1.1) and $p(z)$ defined by (1.14). Then $f \in \Theta S^m_{\sum}(r,x,\gamma)$, if

$$|a_2| \leq \left| \frac{-12[X_1^{(r)}(x)][C_1^\gamma(m)] \pm \sqrt{144[X_1^{(r)}(x)]^2[C_1^\gamma(m)]^2 - M_1}}{2\left\{2[X_2^{(r)}(x)] + 3[X_1^{(r)}(x)]^2\right\}} \right| \tag{2.1}$$

$$|a_3| \leq \left| \frac{M_2 + 2[X_1^{(r)}(x)]^2\left\{8[C_1^\gamma(m)]^2 - 8[C_2^\gamma(m)] + m_2 - v_2\right\}}{16[X_1^{(r)}(x)]^2[X_2^{(r)}(x)]} \right| \tag{2.2}$$

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{1}{16[X_1^{(r)}(x)]^2[C_2^{(r)}(x)]}\left|M_3 - \sigma\left\{m_1^2 + 16[C_1^\gamma(m)]^2 - 8[C_1^\gamma(m)]m_1\right\}\right| \tag{2.3}$$

where

$$M1 = \left\{2[X_2^{(r)}(x)] + 3[X_1^{(r)}(x)]^2\right\}\left\{28[C_1^\gamma(m)]^2 - m_2 - v_2\right\},$$

$$M_2 = \left\{m_1^2 + 16[C_1^\gamma(m)]^2 - 8[C_1^\gamma(m)]m_1\right\}[X_2^{(r)}(x)],$$

$$M_3 = M_2 + 2[X_1^{(r)}(x)]^2\left\{8[C_1^\gamma(m)]^2 - 8[C_2^\gamma(m)] + m_2 - v_2\right\}.$$

Proof. In view of (1.21) and (1.22) and that $G(m,z)$ is symmetric and commutative with $G(z,m)$ it shows that

$$\frac{G(m,\ z)\Phi'(z)}{\Phi(z)} = \sqrt{1+v(z)} \tag{2.4}$$

and

$$\frac{g^{-1}(w,m)(g^{-1}(w))'}{g^{-1}(w)} = \sqrt{1+\varphi(w)}. \tag{2.5}$$

Thus

$$\frac{G(m,\ z)\Phi'(z)}{\Phi(z)} = 1 + \left\{[C_1^\gamma(m)] + [X_1^{(r)}(x)]a_2\right\}z$$

$$+\left\{[C_2^\gamma(m)] + [C_1^\gamma(m)][X_1^{(r)}(x)]a_2 + [X_2^{(r)}(x)]a_3 - [X_1^{(r)}(x)]^2a_2^2\right\}z^2 + ... \tag{2.6}$$

and

$$\frac{g^{-1}(w,m)(g^{-1}(w))'}{g^{-1}(w)} = 1 - (([C_1^\gamma(m)] + [X_1^{(r)}(x)]a_2)w +$$

$$+\left\{(2[X_2^{(r)}(x)] - [X_1^{(r)}(x)]^2)a_2^2 - [X_2^{(r)}(x)]a_3 + [X_1^{(r)}(x)][C_2^\gamma(m)]a_2 + 2[C_2^\gamma(m)]^2 - [C_1^\gamma(m)]\right\}w^2 + ... \tag{2.7}$$

We notice that

$$\sqrt{1 + \frac{m(z)-1}{m(z)+1}} = 1 + \frac{m_1}{4}z + \left(\frac{m_2}{4} - \frac{5m_1^2}{32}\right)z^2 + ... \tag{2.8}$$

and

$$\sqrt{1+\frac{v(w)-1}{v(w)+1}} = 1+\frac{v_1}{4}z+\left(\frac{v_2}{4}-\frac{5v_1^2}{32}\right)w^2+.... \tag{2.9}$$

We can write from (2.6)-(2.9) that

$$[C_1^\gamma(m)]+[X_1^{(r)}(x)]a_2 = \frac{m_1}{4} \tag{2.10}$$

$$[C_2^\gamma(m)]+[C_1^\gamma(m)][X_1^{(r)}(x)]a_2+[X_2^{(r)}(x)]a_3-[X_1^{(r)}(x)]^2a_2^2 = \frac{m_2}{4}-\frac{5m_1^2}{32} \tag{2.11}$$

$$-([C_1^\gamma(m)]+[X_1^{(r)}(x)]a_2) = \frac{v_1}{4} \tag{2.12}$$

$$(2[X_2^{(r)}(x)]-[X_1^{(r)}(x)]^2)a_2^2-[X_2^{(r)}(x)]a_3+[X_1^{(r)}(x)][C_2^\gamma(m)]a_2+2[C_1^\gamma(m)]^2-[C_2^\gamma(m)] = \frac{v_2}{4}-\frac{5v_1^2}{32}. \tag{2.13}$$

In the light of (2.10) and (2.12)

$$m_1^2-v_1^2=0, \quad m_1^3+v_1^3=0 \tag{2.14}$$

and

$$32([C_1^\gamma(m)]a_2+[X_1^{(r)}(x)])^2 = m_1^2+v_1^2. \tag{2.15}$$

Combination of (2.11) and (2.13) produces

$$(2[X_2^{(r)}(x)]-2[X_1^{(r)}(x)]^2)a_2^2+2[X_1^{(r)}(x)][C_1^\gamma(m)]a_2+2[C_1^\gamma(m)]^2 = \frac{1}{4}[m_2+v_2]-\frac{5}{32}[m_1^2+v_1^2]. \tag{2.16}$$

Application of (2.14) and (2.15) on (2.16) together with simple computation results

$$4(2[X_2^{(r)}(x)]+3[X_1^{(r)}(x)]^2)a_2^2+48[X_1^{(r)}(x)][C_1^\gamma(m)]a_2+(28[C_1^\gamma(m)]^2-m_2-v_2)=0. \tag{2.17}$$

Using appropriate mathematical method on (2.17) yields the desire result.
Further, in view of (2.11) and (2.13)

$$2[X_2^{(r)}(x)]a_3 = 2[X_2^{(r)}(x)]a_2^2++2[C_1^\gamma(m)]^2-2[C_2^\gamma(m)] + \frac{1}{4}(m_2-v_2)-\frac{5}{32}(m_1^2-v_1^2). \tag{2.18}$$

Hence, by (2.10) together with simple computation on the above equation produces the desire result.
Next, for $f \in \Theta_{\sum}^m(r,x,\gamma)$ and $\eta \in R$, then

$$a_3-\eta a_2^2 = \frac{M_3}{16[X_1^{(r)}(x)]^2[X_2^{(r)}(x)]} - \frac{\sigma\left\{m_1^2+16[C_1^\gamma(m)]^2-8[C_1^\gamma(m)]m_1\right\}}{16[X_1^{(r)}(x)]^2}$$

which yields desired result after application of triangular inequality together with simple mathematical computation.

Theorem 2.2: Let $f(z)$ be form (1.1) and $p(z)$ defined by (1.15). Then $f \in \Theta S^m_{\sum}(r, x, \gamma)$, if

$$\left|a_3^2 - a_2^2\right| \le \left|\frac{M_3^2 - 16[X_1^{(r)}(x)]^2[X_2^{(r)}(x)]^2\left\{m_1^2 + 16[C_1^\gamma(m)]^2 - 8[C_1^\gamma(m)]m_1\right\}}{256[X_1^{(r)}(x)]^4[X_2^{(r)}(x)]^2}\right| \tag{2.19}$$

Proof. The proof follows from the the proof of Theorem 1.1 and we omit the detail.

The next two theorems are concerned with the investigation of certain coefficient problems for the class of functions $\Theta S^m_{\sum}(r, x, \gamma)$ involving the Gregory coefficient function given by (1.15).

Theorem 2.3: Let $f(z)$ be form (1.1) and $v(z)$ defined by (1.15). Then $f \in \Theta S^m_{\sum}(r, x, \gamma)$, if

$$|a_2| \le \left|\frac{-136[X_1^{(r)}(x)][C_1^\gamma(m)] \pm \sqrt{18456[X_1^{(r)}(x)]^2[C_1^\gamma(m)]^2 - 2Q_1}}{4\left\{6[X_2^{(r)}(x)] + 25[X_1^{(r)}(x)]^2\right\}}\right| \tag{2.20}$$

$$|a_3| \le \left|\frac{Q_2 + 4[X_1^{(r)}(x)]^2\left\{16[C_1^\gamma(m)]^2 - 16[C_2^\gamma(m)] + c_2 - n_2\right\}}{64[X_1^{(r)}(x)]^2[X_2^{(r)}(x)]}\right| \tag{2.21}$$

$$\left|a_3 - \eta a_2^2\right| \le \frac{1}{64[X_2^{(r)}(x)][X_1^{(r)}(x)]^2}\left|Q_3 - \eta[X_2^{(r)}(x)]\left\{c_1^2 + 64[C_1^\gamma(m)]^2 - 16[C_1^\gamma(m)]c_1\right\}\right| \tag{2.22}$$

where

$$Q1 = \left\{6[X_2^{(r)}(x)] + 25[X_1^{(r)}(x)]^2\right\}\left\{272[C_1^\gamma(m)]^2 - 3c_2 - 3v_2\right\}$$

$$Q_2 = [X_2^{(r)}(x)]\left\{c_1^2 + 64[C_1^\gamma(m)]^2 - 16[C_1^\gamma(m)c_1]\right\},$$

$$Q_3 = Q_2 + 4[X_1^{(r)}(x)]^2\left\{16[C_1^\gamma(m)]^2 - 16[C_2^\gamma(m)] + c_2 - n_2\right\}.$$

Proof. In view of (1.23) and (1.24) we conclude that

$$\frac{G(m,\ z)\Phi'(z)}{\Phi(z)} = \sqrt{\nu(z)} \tag{2.23}$$

and

$$\frac{g^{-1}(w, m)(g^{-1}(w))'}{g^{-1}(w)} = \sqrt{\nu w)}. \tag{2.24}$$

We notice that

$$\sqrt{1 + \frac{1}{2}\left(\frac{c(z)-1}{c(z)+1}\right) - \frac{1}{12}\left(\frac{c(z)-1}{c(z)+1}\right)^2 + \frac{1}{24}\left(\frac{c(z)-1}{c(z)+1}\right)^3 - ..} = 1 + \frac{c_1}{8}z + \left(\frac{c_2}{8} - \frac{31c_1^2}{384}\right)z^2 + ... \tag{2.25}$$

and

$$\sqrt{1+\frac{1}{2}\left(\frac{n(w)-1}{n(z)+1}\right)-\frac{1}{12}\left(\frac{n(w)-1}{n(w)+1}\right)^2+\frac{1}{24}\left(\frac{n(w)-1}{n(w)+1}\right)^3-..}=1+\frac{n_1}{8}w+\left(\frac{n_2}{8}-\frac{31n_1^2}{384}\right)w^2+... \tag{2.26}$$

We can write from (2.8), (2.9), (2.25) and (2.26) that

$$[C_1^\gamma(m)]+[X_1^{(r)}(x)]a_2=\frac{c_1}{8} \tag{2.27}$$

$$[C_1^\gamma(m)]+[C_1^\gamma(m)][X_1^{(r)}(x)]a_2+\frac{1}{2}[X_2^{(r)}(x)]a_3-[X_1^{(r)}(x)]^2a_2^2=\frac{c_2}{8}-\frac{31c_1^2}{384} \tag{2.28}$$

$$-([C_1^\gamma(m)]a_2+[X_2^{(r)}(x)]=\frac{n_1}{8} \tag{2.29}$$

$$(2[X_2^{(r)}(x)]-[X_1^{(r)}(x)]^2)a_2^2-[X_2^{(r)}(x)]a_3+3[X_1^{(r)}(x)][C_2^\gamma(m)]a_2+2[C_2^\gamma(m)]^2-[C_1^\gamma(m)]$$
$$=\frac{n_2}{8}-\frac{31n_1^2}{384}. \tag{2.30}$$

We omit remaining part of proof since its similar to rest of Theorem 1.

Theorem 2.4: Let $f\in\Theta S^m_{\sum}(r,x,\gamma)$ and $v(z)$ defined by (1.15). Then

$$\left|a_3^2-a_2^2\right|\leq\frac{1}{4096[X_2^{(r)}(x)]^4[X_1^{(r)}(x)]^2}\left|Q_3^2-64[X_2^{(r)}(x)]^2[X_2^{(r)}(x)]^2Q_4\right|, \tag{2.31}$$

where

$$Q_4=\left\{c_1^2+64[C_1^\gamma(m)]^2-16[C_1^\gamma(m)]c_1\right\}.$$

Proof. The proof is similar to that of Theorem 2 and we omit the detail.
Remark: The correct implementation

3. Discussion

Varying parameters $\gamma,\ r,\ x$ known and new results will be arrived at.

Corollary 3.1. Let $f(z)$ be form (1.1) and $p(z)$ defined by (1.14). Then $f\in\Theta S^m_{\sum}(1,x,\gamma)$, if

$$|a_2|\leq\left|\frac{-12[x+1][C_1^\gamma(m)]\pm\sqrt{144[x^2+2x+1]^2[C_1^\gamma(m)]^2-M_4}}{2\left\{5x^2+12x+7\right\}}\right| \tag{3.1}$$

$$|a_3|\leq\left|\frac{M_5+2(x^2+2x+1)\left\{8[C_1^\gamma(m)]^2-8[C_2^\gamma(m)]+m_2-v_2\right\}}{16(x^2+2x+1)(x^2+3x+2)}\right| \tag{3.2}$$

$$\left|a_3-\sigma a_2^2\right|\leq\frac{1}{16(x^2+2x+1)(x^2+3x+2)}\left|M_6-\sigma\left\{m_1^2+16[C_1^\gamma(m)]^2-8[C_1^\gamma(m)]m_1\right\}\right| \tag{3.3}$$

where

$$M_4=\left\{5x^2+12x+7\right\}\left\{28[C_1^\gamma(m)]^2-m_2-v_2\right\},$$
$$M_5=\left\{m_1^2+16[C_1^\gamma(m)]^2-8[C_1^\gamma(m)]m_1\right\}(x^2+3x+2),$$
$$M_6=M_5+2(x^2+2x+1)\left\{8[C_1^\gamma(m)]^2-8[C_2^\gamma(m)]+m_2-v_2\right\}.$$

Corollary 3.2. Let $f(z)$ be form (1.1) and $p(z)$ defined by (1.14). Then $f \in \Theta S^m_{\sum}(2,x,\gamma)$, if

$$|a_2| \leq \left| \frac{-12[x+1][C_1^\gamma(m)] \pm \sqrt{144[x^2+2x+1]^2[C_1^\gamma(m)]^2 - M_7}}{2\left\{5x^2+18x+13\right\}} \right| \tag{3.4}$$

$$|a_3| \leq \left| \frac{M_5 + 2(x^2+2x+1)\left\{8[C_1^\gamma(m)]^2 - 8[C_2^\gamma(m)] + m_2 - v_2\right\}}{16(x^2+2x+1)(x^2+3x+2)} \right| \tag{3.5}$$

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{1}{16(x^2+2x+1)(x^2+3x+2)} \left|M_6 - \sigma\left\{m_1^2 + 16[C_1^\gamma(m)]^2 - 8[C_1^\gamma(m)]m_1\right\}\right| \tag{3.6}$$

where

$$M_7 = \left\{5x^2+18x+13\right\}\left\{28[C_1^\gamma(m)]^2 - m_2 - v_2\right\},$$

$$M_8 = \left\{m_1^2 + 16[C_1^\gamma(m)]^2 - 8[C_1^\gamma(m)]m_1\right\}(x^2+4x+3),$$

$$M_9 = M_8 + 2(x^2+2x+1)\left\{8[C_1^\gamma(m)]^2 - 8[C_2^\gamma(m)] + m_2 - v_2\right\}.$$

Corollary 3.3. Let $f(z)$ be form (1.1) and $p(z)$ defined by (1.14). Then $f \in \Theta S^m_{\sum}(1,x,1)$, if

$$|a_2| \leq \left| \frac{-24m^2[x+1] \pm \sqrt{576m^2[x^2+2x+1] - M_{10}}}{2\left\{5x^2+12x+7\right\}} \right| \tag{3.7}$$

$$|a_3| \leq \left| \frac{M_{11} + 2(x^2+2x+1)\left\{8 + m_2 - v_2\right\}}{16(x^2+2x+1)(x^2+3x+2)} \right| \tag{3.8}$$

$$\left|a_3 - \sigma a_2^2\right| \leq \frac{1}{16(x^2+2x+1)(x^2+3x+2)} \left|M_{12} - \sigma\left\{m_1^2 + 64m^2 - 16mm_1\right\}\right| \tag{3.9}$$

where

$$M_{10} = \left\{5x^2+12x+7\right\}\left\{112m^2 - m_2 - v_2\right\},$$

$$M_{11} = \left\{m_1^2 + 64m^2 - 16mm_1\right\}(x^2+3x+2),$$

$$M_{12} = M_{11} + 2(x^2+2x+1)\left\{8 + m_2 - v_2\right\}.$$

Corollary 3.4: Let $f(z)$ be form (1.1) and $p(z)$ defined by (1.14). Then $f \in \Theta S^m_{\sum}(1,x,\gamma)$, if

$$\left|a_3^2 - a_2^2\right| \leq \left| \frac{M_6^2 - 16[x+1]^2[x^2+3x+2]^2\left\{m_1^2 + 16[C_1^\gamma(m)]^2 - 8[C_1^\gamma(m)]m_1\right\}}{256[x+1]^4[x^2+3x+2]^2} \right| \tag{3.10}$$

Corollary 3.5: Let $f(z)$ be form (1.1) and $p(z)$ defined by (1.14). Then $f \in \Theta S^m_{\sum}(2,x,\gamma)$, if

$$\left|a_3^2 - a_2^2\right| \leq \left| \frac{M_9^2 - 16[x+1]^2[x^2+3x+2]^2\left\{m_1^2 + 16[C_1^\gamma(m)]^2 - 8[C_1^\gamma(m)]m_1\right\}}{256[x+1]^4[x^2+3x+2]^2} \right| \tag{3.11}$$

Corollary 3.6: Let $f(z)$ be form (1.1) and $p(z)$ defined by (1.14). Then $f \in \Theta S^m_{\sum}(2, x, 1)$, if

$$\left|a_3^2 - a_2^2\right| \leq \left|\frac{M_{12}^2 - 16[x+1]^2[x^2+3x+2]^2 \{m_1^2 + 64m^2 - 16mm_1\}}{256[x+1]^4[x^2+3x+2]^2}\right| \tag{3.12}$$

Corollary 3.7: Let $f(z)$ be form (1.1) and $v(z)$ defined by (1.15). Then $f \in \Theta S^m_{\sum}(1, x, \gamma)$, if

$$|a_2| \leq \left|\frac{-136[x+1][C_1^\gamma(m)] \pm \sqrt{18456[x+1]^2[C_1^\gamma(m)]^2 - 2Q_4}}{4\{6[x^2+3x+2] + 25[x+1]^2\}}\right| \tag{3.13}$$

$$|a_3| \leq \left|\frac{Q_5 + 4[x+1]^2 \{16[C_1^\gamma(m)]^2 - 16[C_2^\gamma(m)] + c_2 - n_2\}}{64[x+1]^2[x^2+3x+2]}\right| \tag{3.14}$$

$$\left|a_3 - \eta a_2^2\right| \leq \frac{1}{64[x^2+3x+2][x+1]^2}\left|Q_6 - \eta[X_2^{(r)}(x)]\left\{c_1^2 + 64[C_1^\gamma(m)]^2 - 16[C_1^\gamma(m)]c_1\right\}\right| \tag{3.15}$$

where

$$Q_4 = \left\{6[x^2+3x+2] + 25[x+1]^2\right\}\left\{272[C_1^\gamma(m)]^2 - 3c_2 - 3v_2\right\}$$
$$Q_5 = [x^2+3x+2]\left\{c_1^2 + 64[C_1^\gamma(m)]^2 - 16[C_1^\gamma(m)c_1]\right\},$$
$$Q_6 = Q_5 + 4[x+1]^2\left\{16[C_1^\gamma(m)]^2 - 16[C_2^\gamma(m)] + c_2 - n_2\right\}.$$

Corollary 3.8: Let $f(z)$ be form (1.1) and $v(z)$ defined by (1.15). Then $f \in \Theta S^m_{\sum}(2, x, \gamma)$, if

$$|a_2| \leq \left|\frac{-136[x+1][C_1^\gamma(m)] \pm \sqrt{18456[x+1]^2[C_1^\gamma(m)]^2 - 2Q_7}}{4\{6[x^2+4x+3] + 25[x+1]^2\}}\right| \tag{3.16}$$

$$|a_3| \leq \left|\frac{Q_8 + 4[x+1]^2 \{16[C_1^\gamma(m)]^2 - 16[C_2^\gamma(m)] + c_2 - n_2\}}{64[x+1]^2[x^2+4x+3]}\right| \tag{3.17}$$

$$\left|a_3 - \eta a_2^2\right| \leq \frac{1}{64[x^2+4x+3][x+1]^2}\left|Q_9 - \eta[X_2^{(r)}(x)]\left\{c_1^2 + 64[C_1^\gamma(m)]^2 - 16[C_1^\gamma(m)]c_1\right\}\right| \tag{3.18}$$

where

$$Q_7 = \left\{6[x^2+4x+3] + 25[x+1]^2\right\}\left\{272[C_1^\gamma(m)]^2 - 3c_2 - 3v_2\right\}$$
$$Q_8 = [x^2+4x+3]\left\{c_1^2 + 64[C_1^\gamma(m)]^2 - 16[C_1^\gamma(m)c_1]\right\},$$
$$Q_9 = Q_8 + 4[x+1]^2\left\{16[C_1^\gamma(m)]^2 - 16[C_2^\gamma(m)] + c_2 - n_2\right\}.$$

Corollary 3.9: Let $f(z)$ be form (1.1) and $v(z)$ defined by (1.15). Then $f \in \Theta S^m_{\sum}(1, x, 1)$, if

$$|a_2| \leq \left|\frac{-272m[x+1] \pm \sqrt{73824m^2[x+1]^2 - 2Q_{10}}}{4\{6[x^2+3x+2] + 25[x+1]^2\}}\right| \tag{3.19}$$

$$|a_3| \leq \left|\frac{Q_{11} + 4[x+1]^2 \{16 + c_2 - n_2\}}{64[x+1]^2[x^2+3x+2]}\right| \tag{3.20}$$

$$\left|a_3 - \eta a_2^2\right| \leq \frac{1}{64[x^2+4x+3][x+1]^2}\left|Q_{11} - \eta[x^2+4x+3]\left\{c_1^2 + 256m^2 - 32mc_1\right\}\right| \tag{3.21}$$

where

$$Q_{10} = \left\{6[x^2+3x+2] + 25[x+1]^2\right\}\left\{1088m^2 - 3c_2 - 3v_2\right\}$$
$$Q_{11} = [x^2+3x+2]\left\{c_1^2 + 256m^2 - 32mc_1\right\},$$
$$Q_{12} = Q_{11} + 4[x+1]^2\left\{16 + c_2 - n_2\right\}.$$

4. Conclusion

We conducted a comprehensive analysis of normalized one variable Gegenbauer and Bell-Sheffer Polynomials by developing a new class of bi-G-starlike functions defined by Gregory and Caratheodory coefficients. Various properties such as coefficient bounds, Fekete-Szego functional as well as Toeplitz determinant were established and characterized by exploring different parameter choices. Our results improved existing results in literatures.

Acknowledgment. The authors deeply appreciated the efforts of the reviewers and the contributions of everyone that made this article worthwhile.

Authors Contributions. Not applicable.

Authors' Conflicts of interest. The authors declare no conflict of interest.

Funding Statement. The research received no internal and external funding.

References

[1] R. Aghalary, and P. Arjomandinia. *On a first order strong differential subordination and application to univalent functions. Commun.* Korean Math. Soc. 2022, 37, 445454.

[2] A.Amourah, B.A. Frasin, T. Abdeljawad. *Fekete-Szeg inequality for analytic and biunivalent functions subordinate to Gegenbauer polynomials.* J. Funct. Spaces 2021, 2021, 5574673.

[3] A. Amourah, M. Alomari, F. Yousef, A.Alsoboh. *Consolidation of a Certain Discrete Probability Distribution with a Subclass of Bi-Univalent Functions Involving Gegenbauer Polynomials.* Math. Probl. Eng. 2022, 2022, 6354994.

[4] J.A Antonino. *Strong differential subordination and applications to univalency conditions.* J. Korean Math. Soc. 2006, 43, 311322.

[5] J.A. Antonino, and S. Romaguera. *Strong differential subordination to Briot-Bouquet differential equations.* J. Diff. Equ. 1994, 114, 101105.

[6] I.T Awolere. *Hankel determinant for bi-Bazelevic function involing error and Sigmoid function defined by derivative calculus via Chebyshev polynomials.* Journal of Fractional Calculus and Applications, Vol. 11(2) July 2020, PP-208-217.

[7] I.T. Awolere, S. Ibrahim-Tiamiyu. *New classes of bi-univalent pseudo Starlike function using Alhindi-Darus generalized hypergeometric function.* J. Nigerian Assoc. Math. Phys. (2017) 40 65(72).

[8] I. T. Awolere and A.T.Oladipo. *Determinant for bi-univalent of pseudo starlikeness for certain class of analytic univalent functions.* Libertas Mathematica (new series), Volume 39, (2019) No. 2, 27-43.

[9] I.T. Awolere and A.T.Oladipo. *Coefficients of bi-univalent functions involving psedo-starlikeness associated with Chebyshev polynomials.* Khayyam J.Math.5, (2019) no 1, 140-149.

[10] I.T. Awolere, A.T. Oladipo and S. Altinkaya. *Application of Gegenbauer Polynomials with two variables to bi-unvalency of generalized discrete probability distribution via zero-truncated Poisson distribution series*. Sahand Communicationin Mathematical Analysis(SCMA), (2024) Vol. 21 No. 3, 65-88.
[11] K.O. Babalola. *On pseudo-starlike functions*. J. Class. Anal. 2013, 3,137147.
[12] H. Bavinck,G. Hooghiemstra, E. De Waard, *An application of Gegenbauer polynomials in queueing theory*. Int. J. Comput. Appl. Math. 1993, 49, 110.
[13] E. D. Belokolos, A. I Bobenko, V.Z, Enolskii and A.R. *Algebro Geometric Approach to non-linear integrable equations* New York Springer.
[14] L. De Branges. *A proof of the Bieberbach conjecture*. Acta Math. 1985, 154, 137152.
[15] Alexander, J.W. *Functions which map the interior of the unit circle upon simple region*. Ann. Math. 1915, 17, 1222.
[16] D.A. Brannan and S.T. Taha. *On some classes of bi-univalent functions*. Studia Univ. Babes-Bolyai Math. 31(2) (1986), 70-77
[17] L. Bieberbach. *ber die koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln*. Sitzungsber. Preuss. Akad. Wiss. Phys.-Math. Kl. 1916, 138, 940955.
[18] G. Bretti, P. Natalini, P.F. Ricci, *A new set of Sheffer-Bell polynomials and logarithm numbers*. Georgian Math. J. 2018, in print
[19] P. L. Duren. *Univalent functions*. Volume 259 of Grundlehren der mathematischen Wissenschaften. Springer Verlag, New York, NY.
[20] B. A. Frasin. *Suubodination results for certain class of analytic functions defined by a linear operator*. Journal of Inequalities Pure Applied Mathematics, Volume 7, Issue 4 (2006), Article 134, 1-7.
[21] A.M. Gbolagade and I.T. Awolere. *Coefficient Estimates of Certain Subclasses of bi-Bazilevic Functions Associated with Chebyshev Polynomials and Mittag-Leffler Function*. Earthline Journal of Mathematical Sciences, (2021) Vol. 5, No. 2: 365 376.
[22] A. M. Gbolagade and I. T. Awolere. *Generalized distribution for bi-univalent functions defined by error and Poisson distribution via Bell number*. COAST Journal of the School of Science, (2024) Vol. 6(2): 1120-1128
[23] A. M. Gbolagade, T. M. Asiru, and I. T. Awolere. *Coefficient bounds of bi-univalent function involving pseudo-starlikeness associated with Sigmoid function defined by Salagean operator via Chebyshev polynomial*. International Journal of Mathematics and Computer Research. Vol. 12(1): 3955-3965
[24] P. Koebe. *ber die Uniformisierung beliebiger analytischer Kurven. Nachr. Kgl. Ges. Wiss.* Gtt. Math-Phys. Kl. 1907, 1907,191210.
[25] M. Lewin. *On the coefficient problem for bi-univalent functions*. Proc. Amer. Math. Soc. 18 (1967), no. 1, 63(68).
[26] G. Murugusundaramoorthy , K. Vijaya and T. Bulboac. *"Initial Coefficient Bounds for Bi-Univalent Functions Related to Gregory Coefficients"*. Mathematics, MDPI, vol. 11(13), pages 1-16, June 2023.
[27] P. Natalini,P.E. Ricci. *Remarks on Bell and higher order Bell polynomials and numbers*. Cogent Math. (2016), 3, 1-15
[28] P. Natalini, P. E. Ricci. *Higher order Bell polynomials and the relevant integer sequences*. Appl. Anal. Discret. Math. (2017), 11, 327-339.
[29] P. Natalini, and P.E.Ricci, *New Bell-Sheffer polynomial sets*. Axioms 2018, 7,71; doi: 10.3390/axioms7040071, 1-10
[30] S. O. Olatunji, H. Dutta. *Subclasses of multivalent functions of complex order associated with sigmoid function and Bernoulli lemniscate*. TWMS J. App. Eng. Math. 2020, 10, 360369.
[31] S. O. Olatunji, M. O. Oluwayemi, Oros, G.I. *Coefficient Results concerning a New Class of Functions Associated Gegenbauer Polynomials and Convolution in Terms of Subordination*. Axioms 2023, 12, 360

[32] P. E. Ricci, P. Natalini, G. Bretti. *Sheffer and Brenke polynomials associated with generalized Bell numbers*. Jnanabha Vijnana Parishad India (2017), 337-352.
[33] H.M. Srivastava, A.K. Mishira, P. Gochhayat. *Certain subclasses of analytic and biunivalent functions*. Appl. Math. Lett. 23 (2010), no. 10, 11881192
[34] E. C. Titmarsh. *The theory functions* (1939) E.d. Oxford Uni. Press. Amen.House London

Ibrahim Tunji Awolere
Department of Mathematical Sciences, Olusegun Agagu University of Science and Technology, Okitipupa, Ondo State, Nigeria.
E-mail address: it.awolere@oaustech.edu.ng; awolereibrahim01@gmail.com

Adeniyi Musibau Gbolagade*
Department of Mathematics Education, Emmanuel Alayande University of Education, Oyo, Oyo State, Nigeria.
E-mail address: gbolagadeam@eauedoyo.edu.ng