Unilag Journal of Mathematics and Applications,
Volume 5, Issue 1 (2025), Pages 1–12.
ISSN: 2805 3966. URL: http://lagjma.edu.ng

NEW VARIETIES OF BCI ALGEBRAS

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Abstract. A BCI algebra is said to form a variety if it satisfies certain axioms. New varieties of BCI algebras, namely; palindromic, almost palindromic, hyper-palindromic and point-wise palindromic BCI algebras are introduced in this paper. These algebras are characterized by distinct axiomatic configurations. Their algebraic properties are investigated. The relationships that exist between the varieties are discussed. Necessary and sufficient conditions for the transformation of one variety to another are presented. Moreover, given any arbitrary algebras of type (2,0), we present conditions for such algebras to belong to any of the new varieties of BCI algebras.

1. Introduction

BCI algebras were introduced by Imai and Iseki [16]. Iseki [17] introduced the notion of BCK algebras which is a generalization of BCI algebras. These two algebras originated from two different sources. One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations. They are union, intersection and set difference. If we consider these operations and their properties, then as a generalization of them, we have the notion of Boolean algebras. If we take both union and intersection, then as a general algebra, the notion of distributive lattices is obtained. Moreover, if we consider union or intersection alone, we have the notion of upper semilattices or lower semilattices. However, the set difference together with its properties had not been considered systematically before the works of Imai and Iseki.
Another motivation is from propositional calculi. There are some systems which contain only the implicational functor among logical functors, such as the system of positive implicational calculus, the system of weak positive implicational calculus, BCK- systems and BCI- systems. Undoubtedly, there are common properties among these systems. It is well known that there are close relationships between

2010 *Mathematics Subject Classification.* Primary: 20N02. Secondary: 20N05, 06F35.
Key words and phrases. Palindromicity; almost palindromicity; hyper-palindromicity; point-wise palindromicity.
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Submitted: January 17, 2025. Revised: Apri 21, 2025. Accepted: May 11, 2025.
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the notion of set difference in set theory and implication functor in logical systems. Some questions were therefore raised. What are the most essential and fundamental properties of these relationships? Can there be a formulation of a general algebra from the above consideration? How would an axiom system be obtained that establishes a theory of the general algebra. It was while answering these pertinent questions that the notion of BCI algebras was birthed.
Several generalizations of BCI algebras have been studied. For instance, the notion of BCH algebras was introduced in Hu and Li [4]. In Neggers and Kim [24], d algebras were studied. In Kim and Kim [21], the notion of BE algebras was introduced. Ideals and upper sets in BE algebras were investigated in Ahn and So [1] and Ahn and So [2]. Pre- commutative algebras were studied in Kim *et. al* [22]. Fenyves algebras were studied in Jaiyeola *et. al* [18], Jaiyeola *et. al* [19] and Ilojide *et. al* [15]. In Neggers *et. al* [23], Q algebras were introduced. Homomorphisms of Q algebras were studied in Ilojide [14]. Lojid algebras were studied in Ilojide [20]. Over the years, many varieties of BCI algebras have been studied mainly due to their diverse applications in coding theory. Motivated by this, more research interest has been given to the study of varieties of BCI algebras. For instance, obic algebras were introduced in Ilojide [6]. In Ilojide [11], torian algebras were studied. It was shown that the class of torian algebras is a wider class than the class of obic algebras. Ideals of torian algebras were investigated in Ilojide [8]. The dual and nuclei of ideals as well as congruences developed on ideals of torian algebras were studied. In Ilojide [12], right distributive torian algebras were studied. Isomorphism Theorems of torian algebras were studied in Ilojide [7]. Kreb algebras were studied in Ilojide [13]. Stabilizers of lojid algebras were studied in Ilojide [10]. In Rezaei *et.al* [25], fuzzy congruence relations on pseudo BE-algebras were studied. Positive implicative BE-filters of BE-algebras based on Lukasiewicz fuzzy sets were studied in Jun [20].
There is the need for further improvement on the varieties of algebras of type (2,0) in order to enhance their applications in coding theory. In this regard, new varieties of BCI algebras are introduced in this paper. They are palindromic, almost palindromic, hyper-palindromic and point-wise palindromic BCI algebras. These algebras are characterized by distinct axiomatic configurations. Their algebraic properties are investigated. The relationships that exist between the varieties are discussed. Necessary and sufficient conditions for the transformation of one variety to another are presented. Moreover, given any arbitrary algebras of type (2,0), we present conditions for such algebras to belong to any of the new varieties of BCI algebras.

2. New Varieties of BCI Algebras

In this section, we introduce palindromic, almost palindromic, hyper-palindromic and point-wise palindromic BCI algebras. Their algebraic properties are presented. The axiomatic configurations of these algebras are also presented.

Definition 2.1. An algebra $(X;*,0)$; where X is a non-empty set, $*$ a binary operation defined on X, and 0 a constant element of X is called a BCI algebra if the following hold for all $x, y, z \in X$:

(1) $((x * y) * (x * z)) * (z * y) = 0$;
(2) $(x * (x * y)) * y = 0$;
(3) $x * x = 0$;
(4) $x * y = 0,\ y * x = 0 \Rightarrow x = y$;
(5) $x * 0 = x$.

Define a binary relation $\leq$ on a BCI algebra $(X; *, 0)$ by $x \leq y$ if and only if $x * y = 0$. Then $(X; \leq)$ is a partially ordered set. Let $(X; *, 0)$ be a BCI algebra. Then $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

We shall denote a BCI algebra $(X; *, 0)$ by X unless there is the need to specify the binary operation and constant element.

Definition 2.2. Let X be a BCI algebra. Then X is said to be

1 . palindromic if $(x * (x * y)) * (y * x) = x * (x * (y * (y * x)))$ for all $x, y \in X$;
2 . almost palindromic if $(x * y) * z = ((x * z) * z) * (y * z)$ for all $x, y, z \in X$;
3 . hyper-palindromic if $(x * (x * y)) * (y * x) = y * (y * x)$ for all $x, y \in X$;
4 . point-wise palindromic if $(x * (y * x)) * (0 * (y * x)) = x$. for all $x, y \in X$.

The following Lemmas are obvious from definition.

Lemma 2.3. *Let X be a palindromic BCI algebra. Then the following hold for all $x, y \in X$:*

1 . $(0 * (0 * y)) * y = 0$;
2 . $0 * (0 * x) = x * (x * (0 * (0 * x)))$;
3 . $(y * (y * (0 * (0 * y)))) * y = 0$;
4 . $((0 * (0 * x)) * x) * (0 * x) = x * (x * (0 * (0 * x)))$;
5 . $0 * (((0 * (0 * y)) * y) * x) = x * (x * (0 * (0 * x)))$;
6 . $0 * (0 * x) = x * (x * ((0 * (0 * y)) * y) * (0 * x))$;
7 . $0 * (0 * x) = x * (x * (0 * ((0 * y)) * y) * x)$.

Lemma 2.4. *Let X be an almost palindromic BCI algebra. Then the following hold for all $x, y, z \in X$:*

1 . $(0 * y) * z = ((0 * z) * z) * (y * z)$;
2 . $x * z = ((x * z) * z) * (0 * z)$;
3 . $0 * z = ((y * z) * z) * (y * z)$;
4 . $(0 * y) * z = ((((y * z) * z) * (y * z)) * z) * (y * z)$;
5 . $x * z = ((x * z) * z) * (((y * z) * z) * (y * z))$.

Lemma 2.5. *Let X be hyper palindromic BCI algebra. Then $(0 * (0 * y)) * y = 0$ for all $y \in X$.*

Lemma 2.6. *Let X be a point-wise palindromic BCI algebra. Then $(x * (0 * x)) * (0 * (0 * x)) = x$ for all $x \in X$.*

Example 2.7. Let $X = \{0, 1, 2\}$. Define a binary operation $*$ on X by the following table:

$*$	0	1	2
0	0	0	2
1	1	0	2
2	2	2	0

TABLE 1.

In the table above, for every $x, y \in X$, we have $(x * (x * y)) * (y * x) = x * (x * (y * (y * x)))$. For instance, $2 = (2 * (2 * 1)) * (1 * 2) = 2 * (2 * (1 * (1 * 2)))$. Then $(X; *, 0)$ is a palindromic BCI algebra.

Example 2.8. Let $X = \{0, 1, 2, 3, 4\}$. Define a binary operation $*$ on X by the following table:

$*$	0	1	2	3	4
0	0	0	0	3	3
1	1	0	1	4	3
2	2	2	0	3	3
3	3	3	3	0	0
4	4	3	4	1	0

TABLE 2.

In the table above, for every $x, y \in X$, we have $(x * (x * y)) * (y * x) = y * (y * x)$. For instance, $0 = (0 * (0 * 3)) * (3 * 0) = 3 * (3 * 0)$. Then $(X; *, 0)$ is a hyper palindromic BCI algebra.

Proposition 2.9. *Let X be a BCI algebra. Then X is almost palindromic if and only if $x * y = ((x * y) * y) * (0 * y)$ for all $x, y \in X$.*

Proof. Suppose X is a BCI algebra satisfying $x * y = ((x * y) * y) * (0 * y)$ for all $x, y \in X$. Then $(x * y) * z = (((x * z) * z) * z) * (0 * z) =$

$$(((x * z) * z) * y) * ((y * z) * y) \leq ((x * z) * z) * (y * z) \tag{2.1}$$

Note that $((x * z) * z) * (y * z) \leq (x * z) * y$; which is the same as

$$((x * z) * z) * (y * z) \leq (x * y) * z \tag{2.2}$$

From expressions (2.1) and (2.2), we have $(x * y) * z = ((x * z) * z) * (y * z)$ as required.

The converse follows from Lemma 2.4(2). $\square$

Theorem 2.10. *Let X be an almost palindromic BCI algebra. Then $(x * (x * y)) * (y * x) = (y * (y * x)^2) * (x * y)$ for all $x, y \in X$.*

Proof. Suppose X is an almost palindromic BCI algebra. Then by Proposition 2.9, for all $x, y \in X$, we have

$$x * (x * y) = ((x * (x * y)) * (x * y)) * (0 * (x * y)) \tag{2.3}$$

Now multiply both sides of expression (2.3) on the right by $y * x$ and using the axiom of almost palindromicity, the right hand side of the resulting expression becomes

$$((x * (x * y)^2) * (y * x)^2) * ((0 * (x * y)) * (y * x)) \tag{2.4}$$

Since $(0 * (x * y)) * (y * x) = 0$, expression (4) is the same as $(x * (x*)^2) * (y * x)^2$. So, we have

$$(x * (x * y)) * (y * x) = (x * (x * y)^2) * (y * x)^2 \tag{2.5}$$

Also, since $(x * (x * y)) \leq y$, the right hand side of expression (2.5) equals $((x * (x * y)) * (y * x)^2) * (x * y) \leq (y * (y * x)^2) * (x * y)$. Therefore,

$$(x * (x * y)) * (y * x) \leq (y * (y * x)^2) * (x * y) \tag{2.6}$$

Similarly, since $(y * (y * x)) \leq x$, we have $(y * (y * x)^2) * (x * y) =$

$$((y * (y * x)) * (x * y)) * (y * x) \leq (x * (x * y)) * (y * x) \tag{2.7}$$

From expressions (2.6) and (2.7), we have $(x*(x*y))*(y*x) = (y*(y*x)^2)*(x*y)$ as required. □

3. Relationships between the New Varieties of BCI Algebras

In this section, the relationship between palindromic, almost palindromic, hyper-palindromic and point-wise palindromic BCI algebras are presented. Necessary and sufficient conditions for the transformation of one variety to another are presented. Moreover, conditions for an arbitrary algebra of type (2,0) to be transformed to any of the new varieties of BCI algebras are also presented.

Theorem 3.1. *Let X be a BCI algebra. Then X is palindromic if and only if it is almost palindromic.*

Proof. Suppose X is a palindromic BCI algebra. Then for all $x, y \in X$, we have

$$(x * (x * y)) * (y * x) \tag{3.1}$$

Replacing x with $x * y$ and y with x in expression (3.1) and simplifying the result gives $((x*) * y) * (0 * y) = x * y$; and by Proposition 2.9, X is almost palindromic. Conversely, suppose X is almost palindromic. Then by Theorem 2.10, for all $x, y \in X$, we have

$$(x * (x * y)) * (y * x) = (y * (y * x)^2) * (x * y) \tag{3.2}$$

From the axiom of palindromicity, it now suffices to show that $x*(x*(y*(y*x))) = (y * (y * x)^2) * (x * y)$. Now, let $x * (y * (y * x)) = u$ and let $y * x = v$. So, we now have to show that

$$x * v = (y * v^2) * (x * y) \tag{3.3}$$

Clearly, $y * v \leq x$ and $u * v \leq x * y$. Hence, we have $((y * v) * v) * (x * u) \leq (x * v) * (x * u) \leq u * v \leq x * y$. Thus,

$$(y * v^2) * (x * y) \leq x * u \tag{3.4}$$

It remains to show that $x * u \leq (y * v^2) * (x * y)$. Now, clearly, $y * (y * x) \leq x$ so that $x * (y * (y * x)) \geq 0$. Thus, $u \geq 0$. By Proposition 2.9, we have

$$x * u = (x * u^2) * (0 * u) = x * u^2 \tag{3.5}$$

Now, since $x * (x * (y * (y * x))) \leq y * (y * x)$, then

$$x * u \leq y * v \tag{3.6}$$

By Proposition 2.9 again, we have $y * v = (y * v^2) * (0 * v)$. Comparing this with expression (3.6), we have $x * u \leq (y * v^2) * (0 * v)$. Hence, $(x * u) * ((y * v^2) * (x * y) \leq ((y * v^2) * (0 * v)) * ((y * v^2) * (x * y)) \leq (x * y) * (0 * v)$. Therefore,

$$(x * u) * ((y * v^2) * (x * y)) \leq (x * y) * (0 * v) \tag{3.7}$$

By expression (3.6), we have $(x * y) * u = (x * u) * y \leq (y * v) * y = 0 * v$. Thus,

$$((x * y) * (0 * v)) * u = 0 \tag{3.8}$$

From the fore-going argument, we have $(x * u) * ((y * v^2) * (x * y)) = (x * u^2) * ((y * v^2) * (x * y)) = ((x * u) * ((y * v^2) * (x * y))) * u \leq ((x * y) * (0 * v)) * u = 0$. Therefore, $x * u \leq (y * v^2) * (x * y)$ as required. □

By Theorem 3.1 and Proposition 2.9, we have the following corollary.

Corollary 3.2. *Let X be a BCI algebra. Then X is palindromic if and only if $x * y = ((x * y) * y) * (0 * y)$ for all $x, y \in X$.*

By Theorem 3.1 and Theorem 2.10, we have the following corollary.

Corollary 3.3. *Every palindromic BCI algebra satisfies $(x * (x * y)) * (y * x) = (y * (y * x)^2) * (x * y)$ for all $x, y \in X$.*

Theorem 3.4. *Every point-wise palindromic BCI algebra X satisfying $0 * (x * (x * y)) = 0 * y$ and $((x * (x * y)) * (y * x)) * (y * (y * x)) = 0$ for all $x, y \in X$, is hyper palindromic.*

Proof. Let X be a point-wise palindromic BCI algebra satisfying the hypothesis. Let $x, y \in X$. Then

$$0 * (x * (x * y)) = 0 * y \tag{3.9}$$

and $((x * (x * y)) * (y * x)) * (y * (y * x)) = 0$. Now, consider the axiom of point-wise palindromicity:

$$(y * (x * y)) * (0 * (x * y)) = y \tag{3.10}$$

Put $x * y$ for y in expression (17) and apply expression (3.9). Then it follows that $x * y = ((x * y) * (x * (x * y))) * (0 * (x * (x * y))) = ((x * (x * (x * y))) * y) * (0 * (x * (x * y))) = ((x * y) * y) * (0 * y)$. Therefore, by Proposition 2.9, X is almost palindromic. Now, multiply both sides of expression (3.10) on the right by $y * x$ to get

$$y * (y * x) = ((y * (x * y)) * (0 * (x * y))) * (y * x) \tag{3.11}$$

By the almost palindromicity of X, the right hand side of expression (3.11) equals

$$((y*(x*y))*(y*x)^2)*((0*(x*y))*(y*x) \tag{3.12}$$

Now, since $(0*(x*y))*(y*x) = 0$, expression (3.12) becomes $(y*(x*y))*(y*x)^2$. Thus, $y*(y*x) = (y*(x*y))*(y*x)^2 = ((y*(y*x))*(x*y))*(y*x) \leq (x*(x*y))*(y*x)$. Thus,

$$y*(y*x) \leq (x*(x*y))*(y*x) \tag{3.13}$$

Also, by hypothesis, we have

$$(x*(x*y))*(y*x) \leq y*(y*x) \tag{3.14}$$

By expressions (3.13) and (3.14), the result follows. □

The converse of Theorem 3.4 is not true as we see in the following counter example.

Example 3.5. Let $X = \{0, 1, 2, 3, 4\}$. Define a binary operation $*$ on X by the following table:

$*$	0	1	2	3	4
0	0	0	0	3	3
1	1	0	1	4	3
2	2	2	0	3	3
3	3	3	3	0	0
4	4	3	4	1	0

TABLE 3.

Then $(X; *, 0)$ is a hyper palindromic BCI algebra satisfying $0*(x*(x*y)) = 0*y$ and $((x*(x*y))*(y*x))*(y*(y*x)) = 0$ for all $x, y \in X$. However, it is not point-wise palindromic since $(2*(3*2))*(0*(3*2)) \neq 2$.

The following Lemma is straightforward.

Lemma 3.6. *Let X be a BCI algebra satisfying $x*y = 0 \Rightarrow y*(y*x) = x$ for all $x, y \in X$. Then $x*(x*y) = y*(y*(x*(x*y)))$ for all $x, y \in X$.*

Theorem 3.7. *Let X be a BCI algebra satisfying $x*y = 0 \Rightarrow y*(y*x) = x$ for all $x, y \in X$. Then X is hyper palindromic if and only if it is palindromic.*

Proof. Let X be a BCI algebra satisfying $x*y = 0 \Rightarrow y*(y*x) = x$ for all $x, y \in X$. Then

$$(x*(x*y))*(y*x) = y*(y*x) \tag{3.15}$$

Replacing x with $y*x$ in expression (3.15) and simplifying gives $((y*x)*x)*(0*x) = y*x$. Therefore, by Corollary 3.2, X is palindromic.
Conversely, suppose X is a palindromic BCI algebra satisfying $x*y = 0 \Rightarrow y*(y*x) = x$ for all $x, y \in X$. Then by Lemma 3.6, we have $y*(y*x) = x*(x*(y*(y*x))) = (x*(x*y))*(y*x)$. That is, $y*(y*x) = (x*(x*y))*(y*x)$. So, X is hyper palindromic. □

Theorem 3.8. *Let X be a BCI algebra which satisfies $x*y=0 \Rightarrow y*(y*x)=x$ and $x*(x*(x*y))^n = x*y^n$ for all $x, y \in X$ and for all non-negative integer n. Then X is a hyper palindromic if and only if $(x*(x*)^2)*(0*(x*y)) = y*(y*(x*(x*y)))$ for all $x, y \in X$.*

Proof. Let X be a hyper palindromic BCI algebra satisfying the hypothesis. Then by Theorem 3.7, X is palindromic. By Corollary 3.2, we have

$$(x*(x*y)^2)*(0*(x*y)) = x*(x*y) \tag{3.16}$$

Applying Lemma 2.6 to expression (3.16), we have

$$(x*(x*y)^2)*(0*(x*y)) = y*(y*(x*(x*y))) \tag{3.17}$$

as required.
Conversely, suppose X is a BCI algebra satisfying the hypothesis. Then, from

$$x*(x*(x*y))^2 = (x*y)*y \tag{3.18}$$

and the fact that $x*(x*y) \leq y$, we have $0*(x*(x*y)) = 0*y$ for all $x, y \in X$. Put y for $x*y$ in expression (3.17). Then the left hand side of expression (3.17) equals $(x*(x*(x*y))^2)*(0*(x*(x*y))) = ((x*y)*y)*(0*y)$.
The right hand side of expression (3.17) equals $(x*y)*((x*y)*(x*(x*(x*y)))) = (x*y)*((x*y)*(x*y) = x*y$.
Therefore, $((x*y)*y)*(0*y) = x*y$. Thus, X is palindromic, and by Theorem 3.7, X is hyper palindromic. □

Theorem 3.9. *Let X be an algebra of type (2,0). Then X is a hyper palindromic BCI algebra if and only if the following hold for all $x, y, z \in X$:*

1 . $x*0=x$;
2 . $x*x=0$;
3 . $(x*y)*z=(x*z)*y$;
4 . $(x*z)*(x*y) = ((y*z)*(y*x))*(x*y)$.

Proof. Suppose X is a hyper palindromic BCI algebra. Then, clearly, items (1), (2) and (3) hold for all $x, y, z \in X$. We now show that item (4) also holds. Since X is hyper palindromic, then for all $x, y \in X$, we have

$$x*(x*y) = (y*(y*x))*(x*y) \tag{3.19}$$

Multiplying both sides of expression (3.19) on the right by z and simplifying the result gives $(x*z)*(x*y) = ((y*z)*(y*x))*(x*y)$ as required.
Conversely, suppose X is an algebra of type (2,0) satisfying the hypothesis. Then, to show that X is a BCI algebra, we only need to show that for all $x, y, z \in X$, we have

$$((x*y)*(x*z))*(z*y) = 0 \tag{3.20}$$

and

$$x*y=0,\ y*x=0 \Rightarrow x=y \tag{3.21}$$

Put $z=0$ in item (4) and applying item (1), we have $x*(x*y) = (y*(y*x))*(x*y)$ for all $x, y \in X$. By item (1), we have

$$(x*y)*(x*z) = ((z*y)*(z*x))*(x*z) \tag{3.22}$$

Multiply both sides of expression (3.22) on the right by $z * y$ to get

$$((x * y) * (x * z)) * (z * y) = (((z * y) * (z * x)) * (x * z)) * (z * y) \tag{3.23}$$

By item (3), the right hand side of expression (3.23) is the same as

$$(((z * y) * (z * y)) * (z * x)) * (x * z) \tag{3.24}$$

By item (2), expression (3.24) becomes $(0 * (z * x)) * (x * z)$ or equivalently,

$$((z * z) * (z * x)) * (x * z) \tag{3.25}$$

By item (3), expression (3.25) becomes

$$((z * (z * x)) * (x * z)) * z \tag{3.26}$$

From the axiom of hyper palindromicity, expression (3.26) becomes

$$((x * (x * z))) * z \tag{3.27}$$

By item (3), expression (3.27) becomes $(x*z)*(x*z) = 0$. Thus, expression (3.22) gives $((x * y) * (x * z)) * (z * y) = 0$. Hence, expression (27) holds. From item (1) and the axiom of hyper palindromicity, expression (3.21) holds as required. □

Theorem 3.10. *Let X be a BCI algebra satisfying $x * y = 0 \Rightarrow y * (y * x) = x$ for all $x, y \in X$. Then the following hold for all $x, y \in X$:*

1 . *X is hyper palindromic $\Rightarrow (x * (x * y)) * (0 * (x * y)) = y * (y * x)$;*
2 . *$x * (x * (y * x)) = 0 * (x * y) \Leftrightarrow x * (y * x) = x * (0 * (x * y))$;*
3 . *$x * (y * x) = x * (0 * (x * y)) \Rightarrow (x * (x * y)) * (y * (y * x)) = 0 * (x * y)$;*
4 . *$x * (y * x) = x * (0 * (x * y)) \Rightarrow (x * y) * (0 * (0 * y)) = (x * y) * y$;*
5 . *$(x * y) * (0 * (0 * y)) = (x * y) * y \Rightarrow (x * y) * ((x * y) * y) = 0 * (0 * y)$;*
6 . *$(x * (x * y)) * (0 * (x * y)) = y * (y * x)$ and $(x * y) * ((x * y) * y) = 0 * (0 * y) \Rightarrow x * y = ((x * y) * y) * (0 * y)$;*
7 . *$x * y = ((x * y) * y) * (0 * y) \Rightarrow (x * y) * (0 * (0 * y)) = (x * y) * y$.*

Proof. 1 . Suppose X is hyper palindromic. Then $(x * (x * y)) * (y * x) = y * (y * x)$ for all $x, y \in X$ and since $0 * (x * y) \leq y * x$, we have $(x * (x * y)) * (y * x) \leq (x * (x * y)) * (0 * (x * y))$. Thus,

$$y * (y * x) \leq (x * (x * y)) * (0 * (x * y)) \tag{3.28}$$

By Lemma 2.6, we have $(x * (x * y)) * (y * (y * x)) = 0 * (x * y)$; and so

$$(x * (x * y)) * (0 * (x * y)) \leq y * (y * x) \tag{3.29}$$

From expressions (3.28) and (3.29), the result follows.

2 . Suppose $x * (x * (y * x)) = 0 * (x * y)$ for all $x, y \in X$. Then $x * (y * x) = x * (x * (x * (y * x))) = x * (0 * (x * y))$.
Conversely, suppose $x * (y * x) = x * (0 * (x * y))$. Then $x * (x * (y * x)) = x * (x * (0 * (x * y))) \leq 0 * (x * y)$. Thus, $x * (x * (y * x)) = 0 * (x * y)$.

3 . Since $x * (y * x) = x * (0 * (x * y))$, then $(x * (x * y)) * (0 * (x * y)) = (x * (0 * (x * y))) * (x * y) = (x * (y * x)) * (x * y) \leq y * (y * x)$. Thus, $(x*(x*y))*(y*(y*x)) \leq 0*(x*y)$. Therefore, $(x*(x*y))*(y*(y*x)) = 0 * (x * y)$.

4 . Replacing x with $x * y$ and y with x in $x * (y * x) = x * (0 * (x * y))$ and simplifying, we have the result.

5 . Since $(x*y)*(0*(0*y)) = (x*y)*y$, then $(x*y)*((x*y)*y) \leq 0*(0*y)$ which implies that $(x*y)*((x*y)*y) = 0*(0*y)$.

6 . Let $x*y = p$. Then we have

$$x*y = x*(x*(x*y)) = x*(x*p) \tag{3.30}$$

Now, by hypothesis, we have $x*(x*p) = (p*(p*x))*(0*(p*x))$ and $p*(p*x) = (x*(x*p))*(0*(x*p))$. Thus,

$$x*(x*p) = ((x*(x*p))*(0*(x*p)))*(0*(p*x)) \tag{3.31}$$

From expressions (3.30) and (3.31), we have $x*y = ((x*y)*(0*(x*p)))*(0*(p*x))$. Since $x*y = p$, we also have $0*(x*p) = 0*y$ and $0*(p*x) = 0*(0*y)$. Thus, $x*y = ((x*y)*(0*y))*(0*(0*y))$; giving us

$$x*y = ((x*y)*(0*(0*y)))*(0*y) \tag{3.32}$$

Moreover, by hypothesis again, $(x*y)*((x*y)*y) = 0*(0*y)$. Hence,

$$(x*y)*(0*(0*y)) \leq (x*y)*y \tag{3.33}$$

Multiplying both sides of expression (3.33) on the right by $0*y$, we have

$$((x*y)*(0*(0*y)))*(0*y) \leq ((x*y)*y)*(0*y) \tag{3.34}$$

From expressions (3.32) and (3.34), we have $x*y \leq ((x*y)*y)*(0*y)$. Also, clearly, $((x*y)*y)*(0*y) \leq x*y$. Therefore, $x*y = ((x*y)*y)*(0*y)$ as required.

7 . By hypothesis, since $(((x*y)*y)*(0*y))*((x*y)*y) = 0*(0*y)$, we have $(x*y)*((x*y)*y) = 0*(0*y)$. Hence $(x*y)*(0*(0*y)) \leq (x*y)*y$. Also, since $0*(0*y) \leq y$, we have $(x*y)*y \leq (x*y)*(0*(0*y))$. Therefore, $(x*y)*(0*(0*y)) = (x*y)*y$ as required.

□

4. Conclusion

We have introduced four new varieties of BCI algebras. We investigated their algebraic properties as well as the relationships between the algebras. In particular, we presented necessary and sufficient conditions for one variety to be transformed to another variety. This is a remarkable contribution to the application of BCI algebras to coding theory. Moreover, the manner of the axiomatic configurations of these algebras makes them securely reliable in the transmission of codes. Further research can be carried out in the aspect of deriving conditions under which these new varieties of BCI algebras can be transformed to other algebras of type (2,0).

Acknowledgment. I acknowledged authors whose research works were consulted as listed in the reference section.

Authors Contributions. I declare that I was the only one who conducted this research.

Conflicts of interest. I declare that there is no conflict of interests in this research.

Funding. I declare that I did not receive any funding for this research.

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