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BI- UNIVALENT PROBLEM FOR CERTAIN GENERALIZED CLASS OF ANALYTIC FUNCTIONS INVOLVING Q-INTEGRAL OPERATOR ASSOCIATED WITH NEPHROID DOMAIN.

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ABSTRACT. The authors in this article study a new subclass of bi-univalent functions involving q-integral operator associated with nephroid domain by using the concept of subordination and bi-linear fractional principles.We further employed our investigation to determine new coefficient bounds and subsequently obtain the Fekete-Szegö inequalities for functions belonging to the aforementioned subclass were obtained.

1. Introduction

Let Ω stand for the class of all analytic functions $f(z)$ of the form:

$$f(z) = z + a_2z^2 + a_3z^3 + \dots, \tag{1.1}$$

normalized by $f(0) = 0$ and $f'(0) = 1$. Also, let S denote the subclasses of Ω which are univalent in the open unit disk $K = \{z \in C : |z| < 1\}$. Let f be subordinate to analytic function $g(z) = z + b_2z^2 + b_3z^3 + \dots$. This is denoted by $f \prec g$ and this suggest that there exists a Schwarz function $w(z)$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f = g(w(z))$. It also hold for the property if and only if $f(0) = g(0)$ as well as $f(K) \subset g(K)$ for $z \in K$, see [7, 24, 40] for more information.
It is equally worthy to note that for the Schwarz function $w(z) = \sum_{n=1}^{\infty} v_nz^n$, we have $|v_n| \leq 1$, see [20, 27, 48]

In addition, the function $f(z) \in \Omega$ is said to be bi-univaalent see [8, 10, 13, 15, 23, 28, 25, 41, 42, 47, 49] if its inverse $g \in f^{-1}$ is also univalent. Then, $f(f^{-1}(z)) = z$, $(z \in K)$, $f^{-1}(f(w)) = w$, $|w| < r_0(f) : r_0(f) \geq \frac{1}{4}$,

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such that

$$g(w) = f^{-1}(w) = w - a_2 w + (2a_2^2 - a_3)w^2 - (5a_2^3 - 5a_2 a_3 + a_4)w^3 + \ldots = w + \sum_{k=2}^{\infty} b_k w^k, \tag{1.2}$$

where $b_2 = -a_2$, $b_3 = 2a_2^2 - a_3$, $b_4 = -(5a_2^3 - 5a_2 a_3 + a_4)$ etc.
For recent study on bi-univalent function, we refer interested readers to see [2, 4, 5, 9, 11, 21, 22, 28, 25, 30, 35, 43, 50, 53, 54] among others.

However, some of the examples of bi-univalent functions and their inverses are given as follow: (i). $\log \frac{1}{1-z}$ and its inverse $\frac{\exp^w - 1}{\exp^w}$ $(ii)\frac{z}{1-z}$ and its inverse $\frac{w}{w+1}$. For details see [53, 35].
Obviously, we can see that the class of bi-univalent functions are non-empty.

In the recent time, the study of $q-$ calculus in Geometric Function theory is attracting the interest of several researchers in Geometric Function Theory (GFT) and has inspired numerious motivations in so many different ways which have enabled investigations into various choices of domains. One can refer to [6, 11, 14, 36, 37, 52] and [53] to mention a few.

We shall now study the bi-univalent problem which involves the $q-$ integral operator associated with Nephroid domain $(1 + z - \frac{1}{3}z^3)$. For information on the Nephroid function,see [19, 51].

Be that as it may, in this work, standing on the premise that if $0 < q < 1$, then we can consider the investigation of the $q-$ derivative [38, 52] of a complex-valued function f of the form

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & , \quad z \neq 0 \\ f'(z) & , \quad z = 0. \end{cases}$$

Observe that if f is differentiable at z, then the limit as $q \to 1^-$ of $D_q f(z)$ yields $f'(z)$.

The q function known for its crucial significance has its expression in the form

$$\Gamma_q(a) = (1-q)^{1-q} \Pi_{k=0}^{\infty} \frac{1 - q^{k+1}}{1 - q^{k+a}} \quad (a > 0). \tag{1.3}$$

It can as well be expressed equivalently as a special factorial in the form

$$\Gamma_q(a+1) = [a]_q \Gamma(a) = [a]_q!, a \in N$$

and by extension, it can also be conceived in a notation of the form

$$[a]_q! = \begin{cases} [a]_q [a-1]_q \ldots [3]_q [2]_q [1]_q & , a \geq 1 \\ 1 & , a = 0 \end{cases}.$$

it is easily verified that as $q \to 1^-$, $\Gamma_q(a) \to \Gamma(a)$ (see [6, 11, 52, 53]).

Likewise in [52], the $q-$ binomial coefficients can be expresed also in the form

$$\left(\begin{array}{c} k \\ m \end{array} \right)_q = \frac{[k]_q!}{[m]_q![k-m]_q!}. \tag{1.4}$$

The $q-$ beta function $B_q(a,s)$ has the special expression in the form

$$B_q(a,s) = \int_0^1 z^{a-1}(1-qz)_q^{s-1} d_q z, \quad (a, s > 0). \tag{1.5}$$

On the otherhand, the q- analogue of Euler's formular takes the form:

$$B_q(a,s) = \frac{\Gamma_q(a)\Gamma_q(s)}{\Gamma_q(a+s)}. \tag{1.6}$$

For investigation and further consideration see([1, 6, 11, 52, 53]).

Also in [52],the $q-$ integral of function f have interesting properties that shows its connection with analytic and bi-univalent functions that aids the expression of the form:

$$\int_0^z f(a) d_q a = (1-q)z \times \sum_{k=0}^{\infty} q^k f(q^k z)$$

with the provision that the series is convergent.

In [52], the generalized q- integral operator $H_{i,j,q}$: $\Omega \to \Omega$ is written as

$$H_{i,j,q}f(z) = \left(\begin{array}{c} i + +j \\ j \end{array} \right) \frac{[i]_q}{z^j} \int_0^z \left(1 - \frac{q_a}{z}\right)_q^{i-1} a^{j-1} f(a) d_q a$$

for $i > 0$ and $j > -1$.

In view of (1.3), (1.4),(1.5) and (1.6), one can say that

$$H_{i,j,q}f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma_q(j+k)\Gamma(i+j+1)}{\Gamma_q(i+j+k)\Gamma_q(j+1)} a_k z^k.$$

For different choices of parameteers i and j, several integral operators previously studied by various authors are obtained (see[52] and [53]) among others.
It is also crucial to emphasize that coefficient problem usually arise in the investigation of subclasses of analytic univalent functions. These problems have interwoven connections with several fundamental conjectures, one of these conjectures is the Bieberbach conjecture in [12], this was also emphasized in [18, 29, 46] and this shall be sufficient to support the present investigation in order to align with some fundamental properties. The findings of Bieberbach[12] tied around each function of $f(z) \in S$ and he put forward the inequality $|a_n| \leq n$ and emphasized that the equalitty only holds for the Koebe function $k(z) = \frac{z}{(1-z)^2}$, which maps the unit disc K onto the entire complex plane minus the slit along the negative real axis from $\frac{-1}{4} to - \infty$. Also, in [18] , De Branges investigated and solved the Bieberbach conjecture. On the otherhand, Lower[44] and proved that $|a_3| \leq 3$ for

the class S which happens to be a very large class. Now that the known estimates $|a_2| \leq 2$ and $|a_3| \leq 3$ are in view, it is quite interesting to investigate the relation a_3 and a_2^2 for the class S. The class S has special geometric representation in the starlike function structure and it takes the form as follow:

Definition 1.1.1: Suppose $f \in \Omega$ is starlike, then

$$\mathcal{R}_e\Big(\frac{zf'(z)}{f(z)}\Big) > 0.\ z \in K,\ f(z) \neq 0.$$

This is known as the class of starlike function and its notation is S^*. Note, a function f that is analytic and univalent on the unit disk K with the condition $f(0) = 0,\ f'(0) - 1 = 0$ and provided $(1-t)f \prec f$ for every t in $0 \leq t \leq 1$ is considered to be starlike, see [24, 31, 32]. It is this idea that readily prompt the well-known Fekete and Szegö [26] to use the Löwner's method to investigate the classical result for the class S.

The Fekete and Szegö functional [26] has been attracting researchers over the years and takes the form :

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & , \quad if \quad \mu \leq 0 \\ 1 + 2\exp^{\frac{-2\mu}{1-\mu}} & , \quad if 0 \leq \mu \leq 1 \\ 4\mu - 3 & , \quad if \quad \mu \geq 1 \end{cases}.$$

This special inequality plays a very important role in the determination of estimates of higher coefficients for some subclasss of S. For further information see [16, 17, 29, 39, 45, 46].

Barely few years ago, [46] investigated the class of function represented by $\mathcal{S}^*$ which takes the form:

$$\mathcal{S}_s^* = \Big\{f \in A : \frac{\Delta f'(\Delta)}{f(\Delta)} \prec 1 + sin\Delta,\Big\}\ (\Delta \in K)$$

This clearly indicates that the object $\frac{\Delta f'(\Delta)}{f(\Delta)}$ lies in an eight - shaped region in the right-half plane. Also, in [46] studied the class $\mathcal{R}S^*_{sin}(\delta)$ that consists of all analytic functions of the form:

$$(f'(z))^{\delta}\Big(\frac{zf\prime(z)}{f(z)}\Big)^{1-\delta} \prec 1 + sinz = \Phi(z),$$

where $0 \leq \delta \leq 1$. The special case for this class in [46] takes the form

$$\Re S^*_{sin}(0) = \mathcal{S}^*_{sin} = \{f \in A : \frac{zf\prime(z)}{f(z)} \prec 1 + sinz\}$$

and

$$\mathcal{R}S^*_{sin}(0) = \mathcal{R}^*_{sin} = \{f \in A : f'(z) \prec 1 + sinz\}.$$

Definition 1.1.2: Let $\Omega : K \to C$ be a convex univalent function in K and satisfying the conditions:$\Phi(0) = 1$ and $R\{\Phi(z)\} > 0$ $(z \in U)$. Also, let $\Phi(t)$ be defined by

$$\Phi(t) = 1 + \sum_{k=1}^{\infty} B_k t^k.$$

.

Definition 1.1.3: Let $f \in \Omega$ be of the form (1.1). Then for $i > 0$, $j > -1$, $\gamma \in N$, $0 \leq \alpha \leq 1$, the complex-valued function f of the form (1.1) belongs to the class $H_q(i, j, \alpha, \gamma)$ satisfying the geometric condition:

$$Re\left\{\Big(H'_{i,j,q}f(z)\Big)^{\gamma}\Big(\frac{z}{H_{i,j,q}f(z)}\Big)^{\alpha}\right\} \prec 1 + z - \frac{1}{3}z^3 \tag{1.7}$$

and

$$Re\left\{\Big(H'_{i,j,q}g(u)\Big)^{\gamma}\Big(\frac{u}{H_{i,j,q}g(u)}\Big)^{\alpha}\right\} \prec 1 + u - \frac{1}{3}u^3.$$

. Remark: The following are some well known classes subordinate to Nephroid domain.

If $\alpha = 0$ in Definition 1.1.3 we have

$$(i).Re\left\{\Big(H'_{i,j,q}f(z)\Big)^{\gamma}\right\} \prec 1 + z - \frac{1}{3}z^3 \tag{1.8}$$

and

$$Re\left\{\Big(H'_{i,j,q}g(u)\Big)^{\gamma}\right\} \prec 1 + u - \frac{1}{3}u^3.$$

This is known as γ- pseudo-bounded turning.

If $\alpha = 1$ in Definition 1.1.3 we have

$$(ii).Re\left\{\Big(H'_{i,j,q}f(z)\Big)^{\gamma}\Big(\frac{z}{H_{i,j,q}f(z)}\Big)\right\} \prec 1 + z - \frac{1}{3}z^3 \tag{1.9}$$

and

$$Re\left\{\Big(H'_{i,j,q}g(u)\Big)^{\gamma}\Big(\frac{u}{H_{i,j,q}g(u)}\Big)\right\} \prec 1 + u - \frac{1}{3}u^3.$$

This is known as γ- pseudo-starlike.

If $\alpha = 0$, $\gamma = 1$ in Definition 1.1.3 we have

$$(iii).Re\left\{\Big(H'_{i,j,q}f(z)\Big)\right\} \prec 1 + z - \frac{1}{3}z^3 \tag{1.10}$$

and

$$Re\left\{\Big(H'_{i,j,q}g(u)\Big)\right\} \prec 1 + u - \frac{1}{3}u^3.$$

This is bounded turning.

If $\alpha = 1$, $\gamma = 1$ in Definition 1.1.3 and we have

$$(iv).Re\left\{\Big(\frac{zH'_{i,j,q}f(z)}{H_{i,j,q}f(z)}\Big)\right\} \prec 1 + z - \frac{1}{3}z^3 \tag{1.11}$$

and

$$Re\left\{\Big(\frac{uH'_{i,j,q}g(u)}{H_{i,j,q}g(u)}\Big)\right\} \prec 1 + u - \frac{1}{3}u^3.$$

This is starlike.
If $\alpha = 1$, $\gamma = 2$ in Definition 1.1.3 and we have

$$(v)Re\left\{\left(H'_{i,j,q}f(z)\right)\left(\frac{zH'_{i,j,q}f(z)}{H_{i,j,q}f(z)}\right)\right\} \prec 1 + z - \frac{1}{3}z^3 \tag{1.12}$$

and

$$Re\left\{\left(H'_{i,j,q}g(u)\right)\left(\frac{uH'_{i,j,q}g(u)}{H_{i,j,q}g(u)}\right)\right\} \prec 1 + u - \frac{1}{3}u^3.$$

This is known as Geometric combination of starlike and bounded turning. For recent work on various domains, we refer interested readers to [29, 35, 43, 52] to mention a few.

Before we proceed into the results, the following lemmas shall be considered. Here, let $\mathscr{P}$ stand for the collection of $h(z)$ that are analytic with positive part in the open unit disc K which takes the form $p(z) = 1 + \sum_{k=1}^{\infty} P_k z^k \quad z \in K$.

Lemma 1.1.4 [39, 48]**:** Let the function $p \in P$ be given by

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad z \in K.$$

Then $|P_k| \leq 2$, $k \in N$, where $p(0) = 1$ and $Re\{p(z)\} > 0$.

Lemma 1.1.5[48, 50]**:** Let the function $\rho(z) = 1 + \sum_{k=1}^{\infty} b_k z^k$, $z \in U$ be convex in K.
Also, let $y(z)$ be given by $y(z) = 1 + \sum_{k=1}^{\infty} l_k z^k$, be holomorphic in K.
If $y(z) \prec \psi(z)$, $z \in K$,then $|l_k| \leq |b_k|, \quad k \in N$.

Lemma 1.1.6[39]**:** Let $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k \quad z \in K$. Then $f' \in S^*$. then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 3 - 4\mu & , \quad if \quad \mu \leq 0 \\ \\ 1 + 2\exp^{\frac{-2\mu}{1-\mu}} & , \quad if 0 \leq \mu \leq 1 \\ \\ 4\mu - 3 & , \quad if \quad \mu \geq 1 \end{cases} .$$

where $v \leq 0$ or $v \geq 1$, the equality holds if and only if $\frac{(1+z)}{(1-z)}$ or one its rotations. If $0 < v < 1$, then equality holds if and only if $p(z)$ is $\frac{(1+z^2)}{(1-z^2)}$ or one of its rotations. If $v = 0$, the equality holds [46] if and only if $p(z) = \left(\frac{1}{2} + \frac{1}{2}\rho\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\rho\right)\frac{1-z}{1+z}$ $(0 \leq \rho \leq 1)$ or one of its rotations. If $v = 1$, the equality holds if and onlyif p is the reciprocal of one of the functions such that the equality holds in the case of $v = 0$.

Lemma 1.1.7[24, 25]:Let $p(z) = 1 + \sum_{m=1}^{\infty} c_m z^m \quad P$ is an analytic function with positive real part and v is a complex variable, then

$$|c_3 - vc_2^2| \leq \begin{cases} 2-4v & , \quad if \quad v \leq 0 \\ 2 & , \quad if 0 \leq v \leq 1 \\ 4v-2 & , \quad if \quad v \geq 1 \end{cases} .$$

2. Coefficient bounds for the class $H_q(i, j, \alpha, \gamma)$

We begin this section with the statement of theorem for the class $H_q(i, j, \alpha, \gamma)$ and the proof the theorem follows in quick succession.

Theorem 2.1: Let f be of the form given (1.1)that belong to the class $H_q(i, j, \alpha, \gamma)$, then

$$|a_2| \leq \sqrt{\frac{|B_1|(|B_1|+2)}{4\{(3\gamma-\alpha)\lambda_2^{i,j} + ([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1^{i,j} - \frac{\alpha(\alpha-1)}{2})\lambda_1^{i,j}\}}} \tag{2.1}$$

and

$$|a_3| \leq \frac{|B_1|(|B_1|+2)}{4(3\gamma-\alpha)\lambda_2^{i,j}} + \frac{|B_1|^2(3\gamma-\alpha)\lambda_2^{i,j}}{2(2\gamma-\alpha)^2(\lambda^{i,j})_1^2} \tag{2.2}$$

where $\lambda_1^{i,j} = \frac{\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)}$, $\lambda_2^{i,j} = \frac{\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+3)\Gamma_q(j+1)}$ respectively.

Proof: Let $f \in H_q(i, j, \alpha, \gamma)$ and by applying the principle of subordination we have

$$\left(H'_{i,j,q}f(z)\right)^{\gamma}\left(\frac{z}{H_{i,j,q}f(z)}\right)^{\alpha} = 1 + \omega(z) - \frac{1}{3}(\omega(z))^3$$

and

$$\left(H'_{i,j,q}g(u)\right)^{\gamma}\left(\frac{u}{H_{i,j,q}g(u)}\right)^{\alpha} = 1 + \phi(u) - \frac{1}{3}(\phi(u))^3.$$

Notice, $\left(H'_{i,j,q}f(z)\right)^{\gamma} = 1 + \left(\frac{2\gamma\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)}\right)a_2 z$

$$+\left(\frac{3\gamma\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+3)\Gamma_q(j+1)}a_3 + \frac{2\gamma(\gamma-1)\left(\Gamma_q(2+j)\right)^2\left(\Gamma_q(i+j+1)\right)^2}{\left(\Gamma_q(i+j+2)\right)^2\left(\Gamma_q(j+1)\right)^2}a_2^2\right)z^2 +$$

$$\left(\frac{4\gamma\Gamma_q(4+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+4)\Gamma_q(j+1)}a_4 + \frac{6\gamma(\gamma-1)\Gamma_q(2+j)\Gamma_q(i+j+1)\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)\Gamma_q(i+j+3)\Gamma_q(j+1)}a_2 a_3 + \right.$$

$$\left.\frac{4\gamma(\gamma-1)(\gamma-2)\left(\Gamma_q(2+j)\right)^3\left(\Gamma_q(i+j+1)\right)^3}{\left(\Gamma_q(i+j+2)\right)^3\left(\Gamma_q(j+1)\right)^3}a_2^3\right)z^3 + \ldots$$

and $\left(\frac{z}{H_{i,j,q}f(z)}\right)^{\alpha} = 1 - \frac{\alpha\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)}a_2 z - \left(\alpha\left[\frac{\Gamma_q(3+j)\Gamma_q(i=j+1)}{\Gamma_q(3+i+j)\Gamma_q(j+1)}a_3 - \right.\right.$
$\left.\left.\frac{\left(\Gamma_q(2+j)\right)^2\left(\Gamma_q(i+j+1)\right)^2}{\left(\Gamma_q(2+i+j)\right)^2\left(\Gamma_q(j+1)\right)^2}a_2^2\right] - \frac{\alpha(\alpha-1)}{2!}\frac{\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(2+i+j)\Gamma_q(j+1)}a_2^2\right)z^2 - \left(\alpha\left[\frac{\Gamma_q(4+j)\Gamma_q(i+j+1)}{\Gamma_q(4+i+j)\Gamma_q(j+1)}a_4 - \right.\right.$
$\left.\frac{2\Gamma_q(2+j)\Gamma_q(i+j+1)\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(2+i+j)\Gamma_q(j+1)\Gamma_q(3+i+j)\Gamma_q(j+1)}a_2a_3 + \frac{\left(\Gamma_q(2+j)\right)^3\left(\Gamma_q(i+j+1)\right)^3}{\left(\Gamma_q(2+i+j)\right)^3\left(\Gamma_q(j+1)\right)^3}a_2^3\right] - \alpha(\alpha -$
$1)\frac{\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(2+i+j)\Gamma_q(j+1)}\left[\frac{\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(3+i+j)\Gamma_q(j+1)}a_2a_3 - \frac{\left(\Gamma_q(2+j)\right)^2\left(\Gamma_q(i+j+1)\right)^2}{\left(\Gamma_q(2+i+j)\right)^2\left(\Gamma_q(j+1)\right)^2}a_2^3\right] +$
$\left.\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\frac{\left(\Gamma_q(2+j)\right)^3\left(\Gamma_q(i+j+1)\right)^3}{\left(\Gamma_q(2+i+j)\right)^3\left(j+1\right)^3}a_2^3\right)z^3 + \ldots.$

This implies that

$$\left(H'_{i,j,q}f(z)\right)^{\gamma}\left(\frac{z}{H_{i,j,q}f(z)}\right)^{\alpha} = \left(1 + \left(\frac{2\gamma\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)}\right)a_2 z + \right.$$
$$\left(\frac{3\gamma\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+3)\Gamma_q(j+1)}a_3 + \frac{2\gamma(\gamma-1)\left(\Gamma_q(2+j)\right)^2\left(\Gamma_q(i+j+1)\right)^2}{\left(\Gamma_q(i+j+2)\right)^2\left(\Gamma_q(j+1)\right)^2}a_2^2\right)z^2 +$$
$$\left(\frac{4\gamma\Gamma_q(4+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+4)\Gamma_q(j+1)}a_4 + \frac{6\gamma(\gamma-1)\Gamma_q(2+j)\Gamma_q(i+j+1)\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)\Gamma_q(i+j+3)\Gamma_q(j+1)}a_2a_3 + \right.$$
$$\left.\left.\frac{4\gamma(\gamma-1)(\gamma-2)\left(\Gamma_q(2+j)\right)^3\left(\Gamma_q(i+j+1)\right)^3}{\left(\Gamma_q(i+j+2)\right)^3\left(\Gamma_q(j+1)\right)^3}a_2^3\right)z^3 + \ldots\right)\left(1 - \frac{\alpha\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)}a_2 z - \right.$$
$$\left(\alpha\left[\frac{\Gamma_q(3+j)\Gamma_q(i=j+1)}{\Gamma_q(3+i+j)\Gamma_q(j+1)}a_3 - \frac{\left(\Gamma_q(2+j)\right)^2\left(\Gamma_q(i+j+1)\right)^2}{\left(\Gamma_q(2+i+j)\right)^2\left(\Gamma_q(j+1)\right)^2}a_2^2\right] - \frac{\alpha(\alpha-1)}{2!}\frac{\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(2+i+j)\Gamma_q(j+1)}a_2^2\right)z^2 -$$
$$\left(\alpha\left[\frac{\Gamma_q(4+j)\Gamma_q(i+j+1)}{\Gamma_q(4+i+j)\Gamma_q(j+1)}a_4 - \frac{2\Gamma_q(2+j)\Gamma_q(i+j+1)\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(2+i+j)\Gamma_q(j+1)\Gamma_q(3+i+j)\Gamma_q(j+1)}a_2a_3 + \right.\right.$$
$$\left.\frac{\left(\Gamma_q(2+j)\right)^3\left(\Gamma_q(i+j+1)\right)^3}{\left(\Gamma_q(2+i+j)\right)^3\left(\Gamma_q(j+1)\right)^3}a_2^3\right] - \alpha(\alpha-1)\frac{\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(2+i+j)\Gamma_q(j+1)}\left[\frac{\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(3+i+j)\Gamma_q(j+1)}a_2a_3 - \right.$$
$$\left.\left.\left.\frac{\left(\Gamma_q(2+j)\right)^2\left(\Gamma_q(i+j+1)\right)^2}{\left(\Gamma_q(2+i+j)\right)^2\left(\Gamma_q(j+1)\right)^2}a_2^3\right] + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}\frac{\left(\Gamma_q(2+j)\right)^3\left(\Gamma_q(i+j+1)\right)^3}{\left(\Gamma_q(2+i+j)\right)^3\left(j+1\right)^3}a_2^3\right)z^3 + \ldots\right).$$

Further simplification gives:

$$\left(H'_{i,j,q}f(z)\right)^{\gamma}\left(\frac{z}{H_{i,j,q}f(z)}\right)^{\alpha} = 1 + (2\gamma-\alpha)\lambda_1^{i,j}a_2z+$$

$$(3\gamma-\alpha)\lambda_2^{i,j}a_3 + ([2\gamma(\gamma-1)-2\gamma\alpha+\alpha]\lambda_1^{i,j} - \frac{\alpha(\alpha-1)}{2})\lambda_1^{i,j}a_2^2z^2+$$

$$\{(4\gamma-\alpha)\lambda_3^{i,j}a_4 + (6\gamma(\gamma-1)-5\alpha\gamma+2\alpha+\alpha(\alpha-1))\lambda_1^{i,j}\lambda_2a_2a_3$$

$$+(\gamma\alpha(\alpha-1)+[4\gamma(\gamma-1)-2\alpha\gamma(\gamma-1)-\alpha-\alpha(\alpha-1)-\frac{\alpha(\alpha-1)(\alpha-2)}{3!}]\lambda_1^{i,j})(\lambda^{i,j})_1^2a_2^3\}z^3+\ldots \tag{2.3}$$

where

$$\lambda_1^{i,j} = \frac{\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)} \tag{2.4}$$

$$\lambda_2^{i,j} = \frac{\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+3)\Gamma_q(j+1)} \tag{2.5}$$

$$\lambda_3^{i,j} = \frac{\Gamma_q(4+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+4)\Gamma_q(j+1)} \tag{2.6}$$

Similarly,

$$\left(H'_{i,j,q}g(u)\right)^{\gamma}\left(\frac{u}{H_{i,j,q}g(u)}\right)^{\alpha} = 1 + (2\gamma-\alpha)\lambda_1^{i,j}b_2u+$$

$$(3\gamma-\alpha)\lambda_2^{i,j}b_3 + ([2\gamma(\gamma-1)-2\gamma\alpha+\alpha]\lambda_1^{i,j} - \frac{\alpha(\alpha-1)}{2})\lambda_1^{i,j}b_2^2u^2+$$

$$\{(4\gamma-\alpha)\lambda_3^{i,j}b_4 + (6\gamma(\gamma-1)-5\alpha\gamma+2\alpha+\alpha(\alpha-1))\lambda_1^{i,j}\lambda_2^{i,j}b_2b_3$$

$$+(\gamma\alpha(\alpha-1)+[4\gamma(\gamma-1)-2\alpha\gamma(\gamma-1)-\alpha-\alpha(\alpha-1)-\frac{\alpha(\alpha-1)(\alpha-2)}{3!}]\lambda_1^{i,j})(\lambda^{i,j})_1^2b_2^3\}z^3+\ldots \tag{2.7}$$

Next, define $p(z) = \frac{1+\omega}{1-\omega} = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots$ and
$q(u) = \frac{1+\phi(u)}{1-\phi(u)} = 1 + d_1u + d_2u^2 + d_3u^3 + \ldots$
This simply implies that $\omega(z) = \frac{p(z)-1}{p(z)=1}$.

Further computation gives: $\omega(z) = \frac{c_1z}{2} + \frac{1}{2}\left(c_2 - \frac{c_2^2}{2}\right)z^2 + \frac{1}{2}\left(c_3 - c_1c_2 + \frac{c_1^3}{4}\right)z^3 + \ldots$

Since $\Big(\omega(z)\Big)^2 = \frac{c_1^2}{4}z^2 + \frac{1}{2}\Big(c_1c_2 - \frac{c_1^3}{2}\Big)z^3 + \ldots$ and $\Big(\omega(z)\Big)^3 = \frac{1}{8}c_1^3z^3 + \ldots$
Then

$$1+\omega(z)-\frac{1}{3}(\omega(z))^3 = 1+\frac{c_1z}{2}+\frac{1}{2}\Big(c_2-\frac{c_1^2}{2}\Big)z^2+\frac{1}{2}\Big(c_3-c_1c_2+\frac{1}{6}c_1^3\Big)z^3+\ldots. \tag{2.8}$$

Similarly, we have

$$1+\phi(u)-\frac{1}{3}(\phi(u))^3 = 1+\frac{d_1u}{2}+\frac{1}{2}\Big(d_2-\frac{d_1^2}{2}\Big)u^2+\frac{1}{2}\Big(d_3-d_1d_2+\frac{1}{6}d_1^3\Big)u^3+\ldots \tag{2.9}$$

By comparing (2.3) with(2.8) and also (2.3) with (2.8) we have

$$(2\gamma-\alpha)\lambda_1a_2 = \frac{c_1}{2} \tag{2.10}$$

$$(3\gamma-\alpha)\lambda_2a_3+\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\lambda_1a_2^2 = \frac{c_2}{2}-\frac{c_1^2}{4} \tag{2.11}$$

$$(4\gamma-\alpha)\lambda_3a_4+(6\gamma(\gamma-1)-5\alpha\gamma+\alpha(\alpha-1))\lambda_1\lambda_2a_2a_3+$$

$$(\alpha\gamma(\alpha-1)+\Big[2\gamma(\gamma-1)(2-\alpha)+\alpha(2\gamma-\alpha)-\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Big]\lambda_1)\lambda_1^2a_2^3 = \frac{c_3}{2}-\frac{c_1c_2}{2}+\frac{c_1^3}{12} \tag{2.12}$$

$$(2\gamma-\alpha)\lambda_1b_2 = \frac{d_1}{2} \tag{2.13}$$

$$(3\gamma-\alpha)\lambda_2b_3+\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\lambda_1b_2^2 = \frac{d_2}{2}-\frac{d_1^2}{4} \tag{2.14}$$

$$(4\gamma-\alpha)\lambda_3b_4+(6\gamma(\gamma-1)-5\alpha\gamma+\alpha(\alpha-1))\lambda_1\lambda_2b_2b_3+$$

$$(\alpha\gamma(\alpha-1)+\Big[2\gamma(\gamma-1)(2-\alpha)+\alpha(2\gamma-\alpha)-\frac{\alpha(\alpha-1)(\alpha-2)}{3!}\Big]\lambda_1)\lambda_1^2b_2^3 = \frac{d_3}{2}-\frac{d_1d_2}{2}+\frac{d_1^3}{12} \tag{2.15}$$

From (2.10) and (2.13), it follows that

$$c_1 = -d_1 (since\ b_2 = -a_2) \tag{2.16}$$

Squaring (2.10) and (2.13), and then adding we have

$$(2\gamma-\alpha)^2\lambda_1^2a_2^2+(2\gamma-\alpha)^2\lambda_1^2a_2^2 = \frac{1}{4}(c_1^2+d_1^2) \tag{2.17}$$

This implies that

$$a_2^2 = \frac{(c_1^2+d_1^2)}{8(2\gamma-\alpha)^2\lambda_1^2} \tag{2.18}$$

Adding (2.10) and (2.14), we have

$$(3\gamma-\alpha)\lambda_2a_3+\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\lambda_1a_2^2+(3\gamma-\alpha)\lambda_2(2a_2^2-a_3)+$$

$$\Big[(2\gamma(\gamma-\alpha-1)+\alpha)\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big]\lambda_1 a_2^2=\frac{c_2}{2}-\frac{c_1^2}{4}+\frac{d_2}{2}-\frac{d_1^2}{4}$$

$$\Big\{2\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\lambda_1+2\lambda_2(3\gamma-\alpha)\Big\}a_2^2=\frac{1}{4}[2c_2-c_1^2+2d_2-d_1^2]$$

This further becomes

$$a_2^2=\frac{2(c_2+d_2)-(c_1^2+d_1^2)}{8\Big\{2\lambda_2(3\gamma-\alpha)+\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\Big\}}=$$
$$\frac{2|B_1|+2|B_1|+|B_1|^2+|B_1|^2}{8\Big\{2\lambda_2(3\gamma-\alpha)+\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\Big\}}. \tag{2.19}$$

Applying Lemma (1.1.5) we have

$$|a_2|\le\sqrt{\frac{|B_1|(|B_1|+2)}{4\Big\{2\lambda_2(3\gamma-\alpha)+\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\Big\}}}$$

Also, subtractin (2.14) from (2.10), then

$$(3\gamma-\alpha)\lambda_2 a_3+\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\lambda_1 a_2^2-$$

$$(3\gamma-\alpha)\lambda_2(2a_2^2-a_3)-\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\Big)\lambda_1 a_2^2=\frac{c_2}{2}-\frac{d_2}{2}-\frac{c_1^2}{4}+\frac{d_1^2}{4}$$

This simply becomes

$$2(3\gamma-\alpha)\lambda_2 a_3-2(3\gamma-\alpha)\lambda_2 a_2^2=\frac{1}{4}\Big(2(c_2+d_2)-(c_1^2-d_1^2)\Big) \tag{2.20}$$

Using (2.18) in (2.20) and by further simplification we have

$$2(3\gamma-\alpha)\lambda_2 a_3=\frac{1}{4}\Big(2(c_2+d_2)-(c_1^2-d_1^2)\Big)+\frac{2(3\gamma-\alpha)\lambda_2(c_1^2+d_1^2)}{8(2\gamma-\alpha)^2\lambda_1^2}.$$

Appealing to Lemma (1.1.5) we have,

$$|a_3|\le\frac{4|B_1|+2|B_1|^2}{8(3\gamma-\alpha)\lambda_2}+\frac{4|B_1|^2(3\gamma-\alpha)\lambda_2}{8(2\gamma-\alpha)^2\lambda_1^2}=\frac{|B_1|(|B_1|+2)}{4(3\gamma-\alpha)\lambda_2}+\frac{|B_1|^2(3\gamma-\alpha)\lambda_2}{2(2\gamma-\alpha)^2\lambda_1^2}$$

This completes the proof of Theorem 2.1.
Now, we cast attention on some noteworthy Corollaries which are obtained by specializing some values of the parameters i, j, α, γ in the Theorem 2.1.
Corollary 2.1.2: Setting $i=1$ in Theorem 2.1 then we have

$$|a_2|\le\sqrt{\frac{|B_1|(|B_1|+2)}{4\Big\{(3\gamma-\alpha)\Big(\frac{1+j}{3+j}\Big)+\Big([2\gamma(\gamma-\alpha-1)+\alpha]\Big(\frac{1+j}{2+j}\Big)-\frac{\alpha(\alpha-1)}{2}\Big)\Big(\frac{1+j}{2+j}\Big)\Big\}}}$$

and

$$|a_3| \leq \frac{|B_1|(|B_1|+2)}{4(3\gamma-\alpha)\left(\frac{1+j}{3+j}\right)} + \frac{|B_1|^2(3\gamma-\alpha)\left(\frac{1+j}{3+j}\right)}{2(2\gamma-\alpha)^2\left(\frac{1+j}{2+j}\right)^2}. \tag{2.21}$$

Corollary 2.1.3: Setting $i = 1$, $j = 0$ in Theorem 211 then we have

$$|a_2| \leq \sqrt{\frac{|B_1|(|B_1|+2)}{4\left\{\left(\frac{(3\gamma-\alpha)}{3}\right) + \frac{1}{2}\left([\gamma(\gamma-\alpha-1)+\alpha] - \frac{\alpha(\alpha-1)}{2}\right)\right\}}}$$

and

$$|a_3| \leq \frac{3|B_1|(|B_1|+2)}{4(3\gamma-\alpha)} + \frac{2|B_1|^2(3\gamma-\alpha)}{3(2\gamma-\alpha)^2}. \tag{2.22}$$

Corollary 2.1.4: Setting i=1, and $\alpha = 0$ in Theorem 2.1 then we have

$$|a_2| \leq \sqrt{\frac{|B_1|(|B_1|+2)}{\left\{(12\gamma)\left(\frac{1+j}{3+j}\right) + \left([8\gamma(\gamma-1)]\left(\frac{1+j}{2+j}\right)2\right)\left(\frac{1+j}{2+j}\right)\right\}}}$$

and

$$|a_3| \leq \frac{|B_1|(|B_1|+2)}{12(\gamma)\left(\frac{1+j}{3+j}\right)} + \frac{3|B_1|^2\gamma\left(\frac{1+j}{3+j}\right)}{8(\gamma)^2\left(\frac{1+j}{2+j}\right)^2}. \tag{2.23}$$

Corollary 2.1.5: Setting i=1, $\alpha = 0$ and $\gamma = 1$ in Theorem 2.1 then we have

$$|a_2| \leq \sqrt{\frac{|B_1|(|B_1|+2)}{12\left(\frac{1+j}{3+j}\right)}}$$

and

$$|a_3| \leq \frac{|B_1|(|B_1|+2)}{12\left(\frac{1+j}{3+j}\right)} + \frac{3|B_1|^2\lambda_2}{8\left(\frac{1+j}{2+j}\right)^2}. \tag{2.24}$$

Corollary 2.1.6: Setting $\alpha = 0$, $\gamma = i = 1$ in Theorem 2.1 then we have

$$|a_2| \leq \sqrt{\frac{|B_1|(|B_1|+2)(3+j)}{12(1+j)}}$$

and

$$|a_3| \leq \frac{|B_1|(|B_1|+2)(3+j)}{12(1+j)} + \frac{3|B_1|^2(2+j)^2}{8(3+j)(1+j)}. \tag{2.25}$$

Corollary 2.1.7: Setting $\alpha = j = 0$, $\gamma = i = 1$ then we have

$$|a_2| \leq \sqrt{\frac{|B_1|(|B_1|+2)}{4}}$$

and

$$|a_3| \leq \frac{|B_1|(|B_1|+2)}{4} + \frac{|B_1|^2}{2}. \tag{2.26}$$

3. Fekete-Szegö inequalities for the class $H_q(i,j,\alpha,\gamma)$

This section is concerned with the Fekete-Szegö inequalities for the class $H_q(i,j,\alpha,\gamma)$. The Taylor-Maclaurin coeficients $|a_2|$ $|a_3|$ of the function $f \in H_q(i,j,\alpha,\gamma$ obtained in Theorem 2.1 were used to investigate the Fekete-Szegö functional.

Theorem 3.1: Let f be of the form given in (1.1) that belongs to the class $H_q(i,j,\alpha,\gamma)$, then

$$|a_3 - va_2^2| \leq \begin{cases} \dfrac{(4p\lambda_2^{i,j}(\lambda_1^{i,j})^2-(q(\lambda_1^{i,j})^2-r)(\lambda_2^{i,j})^2)\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1^{i.j}-\frac{\alpha(\alpha-1)}{2}\right)\lambda_1^{i,j}+2\lambda_2^{i,j}p_1\right]-p\Omega[2s-t]}{2p\lambda_2^{i,j}(\lambda_1^{i,j})^2[[(2\gamma(\gamma-\alpha-1)+\alpha)]\lambda_1^{i,j}+2\lambda_2^{i,j}p_1]}, \\ \quad if \quad v \leq k_1^{i,j,\alpha,\gamma} \\ 2, \quad if \quad k_1^{i,j,\alpha,\gamma} \leq v \leq k_2^{i,j,\alpha,\gamma} \\ \dfrac{((q(\lambda_1^{i,j})^2-r(\lambda_2^{i,j})^2)-4p\lambda_2^{i,j}(\lambda_1^{i,j})^2)\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1-\frac{\alpha(\alpha-1)}{2}\right)\lambda_1^{i,j}+2\lambda_2^{i,j}p_1\right]+p\Omega[2s-t]}{2p\lambda_2^{i,j}(\lambda_1^{i,j})^2[[(2\gamma(\gamma-\alpha-1)+\alpha)]\lambda_1^{i,j}+2\lambda_2^{i,j}p_1]}, \\ \quad if \quad v \geq k_2^{i,j,\alpha,\gamma} \end{cases} \tag{3.1}$$

where $\Omega^{i,j} = \mu\lambda_2\lambda_1^2$, $p = (3\gamma-\alpha)(2\gamma-\alpha)^2$, $p_1 = 3\gamma-\alpha$, $q = c_1^2 - d_1^2$, $r = c_1^2 + d_1^2$, $s = (c_2 + d_2)$, $t = (c_1^2 + d_1^2)$

$$k_1^{i,j,\alpha,\gamma} = \frac{-(q(\lambda_1^{i.j})^2-r(\lambda_2^{i,j})^2)\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1^{i,j}-\frac{\alpha(\alpha-1)}{2}\right)\lambda_1^{i,j}+2\lambda_2^{i,j}p_1\right]}{p\lambda_2^{i,j}(\lambda_1^{i,j})^2[2s-t]},$$

$$k_2^{i,j,\alpha,\gamma} = \frac{(8p\lambda_2^{i,j}(\lambda_1^{i,j})^2-(q(\lambda_1^{i,j})^2-r(\lambda_2^{i,j})^2))\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1^{i,j}-\frac{\alpha(\alpha-1)}{2}\right)\lambda_1^{i,j}+2\lambda_2^{i.j}p_1\right]}{p\lambda_2^{i,j}(\lambda_1^{i,j})^2[2s-t]}$$

$\lambda_1 = \frac{\Gamma_q(2+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+2)\Gamma_q(j+1)}$, $\lambda_2 = \frac{\Gamma_q(3+j)\Gamma_q(i+j+1)}{\Gamma_q(i+j+3)\Gamma_q(j+1)}$.

Proof.: Assuming Theorem 2.1 holds. Then we have

$$a_3 = \frac{1}{8}\Big\{\frac{2(c_2+d_2)-(c_1^2-d_1^2)}{(3\gamma-\alpha)\lambda_2} + \frac{(c_2^2+d_1^2)\lambda_2}{(2\gamma-\alpha)^2\lambda_1^2}\Big\}. \tag{3.2}$$

$$a_2^2 = \frac{1}{4}\Big\{\frac{2(c_2+d_2)-(c_1^2+d_1^2)}{2([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1 - \frac{\alpha(\alpha-1)}{2})\lambda_1 + 2\lambda_2(3\gamma-\alpha)}\Big\}. \tag{3.3}$$

With (3.2) and (3.3)we obtain the equation of the form

$$a_3 - va_2^2 = \frac{1}{4}\frac{c_2+d_2}{(3\gamma-\alpha)\lambda_2} + \frac{(c_1^2-d_1^2)(2\gamma-\alpha)^2\Psi\lambda_1^2 + (c_1^2+d_1^2)(3\gamma-\alpha)\lambda_2^2\Psi - \Omega}{8(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2\Psi} \tag{3.4}$$

where

$$\Psi = \Big[\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1 - \frac{\alpha(\alpha-1)}{2}\Big)\lambda_1 + 2\lambda_2(3\gamma-\alpha)\Big]$$
$$\Omega = \mu(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2[2(c_2+d_2)-(c_1^2+d_1^2)]$$

Equivalently it can be re-express in the form:

$$a_3 - va_2^2 = \frac{1}{4}\frac{c_2+d_2}{(3\gamma-\alpha)\lambda_2} - \frac{((c_1^2-d_1^2)(2\gamma-\alpha)^2\Psi\lambda_1^2 - (c_1^2+d_1^2)(3\gamma-\alpha)\lambda_2^2)\Psi + \Omega}{8(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2\Psi}$$

where

$$\Psi = \Big[\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1 - \frac{\alpha(\alpha-1)}{2}\Big)\lambda_1 + 2\lambda_2(3\gamma-\alpha)\Big]$$
$$\Omega = \mu(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2[2(c_2+d_2)-(c_1^2+d_1^2)]$$

Here

$$v = \frac{((c_1^2-d_1^2)(2\gamma-\alpha)^2\Psi\lambda_1^2 - (c_1^2+d_1^2)(3\gamma-\alpha)\lambda_2^2)\Psi + \Omega}{8(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2\Psi} \tag{3.5}$$

where

$$\Psi = \Big[\Big([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_1 - \frac{\alpha(\alpha-1)}{2}\Big)\lambda_1 + 2\lambda_2(3\gamma-\alpha)\Big]$$
$$\Omega = \mu(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2[2(c_2+d_2)-(c_1^2+d_1^2)]$$

Now let $v \leq 0$ then we have

$$\mu \leq \frac{-((c_1^2-d_1^2)(2\gamma-\alpha)^2\Psi\lambda_1^2 - (c_1^2+d_1^2)(3\gamma-\alpha)\lambda_2^2)\Psi}{(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2[2(c_2+d_2)-(c_1^2+d_1^2)]} \tag{3.6}$$

This then implies that we now have

$$\mu \leq k_1.$$

Also, let us set $v \geq 1$, we then have

$$\mu \geq \frac{8(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2 - M}{(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2[2(c_2+d_2)-(c_1^2+d_1^2)]}. \tag{3.7}$$

Where $M = ((c_1^2-d_1^2)(2\gamma-\alpha)^2\Psi\lambda_1^2 - (c_1^2+d_1^2)(3\gamma-\alpha)\lambda_2^2)\Psi$.
This simply become the form

$$\mu \geq k_2.$$

newline From here we now seek $2-4v$ in the form:

$$-4v+2 = \frac{4(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2\Psi - M - \Omega}{2(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2\Psi}. \tag{3.8}$$

Where $M = (c_1^2-d_1^2)\lambda_1^2 - ((c_1^2-d_1^2)(2\gamma-\alpha)^2\Psi\lambda_1^2 - (c_1^2+d_1^2)(3\gamma-\alpha)\lambda_2^2)\Psi$
Similarly, we also seek $4v-2$ in the form:

$$4v-2 = \frac{((c_1^2-d_1^2)(2\gamma-\alpha)^2\Psi\lambda_1^2 - (c_1^2+d_1^2)(3\gamma-\alpha)\lambda_2^2)\Psi - 4(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2\Psi + \Omega}{2(3\gamma-\alpha)(2\gamma-\alpha)^2\lambda_2\lambda_1^2\Psi}. \tag{3.9}$$

By applying Lemm 1.1.7 with the appropriate use of eqns:(3.5),(3,6),(3,7),(3.8) and (3.9) we achieve the desire result.
After obtaining the reuslt of Fekete-Szegö functional in Theorem 3.1, the following corolllaries were investigated by varying the parameters involved so that special cases of the result in Theorem 3.1 could be pointed out.
Setting $i=1$ in Theorem 3.1, Corollary 3.1.2 follows accordingly.

Corollary 3.1.3 Let i= 1 and f be of the form given in (1,1) that belongs to the class $H_q(1,j,\alpha,\gamma)$, then

$$|a_3 - va_2^2| \leq \begin{cases} \dfrac{\lambda_a\lambda_b^4(4p\lambda_a^2\lambda_c-(q\lambda_c^3-r\lambda_a\lambda_b^2))\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_a-\frac{\alpha(\alpha-1)\lambda_b\lambda_c^2}{2}\right)+2\lambda_b^2 b\lambda_c p_1\right]-\mu p\lambda_1\lambda_b^2\lambda_c[2s-t]}{2p\lambda_a^2\lambda_c\left[\left(\lambda_a[2\gamma(\gamma-\alpha-1)+\alpha]-\frac{\alpha(\alpha-1)}{2}\lambda_a\right)+2p_1\lambda_b^2\right]}, \\ \quad if\ v \leq k_1^{(1,j,\alpha,\gamma)} \\ 2, if\ k_1^{(1,j,\alpha,\gamma)} \leq v \leq k_2^{(1,j,\alpha,\gamma)} \\ \dfrac{\lambda_a\lambda_b^4((q\lambda_a\lambda_c-r\lambda_a\lambda_b^2)-4p\lambda_b^2\lambda_c)\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_a-\frac{\alpha(\alpha-1)\lambda_b\lambda_c^2}{2}\right)+2\lambda_b^2\lambda_c p_1\right]+\mu p\lambda_a\lambda_b^2\lambda_c[2s-t]}{2p\lambda_a^2\lambda_c\left[\left(\lambda_a[2\gamma(\gamma-\alpha-1)+\alpha]-\frac{\alpha(\alpha-1)}{2}\lambda_a\right)+2\lambda_b^2 p_1\right]}, \\ \quad if\ v \geq k_2^{(1,j,\alpha,\gamma)} \end{cases} \tag{3.10}$$

where $k_1^{(1,j,\alpha,\gamma)} = \dfrac{-(q\lambda_a\lambda_c^2-r\lambda_a\lambda_b^4)\left[[2\gamma(\gamma-\alpha-1)+\alpha]\lambda_a\lambda_c-\frac{\alpha(\alpha-1)}{2}\lambda_b\lambda_c-2\lambda_b^2 p_1\right]}{p\lambda_a^2[2s-t]}$,

$$k_2^{(1,j,\alpha,\gamma)} = \frac{(8p\lambda_a^2\lambda_b\lambda_c-(q\lambda_a\lambda_c^2-r\lambda_a\lambda_b^2))\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]\lambda_a\lambda_c^2-\frac{\alpha(\alpha-1)}{2}\lambda_a\lambda_b\lambda_c^2\right)+2\lambda_b^2\lambda_c p_1\right]}{p\lambda_a^2\lambda_c[2s-t]}.$$

Setting $i = 1, j = 0$ in Theorem 3.1, Corolarry 3.1.4 follows accordingly.
Corollary 3.1.4 Let i =1, j= 0 and f be of the form given in (1.1) that belongs to the class $H_q(1, 0, \alpha, \gamma)$, then

$$|a_3 - va_2^2| \leq \begin{cases} \frac{((48p-(36q-16r)))\left[[2\gamma(\gamma-\alpha-1)+\alpha]12-12\alpha(\alpha-1)+32p_1\right]-\mu 4p[2s-t]}{2p\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]-\frac{\alpha(\alpha-1)}{2}\right)+8p_1\right]} & , if\ v \leq k_1^{i=1,j=o} \\ 2 & , if\ k_1^{i=1,j=o} \leq v \leq k_2^{i=1,j=o} \\ \frac{((36q-16r)-48p)\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]12-12\alpha(\alpha-1)\right)+32p_1\right]+\mu 4p[2s-t]}{2p_1\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]-\frac{\alpha(\alpha-1)}{2}\right)+8(3\gamma-\alpha)\right]} & , if\ v \geq k_2^{(i=1,j=0)} \end{cases} \quad (3.11)$$

where $k_1^{i=1,j=0} = \frac{-(9q-4r)\left[[2\gamma(\gamma-\alpha-1)+\alpha]3-\alpha(\alpha-1)3-8p_1\right]}{p[2s-t]}$,
$k_2^{i=1,j=0} = \frac{(48p-(9q-4r))\left[\left([2\gamma(\gamma-\alpha-1)+\alpha]3-\alpha(\alpha-1)3\right)+8p_1\right]}{p[2s-t]}$.

Setting $i = 1, j = \alpha = 0$ in Theorem 3.1, Corolarry 3.1.5 follows accordingly.
Corollary 3.1.5 Let $j = \alpha = 0, i = 1$ and f be of the form given in (1.1) that belongs to the class $H_q(1, j, 0, \gamma)$,. For $i = 1$ then

$$|a_3 - va_2^2| \leq \begin{cases} \frac{(144\gamma^3-(9q-4r))\left[[2(\gamma-1)]+8\right]-\mu\gamma^2[2s-t]}{\gamma^3\left[\left([(\gamma-1)]\right)+12\right]} & , if\ v \leq k_1^{i=1,j=\alpha=o} \\ 2 & , if\ k_1^{i=1,j=\alpha=o} \leq v \leq k_2^{i=1,j=\alpha=o} \\ \frac{((9q-4r)-144\gamma^3)\left[\left([2(\gamma-1)]\right)+8\right]+\mu[2s-t]}{(\gamma)^3\left[\left([(\gamma-1)]\right)+12\right]} & , if\ v \geq k_2^{i=1,j=\alpha=o} \end{cases} \quad (3.12)$$

where $k_1^{i=1,j=\alpha=o} = \frac{-(9q-4r)\left[[(\gamma-1)]-4\right]}{2\gamma^2[2s-t]}$,

$k_2^{i=1,j=\alpha=o} = \frac{(576\gamma^2\gamma^3-(9q-4r))\left[\left([3(\gamma-1)]\right)+4\right]}{2\gamma^2[2s-t]}$.

Setting $i = \gamma = 1, j = \alpha = 0$ in Theorem 3.1, Corolarry 3.1.6 follows accordingly.

Corollary 3.1.6 $j = \alpha = 0, \gamma = i = 1$ and f be of the form given in (1.1) that belongs to the class $H_q(1,0,0,1)$, then

$$|a_3 - va_2^2| \leq \begin{cases} \frac{(144-(9q-4r))[8]-\mu[2s-t]}{12} & , if\ v \leq k_1^{i=\gamma=1,j=\alpha=o} \\ 2 & , if\ k_1^{i=\gamma=1,j=\alpha=o} \leq v \leq k_2^{i=\gamma=1,j=\alpha=o} \\ \frac{((9q-4r)-144)[8]+\mu[2s-t]}{12} & , if\ v \geq k_{2(i=\gamma=1,j=\alpha=0)} \end{cases} \tag{3.13}$$

where $k_1^{i=\gamma=1,j=\alpha=o} = \frac{-(9q-4r)[2]}{[2s-t]}$,

$k_2^{i=\gamma=1,j=\alpha=o} = \frac{(24^2-(9q-4r))[2]}{[2s-t]}$.

4. **Conclusion**

Finally, in this work, the authors have successfully studied a new subclass of bi-univalent function involving q-integral operator associated with nephroid domain using the concept of subordination and bi-linear fractional principles. They obtained among others, new coefficient bounds, and sharp estimate bounds on the Fekete-Szego inequalities for functions belonging to the aforementioned subclass of bi-univalent function while some consequences of the results obtained follow as corollaries. For recent studies on the coefficient bounds and Fekete-Szego inequalities we refer to [3] , [31], [33], [32] and [34] among others.

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