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THE NEUTROSOPHIC HOM - GROUPS AND NEUTROSOPHIC HOM - SUBGROUPS I

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ABSTRACT. Hom-groups are the non-associative generalization of a group whose associativity and unitality are twisted by a compatible bijective map. The neutrosophic set is a powerful tool in dealing with incomplete, indeterminate and inconsistent data that exist in the real world. Neutrosophic set is characterized by the truth membership function in the set (T), indeterminacy membership function in the set (I) and falsity membership function in the set (F) where $0 \leq T + I + F \leq 3+$. In this work, efforts are intensified to clearly exemplify and create distinctions between certain structural (classical) groups , which are neutrosophic Hom groups and those which are not. Some examples of the neutrosophic Hom groups are also carefully constructed with elementary features and characterizations such as the subgroup series as well as their lattices. Finally , the certainty of the Lagranges theorem involving the subgroup of any finite neutrosophic Hom - group G(I)

1. INTRODUCTION

The Hom-Lie algebras including some of its generalizations was believed to have been introduced by Hartwig et.al. This was discovered in one of their seminal papers ([12] and [13]). In the paper, the deformations of Witt and Virasoro algebras was studied. Meanwhile, Jiang et.al. in [18] studied Hom-Lie algebras and Hom-Lie groups and the discoveries were presented as several results. In [19], Laurent-Gengouxa et.al. introduced the notion of a Hom-group and using their work as the foundation, Hassanzadeh [14] took the twisting map to be invertible and then used the invertibility of to study and establish several fundamental properties of the Hom-groups with interesting examples. Specifically, It was established that Lagranges theorem holds in any finite Hom-group and cosets were used to partition a Hom-group. In [15], Hassanzadeh further studied

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Hom-groups, their representations and homological algebras. Several interesting results were presented. Based on the work of Hassanzadeh in [14] , Liang et.al. in [20] extensively studied Hom-groups and Hom-group actions. In their work, new examples of Hom-groups were provided, further properties of Hom-groups were equally presented. Specifically, the first, second and third isomorphism theorems of Hom-groups were presented. They introduced the concept of Hom-group action and as an application of this concept, they established the first Sylow theorem for Hom-groups. The concept of neutrosophy and neutrosophic set was introduced by Smarandache in [21]. Neutrosophic set is the generalization of fuzzy set introduced by Zadeh in [25] and intuitionistic fuzzy set introduced by Atanassov in [11]. Neutrosophic set is a powerful tool in dealing with incomplete, indeterminate and inconsistent data that exist in the real world. Neutrosophic set is characterized by the truth membership function in the set (T), indeterminacy membership function in the set (I) and falsity membership function in the set (F) where $0T + I + F3+$. Vasantha Kandasamy and Smarandache in [24] introduced the concept neutrosophic algebraic structures. In [22], Smarandache introduced and studied $(T, I, F)-$ neutrosophic structures and presented their properties. In [23] Smarandache studied neutrosophic quadruple numbers, refined neutrosophic quadruple numbers, absorbance law, and the multiplication of neutrosophic quadruple numbers. Since the introduction of neutrosophic algebraic structure. In [24], many neutrosophic researchers have been working on different neutrosophic algebraic structures and hyper structures such as neutrosophic groups, neutrosophic rings, neutrosophic vector spaces, neutrosophic modules, neutrosophic hypergroups, neutrosophic hyperrings, neutrosophic hypervector spaces, neutrosophic hypermodules, etc. see ([1], [2], [3], [4] - [6], [8] - [9], [10], [16] and [17]). In [30] , some forms of morphism relationship that exists between a neutrosophic Hom-group $G(I)$ and a Hom-group $G \times G$ were studied.

Definition 1.1. (see [27]) A Hom-group consists of a set G together with a distinguished member 1 in G, a bijective set map $\alpha : G \to G$, a binary operation $\mu : G \times G \to G$, where these pieces of structure are subject to the following axioms: (1) The product map $\alpha : G \to G$ satisfies the Hom-associativity property $((g), (h, k)) = ((g, h), (k))$.

Explanations Since μ is the binary operation of the group, set $\mu =" \bigodot "$ for a general case. We have, the first axiom given by the following : For every $g, h, k, G, \alpha(g) \bigodot (h \bigodot k) = (g \bigodot h) \bigodot \alpha(k)$ We have the following examples :

- Let $\bigodot = +$, the ordinary addition, we have, the first axiom given by : For every $g, h, k, \in G, \alpha(g) + (h + k) = (g + h) + \alpha(k)$
- Let $\bigodot = x$, the ordinary multiplication, we have, that For every $g, h, k, \in G, \alpha(g)(hk) = (gh)\alpha(k)$

Here, μ indicates the binary operation under which the group G assumes its closure as a set. Hence, for simplicity sake, the multiplication sign μ may be omitted where necessary.

2. The map α is multiplicative $\alpha(gh) = \alpha(g)\alpha(h)$.

3. The element 1 is called unit and it satisfies the Hom-unitarity conditions $g1 = 1g = \alpha(g)$. 4. For every element $g \in G$, there exists an element $g1 \in G$ such that $gg1 = g1g = 1$. Without loss of generality, denote the Hom-group by the pair (G, α).

Definition 1.2. (see [3]) A Hom-group consists of a set G together with a distinguished member $1inG$, a bijective set map $\alpha : G \to G$, a binary operation $\mu : G \times G \to G$, where these pieces of structure are subject to the following axioms:
5. The product map $\alpha : G \to G$ satisfies the Hom-associativity property $\mu(\alpha(g), \mu(h, k)) = \mu(\mu(g, h), \alpha(k))$.

Explanations
Since μ is the binary operation of the group, set $\mu = \circledcirc$" for a general case. We have, the first axiom given by the following : For every $g, h, k, \in G, \alpha(g) \circledcirc (h \circledcirc k) = (g \circledcirc h) \circledcirc \alpha(k)$ We have the following examples :

- Let $\circledcirc = +$, the ordinary addition, we have, the first axiom given by : For every $g, h, k, \in G, \alpha(g) + (h + k) = (g + h) + \alpha(k)$
- Let $\circledcirc = x$, the ordinary multiplication, we have, that For every $g, h, k, \in G, \alpha(g)(hk) = (gh)\alpha(k)$

Here, μ indicates the binary operation under which the group G assumes its closure as a set. Hence, for simplicity sake, the multiplication sign μ may be omitted where necessary.
6. The map α is multiplicative $\alpha(gh) = \alpha(g)\alpha(h)$.
7. The element 1 is called unit and it satisfies the Hom-unitarity conditions $g1 = 1g = \alpha(g)$.
8. For every element $g \in G$, there exists an element $g1 \in G$ such that $gg1 = g1g = 1$. Without loss of generality, denote the Hom-group by the pair (G, α). Definition : A Hom-group (G, α) is called an idempotent Hom-group if $^2 =$

2. Construction of a neutrosophic Hom - group and Hom - subgroups

Definition 2.1. Let $H = (H, \mu) = (G, \alpha)$ be any Hom-group and let $\langle H \bigcup I \rangle = \{a + bI \ni a, b \in G\}$, and I is the indeterminate element . Define $NH(G) = (\langle H \bigcup I \rangle, \mu)$. Then, $NH(G)$ is called a neutrosophic Homgroup which is generated by H and I under the binary operation μI is called the neutrosophic element with the property $I^2 = I$. For an integer $n, n + I$, and nI are neutrosophic elements and $0.I = 0.I1$, the inverse of I is not defined and hence does not exist. Any neutrosophic Hom group $NH(G)$ in which $\forall a, b, \in NH(G), ab = ba$, is said to be commutative.

For the axioms, we have as follows: Recall that, $g, h, k \in NH(G), g = (g1 + g2I), h = (h1 + h2I)$, and $k = (k1 + k_2I)$
1. The product map $\alpha : NH(G) \to NH(G)$ satisfies the Hom-associativity property $\mu(\mu(g_1+g_2I), \mu((h_1+h_2I), (k_1+k_2I))) = \mu(\mu((g_1+g_2I), (h_1+h_2I)), \alpha(k_1+k_2I))$.

$\mu(\alpha(g_1)+\alpha(g_2)I, \mu((h_1+h_2I),(k_1+k_2I))) = \mu(\mu((g_1+g_2I),(h_1+h_2I)), \alpha(k_1)+\alpha(k_2)I)$.

2. The map α is multiplicative $\alpha((g_1+g_2I)(h_1+h_2I)) = \alpha(g_1+g_2I)\alpha((h_1+h_2I))$.

LHS $= \alpha(g_1h_1+g_1h_2I+g_2h_1I+g_2h_2I)\alpha(a_1+b1I) = \alpha(a_1)+\alpha(b_1)I, where\alpha(a_1), \alpha(b_1) \in G$

RHS $= \alpha(g_1)\alpha(h_1) + \alpha(g_1)\alpha(h_2)I + \alpha(g_2)\alpha(h_1)I + \alpha(g_2)\alpha(h_2)I$

$= \alpha(g_1h_1)+\alpha(g_1h_2)I+\alpha(g_2h_1)I+\alpha(g_2h_2)I = \alpha(g_1h_1+g_1h_2I+g_2h_1I+g_2h_2I) \equiv \alpha(a_1 + b_1I) = \alpha(a_1) + \alpha(b_1)I$

Hence, LHS = RHS

3. The element 1 is called unit and it satisfies the Hom-unitarity conditions $(g_1 + g_2I)1 = 1(g_1 + g_2I) = \alpha(g_1 + g_2I)$

4. For every element $(g_1+g_2I) \in NH(G)$, there exists an element $(g_1+g_2I)1 \in NH(G)$ such that $(g_1+g_2I)(g_1+g_2I)1 = (g_1+g_2I)1(g_1+g_2I) = 1$. For this to be valid, g_2 must be equal to zero since $I1$, the inverse of I is not defined. Hence, ths can only work for the neutrosophic Hom groups $NH(G)$ whose indeterminate is zero

Now , given that $x, y, z \in NH(G) = (\prec H \cup I \succ, ^*)$, where $x = (a + bI), y = (c + dI)$ and $z = (e + fI)$. Here , the action of on the element x can be given by : $\alpha(x)\alpha(a + bI) = \alpha(a) + \alpha(b)I$ Whence , $\alpha(x^*y) = \alpha([a + bI]^*[c + dI])$ which should be equal to $\alpha([a + bI]^*\alpha([c + dI])$

Now , LHS is given by $:= \alpha([a + bI]^*[c + dI]) = \alpha(a^*c, [a^*d + b^*c + b^*d]I) = \alpha(a^*c)+\alpha([a^*d+b^*c+b^*d]I) = \alpha(a^*c)+[\alpha(a^*d)+\alpha(b^*c)+\alpha(b^*d)]I = [\alpha(a)^*\alpha(c)+((\alpha(a)^*\alpha(d) + \alpha(b)^*\alpha(c) + \alpha(b)^*\alpha(d))I]$.

And the $RHS = \alpha([a + bI]^*\alpha([c + dI]) = (\alpha(a) + \alpha(b)I)^*(\alpha(c) + \alpha(d)I)$

$= [\alpha(a)^*\alpha(c) + ((\alpha(a)^*\alpha(d) + \alpha(b)^*\alpha(c) + \alpha(b)^*\alpha(d))I]$.

Hence , LHS = RHS

We now establish the composition of the elements of the neutrosophic Hom group as follows : First, we define $x^*y = (a^*c, [a^*d + b^*c + b^*d]I)$ By axiom one, $(\alpha(x), y^*z))$ must be equal to $(x^*y, \alpha(z))$. We have as follows : $LHS = (\alpha(x), y^*z)) = (\alpha(a+bI),(c+dI)^*(e+fI))) = (\alpha(a)+\alpha(b)I,(c^*e,[c^*f+d^*e+d^*f]I)$

$= [(\alpha(a) + \alpha(b)I]^*[c^*e + (c^*f + d^*e + d^*f)I]$

$= \alpha(a)^*c^*e+[\alpha(b)^*c^*e+\alpha(b)^*c^*f+\alpha(b)^*d^*e+\alpha(b)^*d^*f+\alpha(a)^*c^*f+\alpha(a)^*d^*e+\alpha(a)^*d^*f]I$...(1) And by calculations, the RHS is also given by

$:= a^*c^*\alpha(e)+[b^*c^*\alpha(e)+b^*c^*\alpha(f)+b^*d^*\alpha(e)+b^*d^*\alpha(f)+a^*c^*\alpha(f)+a^*d^*\alpha(e)+a^*d^*\alpha(f)]I$..(2) By using the axiom one , comparing . . . (1) and . . . (2) componentwise, we observe that LHS = RHS Just like in the case of the neutrosophic groups, we have the following proposition.

Proposition 2.2. *Suppose that NH(G) is any neutrosophic Hom - group. (i) NH(G) in general is not a group. (ii) NH(G) always contain a group.*

The proof of this proposition is exactly just as we had it for the neutrosophic group. (please see [26] for more details)

Definition 2.3. : Let NH(G) be a neutrosophic Hom - group. (i) A proper subset N(A) of NH(G) is said to be a neutrosophic Hom - subgroup of NH(G) if N(A) is a neutrosophic Hom group. This means that is N(A) contains a proper subset which is a Hom - group.

(ii) N(P) is said to be a pseudo neutrosophic Hom subgroup of the neutrosophic Hom group if it does not contain a proper subset which is a Hom - group.

Example 2.4. Consider the binary operation given on $\mathbb{R}$ by $\ni a \boxplus b = a + b + \frac{1}{2}(a+b)$

This definitely, is a non-associative operation. Hence, $(\mathbb{R}, \boxplus)$ is not a group. It is very easy to show that $(\mathbb{R}, \boxplus, \circ, \alpha)$ is a Hom- group, where $\alpha : \mathbb{R} \to \mathbb{R}$ can be defined as $\alpha(x) = frac12(x)$.

In what follows , we show that $N_H(\mathbb{R}) = (\langle H \cup I \rangle, ^*)$ is a neutrosophic Hom group.

By the associative axiom , we have that for every $x, y, z \in N_H(\mathbb{R})$, where for every $x \in N_H(\mathbb{R}), x = a + bI$, with $a, b \in \mathbb{R}$ and I is the neutrosophic indeterminate number. We show that : $\alpha(x) \boxplus (y \boxplus z) = (x \boxplus y \boxplus)\alpha(z)$

Proof. let $x = a + Ib, y = c + Id$ and $z = e + If$

$LHS = \alpha(x) \boxplus (y \boxplus z) = \frac{3}{2}(x) \boxplus (y + z + \frac{1}{2}(y+z)) = \frac{3}{2}(x) + \frac{3}{2}(y) + \frac{3}{2}(z) + \frac{3}{4}(x) + \frac{3}{4}(y) + \frac{3}{4}(z)$

$= \frac{9}{4}(x+y+z) = \frac{9}{4}(a+c+e+[b+d+f]I)$

$RHS = [x \boxplus y] \boxplus \frac{3}{2}(z) = [x + y + \frac{1}{2}(x+y)] \boxplus \frac{3}{2}(z) = \frac{3}{2}(x) + \frac{3}{2}(y) + \frac{3}{2}(z) + \frac{1}{2}(\frac{3}{2}(x) + \frac{3}{2}(y) + \frac{3}{2}(z))$

$= \frac{9}{4}(x+y+z) = \frac{9}{4}(a+c+e+[b+d+f]I), LHS = RHS$

□

Example 2.5. Let the binary operation be given on $\mathbb{R}$ by : $a \oplus b = 1(a+b)$ where $k \in \mathbb{C}$ This is also a non-associative operation. It follows that, $(\mathbb{R}, \oplus)$ is not a group. Hence, It can be proved that $(\mathbb{R}, \oplus, \circ, \alpha)$ is a Hom-group, where $\alpha : \mathbb{R} \to \mathbb{R}$ is defined as $\alpha(x) = 1(x)$. We have, by the first axiom for the neutrosophic Hom group, we are going to have that : $\alpha(x) \oplus (y \oplus z) = (x \oplus y \oplus)\alpha(z)$

proof : let $x = a + Ib, y = c + Id$ and $z = e + If$ $LHS = \alpha(x) \oplus (y \oplus z) = \frac{1}{k}(x) \oplus (\frac{1}{k}(y+z)) = \frac{1}{k}(\frac{1}{k}(x) + \frac{1}{k}(y+z)) = \frac{1}{k^2}(x+y+z) = \frac{1}{k^2}(a+c+e+[b+d+f]I)$

$RHS = [x \oplus y] \oplus \frac{1}{k}(z) = [\frac{1}{k}(x+y)] \oplus \frac{1}{k}(z) = \frac{1}{k}\frac{(\oplus(x+y)] + \oplus(z))}{k^2} = (\frac{1}{k^2}(x+y+z))$
$= \frac{1}{k^2}(a+c+e+[b+d+f]I) LHS = RHS$

Example 2.6. Suppose that $x^*y = n(\mid x \mid + \mid y \mid), forallx, y, \in \mathbb{R}$, and any $n \in \mathbb{N}$ It can be proved that $(\mathbb{R}, *, \circ, \alpha)$ is a Hom-group, where $\alpha : \mathbb{R} \to \mathbb{R}$ is defined as $\alpha(x) = n \mid x \mid$, and hence, for every $x, y, z \in N_H(\mathbb{R})$, where $x \in N_H(\mathbb{R}) \Longrightarrow x = \{a + bI\}$, with $a, b \in \mathbb{R}$ and I is the neutrosophic indeterminate number.

Proposition 2.7. *Let $x^*y = \bar{xy}$, the conjugate of the product of $x, y, \forall x, y, \in \mathbb{C}$, the set of complex numbers. Then, $(\mathbb{C}, ^*, 1, \alpha)$ is a Hom-group, where $\alpha : \mathbb{C} \to \mathbb{C}$ is defined as $\alpha(x) = \bar{x}$ and hence, for every $x, y, z \in NH(\mathbb{C})$, where $x \in NH(\mathbb{C})x = a + bI$, with $a, b \in \mathbb{C}$ and I is the neutrosophic indeterminate number, $NH(\mathbb{C})$ is a neutrosophic Homgroup*

Proof. Observe that $\alpha(x) = \alpha(a + Ib) = a \bar{+} Ib$. Now, since $a, b \in \mathbb{C}$, set $a = a_1 + ia_2, andb = b_1 + ib_2$.

We have , $\alpha(x) = \alpha(a_1 + ia_2 + I(b_1 + ib_2)) = \alpha((a_1 + Ib_1) + i(a_2 + Ib_2)) = ((a_1 + Ib_1) + i(a_2 + Ib_2)) = ((a_1 + Ib_1) + (a_2 + Ib_2)) = (a_1 + Ib_1) - i(a_2 + Ib_2)$ It is easy to show that : $\alpha(x)(^*y^*z) = (x^*y^*)\alpha(z)$ □

Proposition 2.8. *Given that* $\mathbb{Z}_\ltimes = (\mathbb{Z}_\ltimes, 0, +) = (Z, \alpha)$ *is a Hom-group and let* $\mathbb{Z}_\ltimes \bigcup I = a + bI : a, b \in \mathbb{Z}_\ltimes$ *(the set of integers modulo n), and I is the indeterminate element. Then,* $NH(\mathbb{Z}_\ltimes) = (\mathbb{Z}_\ltimes \bigcup, +)$ *is a neutrosophic Homgroup.*

Proof. Let $(a_1 + b_1I), (a_2 + b_2I)$, and $(a_3 + b_3I) \in NH(\mathbb{Z}_\ltimes)$. Then , we show that :

- $(a + bI) + 0 = 0 + (a + bI) = \alpha(a + bI)$, for $a, b \in \mathbb{Z}_\ltimes$
- $\alpha(a_1 + b_1I)^*((a_2 + b_2I)^*(a_3 + b_3I)) = ((a_1 + b_1I)^*(a_2 + b_2I))^*\alpha(a_3 + b_3I)$, for $a_i, b_i \in \mathbb{Z}_\ltimes$ and I , an indeterminate number.

For the first case , we have that : $\alpha(a + bI) = (a + bI) = (a + bI)$ For the second case , $LHS = \alpha(a_1+b_1I)^*((a_2+b_2I)^*(a_3+b_3I)) = \alpha(a_1+b_1I)^*((a_2+a_3)+(b_2+b_3)I)$
$= (a_1 + b_1I) + ((a_2 + a_3) + (b_2 + b_3)I) = ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)I)$
$RHS = ((a_1+b_1I)^*(a_2+b_2I))^*\alpha(a_3+b_3I) = ((a_1+a_2)+(b_1+b_2)I)+\alpha(a_3+b_3I)$
$= ((a_1 + a_2 + a_3) + (b_1 + b_2 + b_3)I)$ □

We have the following Cayley table for the $\mathbb{Z}_2(I)$ and $\mathbb{Z}_3(I)$ neutrosophic Hom-subgroup.

TABLE 1. Cayley table for the $\mathbb{Z}_2^*(1)$

$\oplus$	0	1	1	1 + I
0	0	1	1	1 + I
1	1	0	1 + I	1
1	1	1 +I	0	1
1 + I	1 + I	1	1	0

TABLE 2. Cayley table for the $\mathbb{Z}_2'^*(1)$

$\otimes$	0	1	1	1 + I
0	0	0	0	0
1	0	1	1	1 + I
1	0	1	1	0
1 + I	0	1 + I	0	1 + I

3. DISCUSSION / EXPLANATION

While tables 1 and 2 are both valid for the neutrosophic Hom Group, in tables 3 and 4, there exist two distinct identities in each case. Hence , we have the following proposition:

Proposition : For any $k \in \mathbb{N}$, it is vividly clear that the subgroup, $(\mathbb{Z}_k(I), +, (0, 0))$ is a neutrosophic HomGroup. For any $k \in \mathbb{N}$, it is vividly clearly that the subgroup clearly, $(\mathbb{Z}_k(I), \otimes, (0, 0))$ would only be a neutrosophic HomGroup if and only if the identities are not unique.

TABLE 3. Cayley table for the $\mathbb{Z}_3^*(1)$

⊕	0	1	2	I	2I	1+ I	2 + I	1 + 2I	2 + 2I
0	0	1	2	I	1+2I	1+I	2+I	1+2I	2+2I
1	1	2	0	1+I	1+2I	2+I	I	2+2I	2I
2	2	0	1	2+I	2+2I	I	1+I	2I	1+2I
I	I	1+I	2+I	2I	0	1+2I	2+2I	1	2
2I	2I	1+2I	2+2I	0	I	I	2	1+I	2+I
1+I	1+I	2+I	I	1+2I	I	2+2I	2I	2	0
2+I	2+I	I	1+I	2+2I	2	2I	1+2I	0	1
1 + 2I	1 + 2I	2+2I	2I	I	1+I	2	0	2+I	I
2+2I	2+2I	2I	1+2I	2	2+I	0	1	1	1+I

TABLE 4. Cayley table for the $\mathbb{Z}_3'^*(1)$

⊕	0	1	2	I	2I	1+ I	2 + I	1 + 2I	2 + 2I
0	0	0	0	0	0	0	0	0	0
1	0	1	2	1	2I	1+I	2+I	1+2I	2+2I
2	0	2	1	2I	I	2+2I	1+2I	2+I	1+I
I	0	I	2I	I	2I	2I	0	0	1
1+I	0	1+I	2+2I	2I	I	1	2+I	1+2I	1
2+I	0	2+I	1+2I	0	0	2+I	1+2I	2+I	1+2I
1+2I	0	1+2I	2+I	0	0	1+2I	2+1	1+2I	2+I
1 + I	0	1+I	2+2I	2I	I	I	2+I	1+2I	2
2+2I	0	2+2I	1+I	I	2I	2	1+2I	2+I	1

Next, we are going to ascertain the validity of the following cayley and the Latin tables for neutrosophic Homgroup in their respective binary operations.

TABLE 5. $G(I) = \{1 + I, a + I, b + I, c + I, d + I, e + I\}$

∗	1+I	a+I	b+I	c + I
1+I	1+I	a+I	b+I	c+I
a+I	a+I	1+I	c+I	b + I
b+I	b+I	c+I	1+I	a+I
c + I	c+I	b + I	a+I	1 + I

TABLE 6. $G(I) = \{1 + I, a + I, b + I, c + I, d + I, e + I\}$

♯	1+I	a+I	b+I	c + I
1	1+I	a+I	b+I	c+I
A	a+I	b+I	c+I	1 + I
B	b+I	c+I	1+I	a+I
C	c+I	1+ I	a+I	b + I

TABLE 7. $G(I) = \{1 + I, a + I, b + I, c + I, d + I, e + I\}$

$*$	1+I	a+I	b+I	c + I	d+I	e+I
1+I	1+I	a+I	b+I	c+I	d+I	e+I
a+I	a+I	b+I	1+I	e+I	c+I	d+I
b+I	b+I	1+I	a+I	d+I	e+I	c+I
c+I	c+I	d+ I	e+I	1+I	a+I	b+I
d+I	d+I	e+ I	c+I	b+I	1+I	a+I
e+I	e+I	c+ I	d+I	a+I	b+I	1+I

TABLE 8. The Latin square for : $G(I) = \{1 + I, a + I, b + I, c + I, d + I\}$

$\blacksquare$	1+I	a+I	b+I	c + I	d+I
1+I	1+I	a+I	b+I	c+I	d+I
a+I	a+I	1+I	d+I	b+I	c+I
b+I	b+I	c+I	1+I	d+I	a+I
c+I	c+I	d+ I	a+I	1+I	b+I
d+I	d+I	b+ I	c+I	a+I	1+I

To prove that the cayley tables are valid as neutrosophic Hom groups, it is necessary to prove that :

(1) $\alpha(x + I) = (x + I)(1 + I) = (1 + I)(x + I) = (x + I)$, and sufficient to prove that :
(2) $\alpha(a + I)[(b + I)(c + I)] = [(a + I)(b + I)]\alpha(c + I)$

Now, from tables 3, 4, and 5 , we have the first condition as : $\alpha(x + I) = (x + I)(1 + I) = (1 + I)(x + I) = x + I + xI + I^2$
$= x(1 + I) + I = x + I$ (since $(1 + I)$ is the identity) for every $(x + I) \in G(I)$

The second case (the sufficient condition) is obviously true. In table 6, even though the necessary condition is true. i.e. $\alpha(x+I) = (x+I)(1+I) = (1+I)(x+I) = (x+I)$, for every $(x+I) \in G(I)$, it is not true that $\alpha(a+I)[(b+I)(c+I)] = [(a + I)(b + I)]\alpha(c + I)$. Hence, the cayley table 6 does not form a neutrosophic Homgroup.

Proposition 3.1. *Suppose that $N_H(V)$ is a nonempty proper subset of a neutrosophic Hom - group $(N_H(G),^*)$.$N_H(V)$ is a neutrosophic Hom - subgroup of $N_H(G)$ if and only if the following conditions hold:*

(1) *every $(a_1, a_2I), (b_1, b_2I), \in N_H(V)(a_1, a_2I)^*(b_1, b_2I) \in N_H(V) \forall a, b \in N_H(V)$*
(2) *There exists a proper subset S of $N_H(V)$ such that $(S, *)$ is a neutrosophic Hom - group.*

Definition 3.2. Let G be any group and x, any element in G. Then, the set $\langle xrangle = \{x^k : k \in \mathbb{Z}\}$ is a subgroup of G. Furthermore, $\langle a\rangle$ is the smallest subgroup of G containing a. By the foregoing definition, if the notation is used, as in the case of the integers under addition, we write $\langle a\rangle = \{nan \in \mathbb{Z}\}$.

Definition 3.3. Let $(N_H(G),^*)$ be a neutrosophic Hom- group . For $(a + bI) \in (N_H(G),^*)$, we call $\langle(a + bI)\rangle$ the cyclic neutrosophic Hom - subgroup generated by $(a+bI)$ if $(a+bI)$ is a single element of $(N_H(G),^*)$, and we write $(N_H(G),^*) = \langle(a+bI)\rangle$, then $(N_H(G),^*)$ is a neutrosophic Hom cyclic group.

Example 3.4. Consider a neutrosophic Hom - subgroup : $A = \{0, I\} \subset \{0, 1, I, 1+I\} = (_2(I), +, (0, 0)) = (N_H(G),^*)$ Clearly, $\{I\}$ generates $(NH(G),^*)$. Hence, we can write $(NH(G),^*) = \{\langle I\rangle\}$. This means that $\langle I\rangle$ generates $(N_H(G),^*)$

Proposition 3.5. *Every neutrosophic Hom - cyclic group is abelian*

Proposition 3.6. *Every neutrosophic Hom - subgroup of a neutrosophic Hom - cyclic group is cyclic.*

Proposition 3.7. *Let $N(H)$ be a nonempty proper subset of a neutrosophic Hom - group $((N_H(G),^*),^*).N_H(V)$ is a pseudo neutrosophic Hom - subgroup of $(N_H(G),^*)$ if and only if the following conditions hold:*

(1) *$a, b \in N_H(V)$ implies that $a^*b \in N_H(V)\forall a, b \in N_H(V)$*
(2) *$N_H(V)$ does not contain a proper subset A such that $(A,^*)$ is a group.*

Definition 3.8. Let $N_H(V)$ and $N_H(U)$ be any two neutrosophic Hom - subgroups of a neutrosophic Hom - group $N_H(G)$. The product of $N_H(V)$ and $N_H(U)$ denoted by $N_H(V).N_H(U)$ is the set $N_H(V).N_H(U) = \{vu : v \in N_H(V), u \in N_H(U)\}$.

Definition 3.9. Let $N_H(V)$ and $N_H(U)$ be any two pseudo neutrosophic Hom - subgroups of a neutrosophic group $N_H(G)$. The product of $N_H(V)$ and $N_H(U)$ denoted by $N_H(V).N_H(U)$ is the set $N_H(V).N_H(U) = \{vu : v \in N_H(V), u \in N_H(U)\}$.

Definition 3.10. Let $N_H(G)$ be a neutrosophic Hom - group. The order of $(N_H(G),^*)$ denoted by $\circ((N_H(G),^*)$ or $|(N_H(G),^*)|$ is the number of distinct elements in $(N_H(G),^*)$. Suppose that the order $\circ((N_H(G),^*))$ of the neutrosophic Hom - group $(NH(G),^*)$ is finite. Then , $(N_H(G),^*)$ is known as a finite neutrosophic Hom - group. and infinite neutrosophic Hom - group if $\circ((N_H(G),^*)) \not< \infty$

SUBGROUPS OF NEUTROSOPHIC HOM - GROUP

Consider the $(\mathbb{Z}_2(I), +, (0, 0)), (\mathbb{Z}_3(I), +, (0, 0)) \in (\mathbb{Z}_n(I), +, (0, 0)) \subset (N_H(G),^*)$ There exist the following chains of subgroups : $\{0\} = \langle 0\rangle \subset (\mathbb{Z}_2(I), +, (0, 0))$

$\{0\} \subset \{0, 1\} = \langle 1\rangle \subset \{0, 1, I, 1 + I\} = \langle 1, I\rangle \subset (\mathbb{Z}_2(I), +, (0, 0))$

$\{0\} \subset \{0, I\} = \langle I\rangle \subset \{0, 1, I, 1 + I\} = \langle 1, I\rangle \subset (\mathbb{Z}_2(I), +, (0, 0))$

$\{0\} \subset \{0, 1 + I\} = \langle 1 + I\rangle \subset \{0, 1, I, 1 + I\} = \langle 1, I\rangle \subset (\mathbb{Z}_2(I), +, (0, 0))$

$\{0\} = \langle 0\rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$

$\{0\} \subset \{0, 1\} = \langle 1\rangle \subset \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} = \langle 1, I\rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$

$\{0\} \subset \{0, 1, 2\} = \langle 1\rangle \subset \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} = \langle 1, I\rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$

$\{0\} \subset \{0, I, 2I\} = \langle 1\rangle \subset \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} = \langle 1, I\rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$

$\{0\} \subset \{0, 1 + I, 2 + 2I\} = \langle 1 + I\rangle \subset \{0, 1, 2, I, 2I, 1 + I, 1 + 2I, 2 + I, 2 + 2I\} = \langle 1, I\rangle \subset (\mathbb{Z}_3(I), +, (0, 0))$

$\{0\} \subset \{0, 2+I, 2+2I\} = \langle 1+2I \rangle = \langle 2+I \rangle \subset \{0, 1, 2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\} = \langle 1, I \rangle \subset (\mathbb{Z}_3(I), +, (0,0))$

Proposition 3.11. *Let $M(I)$ be the maximal subgroup for $(\mathbb{Z}_n(I), +, (0,0)) \subset (NH(G),^*)$. Then , there exist central normal series for the neutrosophic Hom - Group series ending in $M(I)$ series ending in $(\mathbb{Z}_n(I), +, (0,0)) \subset (NH(G),^*)$*

Theorem 3.12. *Let $(\mathbb{Z}_n(I), +, (0,0))$ be a neutrosophic Hom group then :*

(1) *The order of $(\mathbb{Z}_k(I), +, (0,0))$ denoted by: $|\mathbb{Z}_k(I)| = k^2 for 0 \leq k \leq n \in N$*
(2) *The number of the distinct neutrosophic Hom subgroup is a function of $(2k+1)$*
(3) *The number of maximal subgroups of $(\mathbb{Z}_k(I), +, (0,0)$ is given by : $(k-1)$*

Remark 3.13. The usual method of mathematical induction can be invoked for the proof of $(i.)$

The subgroup lattices of the neutrosophic Hom group

Theorem 3.14. *Let $NH(G)$ be a neutrosophic Hom - group and F the family of all subgroups of $NH(G)$. Define a relation $\mathbb{R}$ on F by $NH(U)\mathbb{R}NH(V)$ if $NH(U) \leq NH(V)$. Then, F form a neutrosophic Hom - subgroup lattice for the group $NH(G)$.*

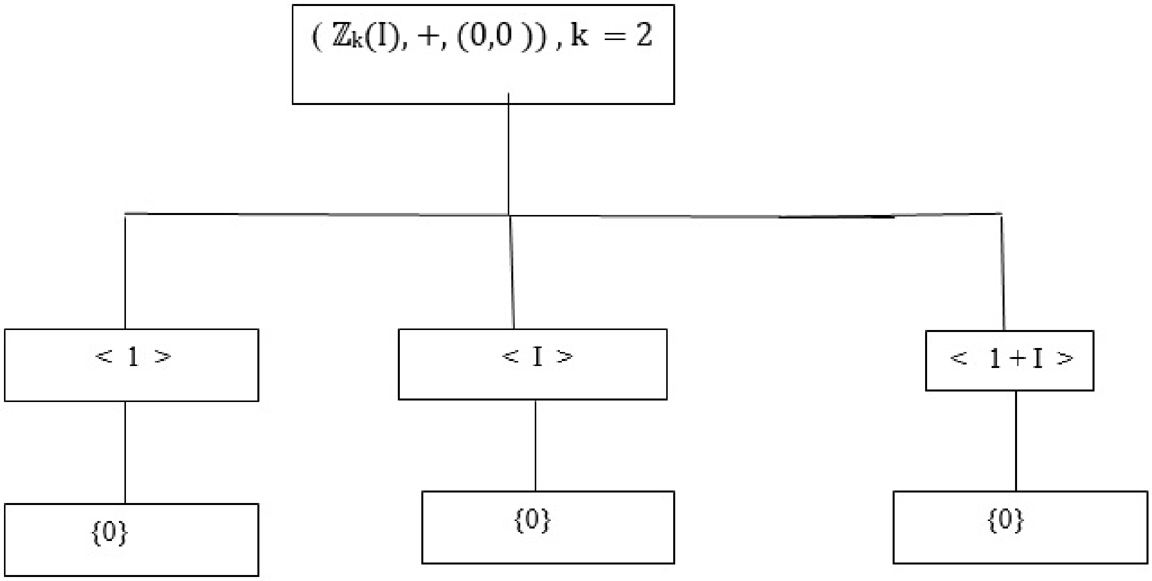
Lemma 3.15. *Let $NH(G)$ be a neutrosophic Hom - group and $NH(V)$, a neutrosophic Hom - subgroup of $NH(G)$. Define the relation $\mathbb{R}$ on $NH(G)$ by $a\mathbb{R}b$ if $ab^{-1} \in NH(V)$. Then $\mathbb{R}$ is an equivalence of relation on $NH(G)$*

Proof.
(1) For $a \in NH(G), aa^{-1} = e \in NH(G)$, since $NH(V)$ is a neutrosophic Hom - subgroup. So, $a\mathbb{R}a$. i.e $\mathbb{R}$ is reflexive.
(2) Also, suppose $a\mathbb{R}b$ i.e $ab^{-1} \in NH(G)$, then $(ab^{-1})^{-1} = ba^{-1} \in NH(V)$. So, $b\mathbb{R}a$. This is symmetric property
(3) Suppose $a\mathbb{R}b$ and $b\mathbb{R}c$, then $ab^{-1} \in NH(V), bc^{-1} \in NH(V), ac^{-1} = ab^{-1}bc^{-1} \in NH(V)$ since closure property holds in $NH(V)$. So $a\mathbb{R}c$ i.e $\mathbb{R}$ is transitive.

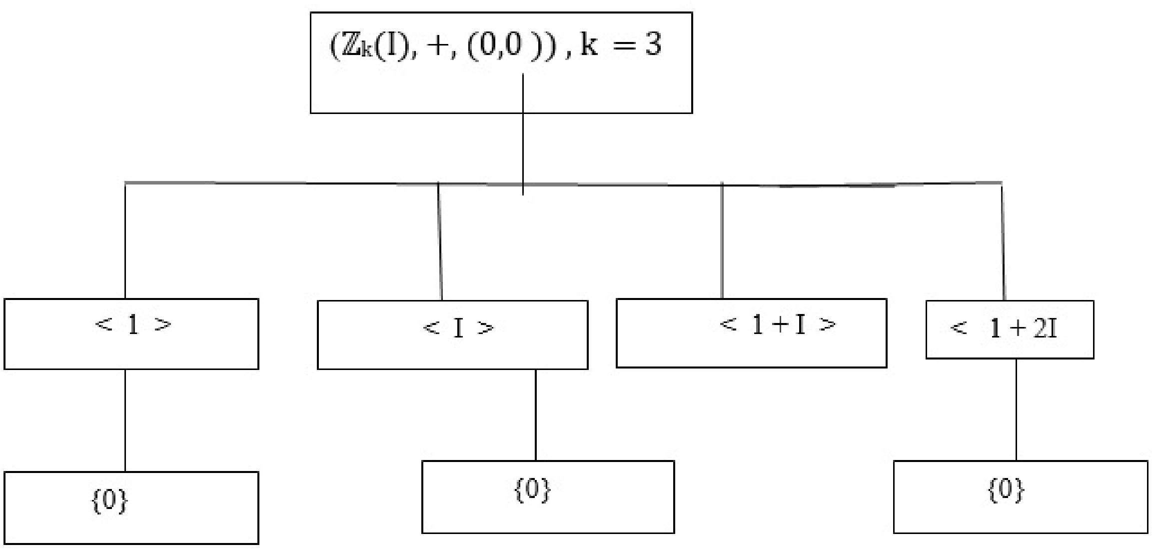
Hence $\mathbb{R}$ is an equivalence relation. □

Let $G(I)$ be a finite abelian neutrosophic Hom - group and $|G(I)| = n = P_1^{r_1} P_2^{r_2} ... P_k^{r_k}$, the decomposition of its order into prime power factors. If $G(I) = G(I)p_1 \oplus G(I)p_2 \oplus ... \oplus G(I)p_k$ is the corresponding primary decomposition, then, denoting by $L(G(I))$ the subgroup lattice of $G(I), L(G(I)) \simeq L(G(I)p_1) \times L(G(I)p_2) \times ... \times L(G(I)p_k)$, the direct product of the corresponding subgroup lattices (see [28], [29]). We denote by $N(G(I))$ the number of subgroups of the group G. Hence $N(G(I)) = \prod_{i=1}^{k} N(G(I)_{p_i})$ and our counting problem is reduced to p-groups.

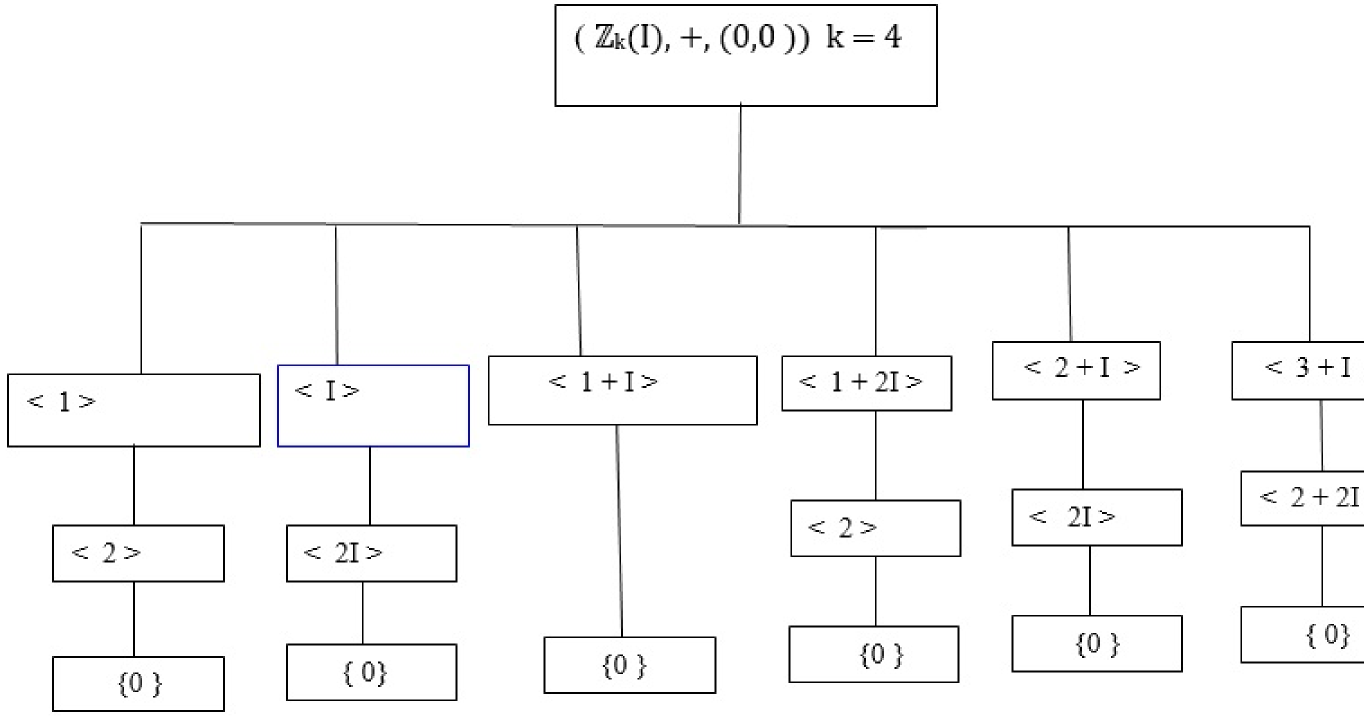
The subgroup lattices of the neutrosophic Hom group $(\mathbb{Z}_k(I), +, (0,0)) for 0 \leq k \leq n \in \mathbb{N}$ [28].



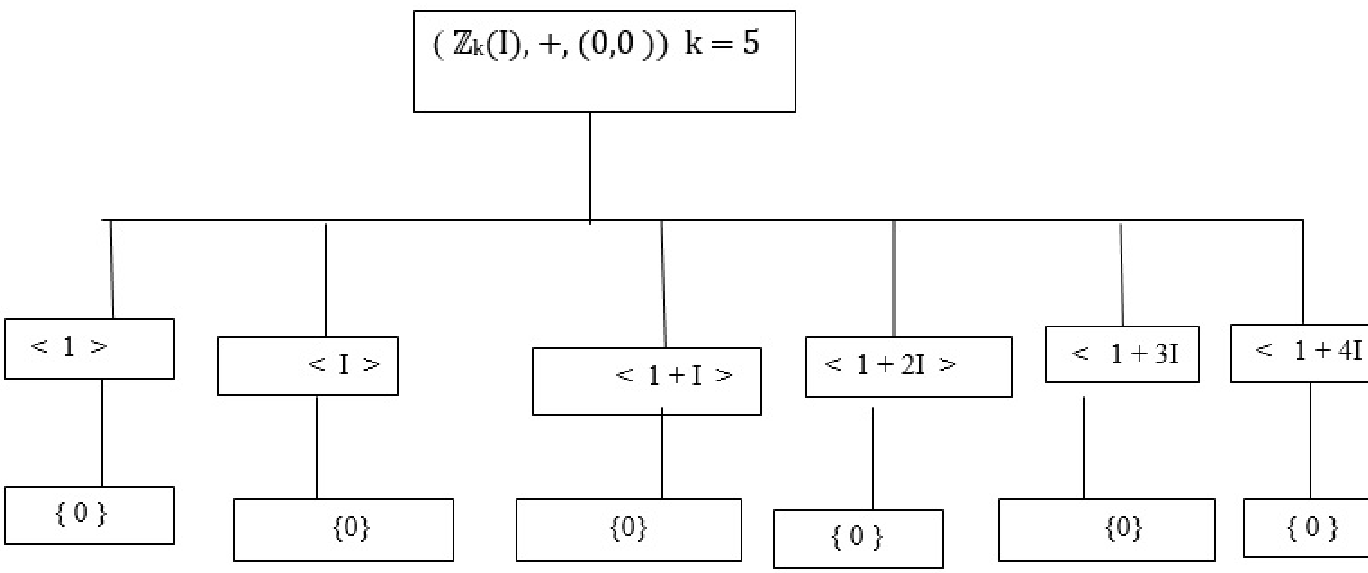
Total number of subgroups = 5



Total number of subgroups = 6



Total number of subgroups = 11



Total number of subgroups = 8

By the Lagranges theorem if H is a subgroup of a finite neutrosophic Hom - group $G(I)$, then the order of $H(I)$ divides the order of $G(I)$ and $|G(I)| = |G(I)/H(I)|$.

Remark 3.16. The converse of this assertion is not true. This can easily be confirmed by the example of the neutrosophic Hom - subgroup $\{(\mathbb{Z}_k(I), +, (0,0)) \ni k = 4\}$. The neutrosophic Hom - group possesses 16 elements but which has factors $\{1, 2, 4, 8, 16\}$. Whereas there exist subgroups of other orders of the factors of 16 but with the exception of the neutrosophic Hom - subgroup of order 8.

4. Conclusion

So far , we have been able to exemplify with clarity, and create distinctions between certain structural (classical) groups , which are neutrosophic Hom groups and those which are not. Some examples of the neutrosophic Hom groups have also been carefully constructed with elementary features and characterizations such as the subgroup series as well as their lattices. The series as well as the lattices have been successfully creatively constructed in order to classify more succinctly , the characteristic features evolving from those algebraic structures under consideration. Finally , the assertions of the usual , well known Lagranges theorem involving the subgroup of any finite groups has also been ascertained and fully confirmed for the neutrosophic Hom - group $G(I)$

SUGGESTIONS FOR FURTHER FUTURE RESEARCHES INVOLVING HOM - GROUPS , NEUTROSOPHIC HOM - GROUPS, AS WELL AS THEIR SUBGROUPS

One of the interesting things about this study is the creation of the subgroup series as well as their lattices. In the nearest future, the aspect needs to be more delved into. This is because, subgroup formations, their series, lattices, as well as their classifications and characterization in general is a versatile ground for breaking and reformations of material entities such that their reformations and escalations pay a great roles in material properties. These are very much applicable in the ground breaking discovery of nano-materials which areas are now very rich in some aspects of engineering , chemistry and other physical sciences.

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