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SOLUTION TO THE PROBLEM OF REGULAR MAPS ON BOUNDED LOJID ALGEBRAS

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Abstract. Lojid algebras are algebras of type (2,0). Regular maps play a vital role in the study of bounded lojid algebras. They are particularly useful in conceptualizing ideals and subalgebras of lojid algebras. It is therefore important to find a set system through which they can be studied. This paper primarily concerns the development of a set system that accounts for the study of all regular maps on bounded lojid algebras and also to solve the open problem that arose in the study of the theory of lojid algebras.

1. Introduction

An algebra of type (2, 0) is a non-empty set, having a constant element, on which is defined a binary operation such that certain axioms are satisfied. There are several of such algebras. BCI algebras were introduced by Imai and Iseki (1966). Iseki (1966) introduced the notion of BCK algebras which is a generalization of BCI algebras. These two algebras originated from two different sources. One of the motivations is based on set theory. In set theory, there are three most elementary and fundamental operations. They are union, intersection and set difference. If we consider these operations and their properties, then as a generalization of them, we have the notion of Boolean algebras. If we take both union and intersection, then as a general algebra, the notion of distributive lattices is obtained. Moreover, if we consider union or intersection alone, we have the notion of upper semilattices or lower semilattices. However, the set difference together with its properties had not been considered systematically before the works of Imai and Iseki. Another motivation is from propositional calculi. There are some systems which contain only the implicational functor among logical functors, such as the system of positive implicational calculus, the system of weak positive implicational calculus, BCK- systems and BCI- systems. Undoubtedly, there are common properties among these systems. It is well known that

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there are close relationships between the notion of set difference in set theory and implication functor in logical systems. Some questions were therefore raised. What are the most essential and fundamental properties of these relationships? Can there be a formulation of a general algebra from the above consideration? How would an axiom system be obtained that establishes a theory of the general algebra. It was while answering these pertinent questions that the notion of BCI algebras was birthed. Several generalizations of BCI algebras have been studied. For instance, the notion of BCH algebras was introduced in Hu and Li (1983). In Neggers and Kim (1999), d algebras were studied. In Kim and Kim (2007), the notion of BE algebras was introduced. Ideals and upper sets in BE algebras were investigated in Ahn and So (2008) and Ahn and So (2009). Precommutative algebras were studied in Kim *et. al*. Fenyves algebras were studied in Jaiyeola *et. al* (2018), Jaiyeola *et. al* (2020) and Ilojide *et. al* (2019). In Neggers *et. al*, Q algebras were introduced. Homomorphisms of Q algebras were studied in Ilojide (2021). Over the years, many algebras of type (2,0) have been studied mainly due to their diverse applications in coding theory. For instance, obic algebras were introduced in Ilojide (2019). In Ilojide (2020), torian algebras were studied. It was shown that the class of torian algebras is a wider class than the class of obic algebras. Ideals of torian algebras were investigated in Ilojide (2020). The dual and nuclei of ideals as well as congruences developed on ideals of torian algebras were studied. In Ilojide (2021), right distributive torian algebras were studied. Isomorphism Theorems of torian algebras were studied in Ilojide (2021). Kreb algebras were studied in Ilojide (2024). In Rezaei *et.al* (2020), fuzzy congruence relations on pseudo BE-algebras were studied. Positive implicative BE-filters of BE-algebras based on Lukasiewicz fuzzy sets were studied in Jun (2023). In Ilojide (2024), lojid algebras were introduced. Properties of classes of lojid algebras were investigated. Associates, commutants and regular maps were introduced and studied. Furthermore, congruences were introduced as a build up to the establishment of quotient lojid algebras. The notion of stabilizers in lojid algebras was studied in Ilojide (2024). It was shown that to every lojid stabilizer, there exists a corresponding lojid adjoint. The properties of stabilizers and their corresponding adjoints were also investigated in some classes of lojid algebras.

Regular maps play a vital role in the study of bounded lojid algebras. They are particularly useful in conceptualizing ideals and subalgebras of lojid algebras. It is therefore important to find a set system through which they can be studied. This paper primarily concerns the development of a set system that accounts for the study of all regular maps on bounded lojid algebras and also to solve the open problem that arose in the study of the theory of lojid algebras.

2. Preliminaries

In this section, some basic concepts necessary for a proper understanding of this paper are discussed.

Definition 2.1. [9]. An algebra $(X; *, 0)$ of type (2, 0) is called a semi-lojid algebra if $x * 0 = x$ for all $x \in X$.

Example 2.2. [9]. Let $X = \{0, 1, 2, 3\}$. Define a binary operation $*$ on X by the following table:

$*$	0	1	2	3
0	0	2	3	1
1	1	0	2	2
2	2	3	0	3
3	3	1	1	0

Table 1.

Then $(X; *, 0)$ is a semi-lojid algebra.

Definition 2.3. [9]. Let a be an element of a semi-lojid algebra $(X; *, 0)$. The mapping $L_a : X \to X$ given by $L_a(x) = a * x$ for all $x \in X$, is called a left translation on $(X; *, 0)$.
Similarly, the mapping $R_a : X \to X$ given by $R_a(x) = x * a$ for all $x \in X$, is called a right translation on $(X; *, 0)$.

Notice that not all the translations on the algebra in Example 2.2 are bijections. This leads us to the following definition.

Definition 2.4. [9]. A semi-lojid algebra $(X; *, 0)$ is called a lojid algebra if all its left and right translations are bijections.

Example 2.5. [9]. Let $X = \{0, 1, 2, 3, 4\}$. Define a binary operation $*$ on X by the following table:

$*$	0	1	2	3	4
0	0	1	3	4	2
1	1	0	4	2	3
2	2	4	1	3	0
3	3	2	0	1	4
4	4	3	2	0	1

TABLE 2.

Clearly, all the translations on the algebra in Example 2.5 are bijections. Hence, $(X; *, 0)$ is a lojid algebra.

Example 2.6. [9]. Let $X = \{0, 1, 2, 3\}$. Define a binary operation $*$ on X by the following table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

TABLE 3.

Clearly, all the translations on the algebra in Example 2.6 are bijections. Therefore, $(X; *, 0)$ is a lojid algebra.

Example 2.7. Let $X = \{0, 1, 2, 3\}$. Define a binary operation $*$ on X by the following table:

$*$	0	1	2	3
0	0	1	3	2
1	1	0	2	3
2	2	3	0	1
3	3	2	1	0

TABLE 4.

Clearly, all the translations on the algebra in Example 2.7 are bijections. Thus, $(X; *, 0)$ is a lojid algebra.
. We shall write X for a lojid algebra $(X; *, 0)$ unless there is the need to emphasize the binary operation and the constant element of $(X; *, 0)$.

Definition 2.8. [9]. A lojid algebra X is said to be bounded if $xL_0 = x$ for all $x \in X$.

Remark 2.9. [9]. Let a be an element of a lojid algebra X. Since L_a and R_a are bijections, they belong to the symmetric group $S(X)$ of all permutations of the set X. Now, consider the following sets:
$L(X) = \{L_a, L_a^{-1} : a \in X\}$,
$R(X) = \{R_a, R_a^{-1} : a \in X\}$ and
$T(X) = \{L_a, R_a, L_a^{-1}, R_a^{-1} : a \in X\}$.
The collection of permutations of X which are members of $L(X)$ (respectively $R(X), T(X)$) forms a subgroup of $S(X)$ called the left (respectively right, centre) group of $S(X)$. They are denoted by $L_S(X), R_S(X)$ and $T_S(X)$ respectively.

Definition 2.10. [9]. Let X be a lojid algebra. A mapping $\alpha \in T_S(X)$ is said to be regular if $\alpha(0) = 0$.

The collection of all regular mappings on a lojid algebra X is denoted by $D(X)$.

Proposition 2.11. [9]. *Let X be bounded lojid algebra. Then the set* $P = \{R_{(x,y)}, L_{(x,y)}, T_{(x)}\}$ *is contained in $D(X)$; where* $R_{(x,y)} = R_x R_y R^{-1}_{(x*y)}$, $L_{(x,y)} = L_x L_y L^{-1}_{(y*x)}$ *and* $T_{(x)} = R_x L_x^{-1}$ *for all* $x, y \in X$.

For more on lojid algebras, see [9].

3. Main Result

. In this section, we solve the open problem posed in Ilojide (2024). We prove that the set of all regular maps on bounded linear algebras can be finitely generated.

Theorem 3.1. *Let X be a bounded lojid algebra. The collection of all regular maps on X is generated by the set* $P = \{R_{(x,y)}, L_{(x,y)}, T_{(x)}\}$*; where* $R_{(x,y)} = R_x R_y R^{-1}_{(x*y)}$, $L_{(x,y)} = L_x L_y L^{-1}_{(y*x)}$ *and* $T_{(x)} = R_x L_x^{-1}$ *for all* $x, y \in X$.

Proof. Let X be a bounded lojid algebra. Let $< P >$ be the group generated by the set P; and let W be the collection of all regular maps on X. We claim that $< P > = W$. Since $R_{(x,y)}, L_{(x,y)}$ and $T_{(x)}$ are bijective translations, they belong to the subgroup V of the permutation group $S(X)$. Let $E = \{\theta \in V : \theta \in \{< P > R_{(\theta(0)}\}\}$. Let $\theta(0) = q$. Then for all $x \in X$, we have

$$R_{(x)}\theta(0) = qx \tag{1}$$

From the definition of E, it is seen that for every $\theta \in E$, there exists $\alpha \in < P >$ such that

$$\theta = R_{\theta(0)}\alpha = R_{(q)}\alpha \tag{2}$$

From (1) and (2) and the definition of $R_{(x,y)}$, we have $\theta R_{(x)} = \alpha R_{(q)} R_{(x)} = \alpha R_{(q,x)} R_{(\theta(0) R_{(x)}}$. Since $\alpha, R_{(q,x)} \in < P >$, then $\theta R_{(x)} \in < P > R_{(\theta(0) R_{(x)})}$; implying that $\theta R_{(x)} \in E$. Therefore, $EV \subseteq E \subseteq V \subseteq EV$. Hence $E = V$. Thus, every $\theta \in V$ is a member of $< P > R_{(\theta(0)}$. So, θ is regular. Similar argument shows that every $\theta \in V$ belongs to $< P > L_{(\theta(0)}$ and $< P > T_{(\theta(0)}$ and θ is regular. Therefore, we have $\theta \in < P > R_{(0)} = < P >, \theta \in < P > L_{(0)} = < P >$ and $\theta \in < P > T_{(0)} = < P >$. Therefore, $W \subseteq < P >$. It is obvious that $< P > \subseteq W$. Consequently, $W = < P >$ as required. □

4. Conclusion

We have considered and solved an open problem in the study of the theory of lojid algebras. We proved that the set $P = \{R_{(x,y)}, L_{(x,y)}, T_{(x)}\}$ generates all regular maps on a bounded lojid algebra; which is a major significant contribution to knowledge since by this discovery, all regular maps on a bounded lojid algebra can be sufficiently studied by studying the elements of P. Future research can be carried out by obtaining a finite set through which all regular maps on lojid algebras in general can be studied. Also, studies can be undertaken in developing set systems which will account for all regular maps in other algebras of type (2,0). Moreover, regular maps on lojid algebras have potential applications in Dyck's triangle groups, Riemann surfaces and algebraic curves.

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