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NUMERICAL APPROXIMATION OF OPTIMAL CONTROL PROBLEMS CONSTRAINED BY DYNAMIC EQUATIONS VIA GALERKIN METHOD

MATTHEW FOLORUNSHO AKINMUYISE*, AKINWUMI SHARIMAKIN, AND ILESANMI FAKUNLE

Abstract. The research investigates the application of the Galerkin method to optimal control problems constrained by coupled dynamic equations. These constrained problems are reformulated into unconstrained ones using the Hamiltonian approach, which facilitates the determination of boundary conditions for both the state and costate variables. By assuming a polynomial solution, the weighted and residual functions were derived. The Orthogonality of the product of these functions leads to the formation of a system of linear equations. Solving these equations provides the solution for the boundary conditions through direct substitution. This scheme was developed for the Lagrange form of optimal control problems to assess its accuracy in approximating exact solutions. Several optimal control problems with known exact solutions were solved using the proposed scheme, and the results were compared to evaluate its effectiveness.

1. Introduction

The objective of numerical optimization is to develop a numerical method that will be taylored towards certain classes of problems (since there is no unique method that can solve all optimization problems) in order to find an approximate solution to some physical problems by using different numerical techniques most especially when the analytic solutions are difficult or not available. Some physical problems that arise from optimal control problems (OCP) are expressed in terms of ordinary differential equations ([5]) due to its dynamic nature and many researchers have used numerous numbers of methods to solve this special class of problems. Akinmuyise et al [4] combined the classical method with the numerical algorithm of Euler by embeddeding each of the boundary conditions from

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the Hamitonian into the algorithm of Euler and allowed the system to undergo its iteraive process until the gradient norm of the objective fuction approaches specified criterion. The variational methods as described in [21] and referenced in [4] were semi-analytical, employing the calculus of variations to obtain two- or multi-point boundary value problems with special structures arising from the derivative of the Hamiltonian: $H(x(t), U(t), \lambda(t), t) = L(x(t), u(t), \lambda, t) + \lambda^T a - N^T b$. The Galerkin method on the other hand has applied to different areas of Mathematics and Economics to solve differential equations related problems; for instance, [14] applied the Galerkin method to find the numerical solution of an integro-differential equations using fourth kind shifted chebyshev polynomials as basis functions to transform the integro-differential equations into a system of linear equations which was solved to obtain an approximate solution. [2] applied Galerkin method to second order ordinary differential equations with mixed boundary conditions by converting the mixed boundary conditions into Neumann type using secand and Runge-kutta methods, Dokuchaev and Zhou [7], developed a Galerkin finite element method to solve a partial differential equation, the Black-D scholes equation arising from pricing European options in which volatility and dividend considered as variable dependent on the state price. Results evidenced convergenceof Galerkinfinite element method, and numerical resultrevealed its appropiateness, efficiency and accuracy.

The scheme combines indirect methods (i.e Pontryagin's Maximum Principle [20] as seen in [2, 6, 19]) with the direct methods ,(discretization) by using the Galerkin method. The state and costate trajectories are discretized by approximating the solution as a linear combination of suitable basis functions and undetermined coefficients. For instance, If $s = \phi_j(t)_{j=1}^{\infty}$ is a basis vector v, a set of linearly independent functions and any function $f(t) \in v$ can be uniquely written as linear combination of the basis function

$$f(t) = \sum_{j=1}^{\infty} c_j w_j(t) \tag{1.1}$$

where $\phi_i = w_j$. The Galerkin's method use a finite number of independant functions $\phi_i(t)_{i=1}^{n}$ as trial functions [19, 21]. Let us suppose that the approximate solution to differential equation: $\frac{d}{dt}f(t)$ is $p(u) = \delta u(t) + g(t) = 0$ on the boundary $A(u) = m$ where $m \in [a, b]$ is in the form:

$$U(t) \equiv \gamma_N(t) = \sum_{i=1}^{N} (c_i w_i(t) + w_0(t), \;\; for \;\; i = 1, ..., N \tag{1.2}$$

where $\gamma_N(t)$ is the approximate solution, $u(t)$ the exact solution, δ a differential operator, g a given function, $w_i(t)$ is the weighted residual which are set of linearly independent functions called weighted functions and c_i are unknown coefficients. The residual $R(t_i, c_j) = p(t) - (\delta(u(t) + g(t))$ and c_j are determined by the orthogonal definition

$$\int_{t_0}^{t_f} w_i(t) R(t_i, c_j) dt = 0, \; for \; j = 1, ..., n \tag{1.3}$$

1.1. **Literature Review.** Due to the complexity of the algorithmic framework of this Research and its relationship with other methods, derivation of Lagrange form of Optimal control problems and brief introduction to the theory of first order differential equations will be discussed.

1.2. **The Lagrange type of Optimal control Problems.** Let the system of a plant be described by the first order differential equation:

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1.4}$$

with a bolza form of performance measure

$$J(\cdot) = s(x(t))|_{t=t_f} + \int_{t_0}^{t_f} v(x(t), u(t), t)dt \tag{1.5}$$

if $s(x(t))|_{t=t_f} = 0$ then equation (1.5) can be written as

$$J(\cdot) = \int_{t_0}^{t_f} v(x(t), u(t), t)dt \tag{1.6}$$

Transformation of constraints equation (1.4) and (1.6) to unconstrained one via Lagrange Multiplier method, give:

$$J(\cdot) = \int_{t_0}^{t_f} v(x(t), u(t), t) + \int_{t_0}^{t_f} \lambda^T(t)[f(x(t), u(t), t) - \dot{x(t)}dt \tag{1.7}$$

According to [4], equation (1.7) can be written in Lagrange form as

$$\int_{t_0}^{t_f} L(x(t), u(t), \lambda(t), t)dt \tag{1.8}$$

where (1.8) can be defined as

$$\int_{t_0}^{t_f} L(x(t), u(t), \lambda(t), t)dt = v(x(t), u(t), \lambda(t), t) + \lambda^T(t)[f(x(t), u(t), t) - \lambda^T(t)\dot{x(t)} \tag{1.9}$$

which can be written in Hamiltonian form as $H(x(t), u(t), \lambda(t), t) - \lambda(t)^T \dot{x(t)}$

According to [22] and [16]. If the objective function is perturbed, (1.7) becomes

$$\begin{aligned} J_a(\cdot) &= \int_{t_0}^{t_f+\delta t_f} v(x^*(t) + \delta x(t), u^*(t) + \delta u(t), t)dt \\ &\int_{t_0}^{t_f+\delta t_f} \lambda^T(t)[f(x_*(t) + \delta x(t), u^*(t) + \delta u(t), t) - \dot{x^*(t)} + \delta \dot{x}(t)]dt \\ &= \int_{t_0}^{t_f+\delta t_f} L_p(\cdot)dt \\ &= \int_{t_0}^{t_f} L_p(\cdot)dt + \int_{t_f}^{t_f+\delta t_f} L_p(\cdot)dt \quad \equiv \int_{t_0}^{t_f} L_p(\cdot)dt + \int_{t_0}^{t_f} L(\cdot)|_{t=t_f}\delta t_f dt \end{aligned} \tag{1.10}$$

where $L_p(\cdot)$ is the perturbed model of the Lagrange multiplier. if the variational form of the function in (1.10)

$$\Delta J_a = J_a(\cdot) - J(\cdot) = \int_{t_0}^{t_f+\delta t_f} L_p(\cdot)dt - \int_{t_0}^{t_f} L(\cdot)dt \approxeq \int_{t_0}^{t_f} L_p(\cdot)dt + L(\cdot)|_{t=t_f}\delta t_f - \int_{t_0}^{t_f} L(\cdot)dt \tag{1.11}$$

Therefore, the application of (1.11) to (1.10) with reference to (1.9) give

$$\int_{t_0}^{t_f} L_p(\cdot) - L(\cdot)dt = \int_{t_0}^{t_f} L_p(x^*(t) + \delta x(t), u^*(t) + \delta u(t), t) - L(x^*(t), u^*(t), \lambda^*, t)dt \\ + L(x^*(t), u^*(t), \lambda^*, t)|_{t=t_f}\delta t_f \tag{1.12}$$

According to [24], application of the Taylor series expansion with integration by part to (1.12) gives

$$\delta J = \int_{t_0}^{t_f} [\frac{\partial L(\cdot)}{\partial x(t)} - \frac{d}{dt}[\frac{\partial L(\cdot)}{\partial \dot{x}(t)}]_*^T \delta x(t)]dt + \int_{t_0}^{t_f} [\frac{\partial L(\cdot)}{\partial u(t)}]^T \delta u(t)dt \\ + [\frac{\partial L(\cdot)}{\partial \dot{x}(t)}]_*^T \delta x(t)/_{t=tf} + L(\cdot)_*/_{t=tf}\delta tf \tag{1.13}$$

Lemma :Let $g(t)$ and $x(t)$ be continuous and integrable over a close interval t_0 and t_f then $\int_{t_0}^{t_f} g(t)\delta x(t)dt = 0$ at every point over the integral $[t_0, t_f]$.

Applying Lemma 1.2 to equation (1.13) yields

$$\frac{\partial L(\cdot)}{\partial x(t)} - \frac{d}{dt}(\frac{\partial L(\cdot)}{\partial \dot{x}(t)})_* = 0 \tag{1.14}$$

and

$$(\frac{\partial L(\cdot)}{\partial u(t)})_* = 0 \tag{1.15}$$

where (1.14) and (1.15) are gotten from the first and second part of (1.13) using lemma 1.2.

Finally,

$$\delta J \approx L(\cdot)_*|_{t=t_f}\delta t_f + (\frac{\partial L(\cdot)}{\partial \dot{x}(t)})_*\delta x(t)|_{t=t_f} \tag{1.16}$$

$$\delta x_f = \delta x_{t_f} + \dot{x}|_{t=t_f}\delta t_f = \delta x_{t_f} + (\dot{x}^*(t) + \delta\dot{x}|_{t=t_f})\delta t_f \approxeq \delta x_{t_f} + (\dot{x}(t_f)^*)\delta t_f \tag{1.17}$$

substituting (1.17) into (1.16), we have

$$\delta J = L(\cdot)_*|_{t=t_f}\delta t_f + [\frac{\partial L(\cdot)}{\partial \dot{x}(t)}]_*|_{t=t_f}(\delta x_f - \dot{x}(t_f))\delta t_f \tag{1.18}$$

Simplication of (1.18) give

$\delta J \approxeq L(\cdot) - [\frac{\partial L(\cdot)}{\partial \dot{x}(t)}\dot{x}(t)]_*|_{t=t_f}\delta t_f + [\frac{\partial L(\cdot)}{\partial \dot{x}(t)}]_*|_{t=t_f}(\delta x_f$

If $\delta J = 0$ in (1.18) which is regarded as necessary condition, then

$$\delta J \approxeq L(\cdot) - [\frac{\partial L(\cdot)}{\partial \dot{x}(t)}\dot{x}(t)]_*|_{t=t_f}\delta t_f + [\frac{\partial L(\cdot)}{\partial \dot{x}(t)}]_*|_{t=t_f}(\delta x_f = 0 \tag{1.19}$$

where

$$L(\cdot) = H(x(t), u(t), \lambda(t), t) + \frac{\partial s(\cdot)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(\cdot)}{\partial t} - \lambda^T(t)\dot{x}(t) \tag{1.20}$$

Using (1.18) in (1.14),(1.15) and (1.19) From (1.14) we have

$$\begin{aligned}\frac{\partial}{\partial x(t)}[H(x(t), u(t), \lambda(t), t) + \frac{\partial s(\cdot)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(\cdot)}{\partial t} - \lambda^T(t)\dot{x}(t)]\\ - \frac{d}{dt}[\frac{\partial[H(x(t), u(t), \lambda(t), t) + \frac{\partial s(\cdot)}{\partial \dot{x}(t)}\dot{x}(t) + \frac{\partial s(\cdot)}{\partial t} - \lambda^T(t)\dot{x}(t)]}{\partial \dot{x}(t)}]\end{aligned} \tag{1.21}$$

If $f(x(t), y(t), z(t))$, is a multivariate function of x, y and z then the application of chain rule give [**?**]

$$\frac{d}{dt}f(\cdot) = \frac{\partial f(\cdot)}{\partial x(t)}\dot{x}(t) + \frac{\partial f(\cdot)}{\partial y(t)}\dot{y}(t) + \frac{\partial f(\cdot)}{\partial z(t)}\dot{z}(t) \tag{1.22}$$

Application of (1.22) in (1.21) give
$\frac{\partial}{\partial x(t)}[H(\cdot) + \frac{ds(\cdot)}{dt} - \lambda^T(t)\dot{x}(t)] - \frac{d}{dt}[\frac{\partial s(\cdot)}{\partial x(t)} - \lambda^T(t)]$ where (1.21),$\frac{\partial s(\cdot)}{\partial \dot{x}(t)}$ and $\frac{\partial s(\cdot)}{\partial t}$ can be written combinely as $\frac{d}{dt}s(\cdot)$
from our last equation,we have

$$(\frac{\partial H(\cdot)}{\partial x(t)})_* = -\dot{\lambda}(t) \tag{1.23}$$

Equation (1.23) is called the co-state equation and free from $\dot{x}(t)$ Also from (1.15)

$$(\frac{\partial L(\cdot)}{\partial u(t)})_* = 0 \Rightarrow (\frac{\partial L(\cdot)}{\partial u(t)})_* = (\frac{\partial H(\cdot)}{\partial u(t)})_* = 0 \tag{1.24}$$

where $L(\cdot)$ remain as defined in (1.20)
The similar version of (1.23) which is

$$(\frac{\partial H(\cdot)}{\partial \lambda}(t))_* = \dot{x}(t) \tag{1.25}$$

Equation (1.25) is called the state equation if the system is express in state pace form.

Finally, the boundary condition (1.19) can be written in Hamiltonian form as

$$[H(\cdot)+(\frac{\partial s}{\partial x(t)})^T-\lambda^T(t)\dot{x}(t)-[\frac{\partial s}{\partial x(t)}-\lambda(t)]\dot{x}(t)]_*|_{t=t_f}\delta t_f+[\frac{\partial s}{\partial x(t)}-\lambda(t)]_*|_{t=t_f}\delta x_f = 0 \tag{1.26}$$

Therefore,

$$[H(\cdot)+\frac{\partial s}{\partial x(t)}]_*|_{t=t_f}\delta t_f+[\frac{\partial s}{\partial x(t)}-\lambda(t)]_*|_{t=t_f}\delta x_f=0 \tag{1.27}$$

if equation (1.23) to (1.25) and (1.27) are solved, we get our trajectory $x(t)$ and the control input $u(t)$ which minimizes the performance measure(or performance index) (1.5)

$$\frac{\partial H(x_i(t),u(t),\lambda_i(t),t)}{\partial\lambda_i(t)}=\dot{x}_i(t) \tag{1.28}$$

and

$$-\frac{\partial H(x_i(t),u(t),\lambda_i(t),t)}{\partial x_i(t)}=\dot{\lambda_i}(t) \tag{1.29}$$

(1.28) and (1.29) are are referred to as the state and co-state equations respectively

2. Materials and Methods

Consider an optimal control problem of the type

$$\underset{(x,u,\lambda)}{\text{minimize}} J=\int_{t_0}^{t_f}[x^T(t)Px(t)+u^T(t)Qu(t)]dt$$

$$\text{subject to} \qquad \dot{x}_i(t)=f(x_i(t),u_i(t),t)$$

$$\text{where } P \text{ is a positive definite matrix } Q=1, \tag{2.1}$$

Applying the Hamiltonian function give

$$H(x(t),u(t),\lambda(t))=x_i^T(t)Px_i(t)+u_i^T(t)Ru_i(t)+\lambda_i^T[f(x_i(t),u_i(t),t)] \tag{2.2}$$

Aplication of equation (1.24), (1.28) and (1.29) to (2.2) yields

$$\frac{\partial H(\cdot)}{\partial u_i(t)}=u_i^T(t)R\dot{u}_i+\lambda_i^T[f(x_i(t),\dot{u}_i(t),t)]=0 \tag{2.3}$$

$$\frac{\partial H(\cdot)}{\partial x_i(t)}=-\dot{\lambda}_i(t)=x_i^T(t)P\dot{x}_i(t)+\lambda_i^T[f(\dot{x}_i(t),u_i(t),t)] \tag{2.4}$$

and

$$\frac{\partial H(\cdot)}{\partial\lambda_i(t)}=\dot{x}_i(t)=\dot{\lambda}_i^T[f(\dot{x}_i(t),u_i(t),t)] \tag{2.5}$$

For simplicity, (2.4) and (2.5) can be written as

$$\dot{x}_i(t)=\alpha_0x_i(t)+\alpha_1\lambda_i(t) \tag{2.6}$$

and

$$\dot{\lambda}_i(t)=\beta_0x_i(t)+\beta_1\lambda_i(t) \tag{2.7}$$

Asumme polynomial solutions of order two for (2.6) and (2.7) i.e the state and costate equations, with their imposed boundary conditions (BC) to get Let

$$x_i(t) = \sum_{k=1}^{n} a_k t^k = x_i^0 + a_k(-t_0 + t) + a_{k+1}(-t_0^2 + t^2) \text{ for } x_i(t_0) = x_i^0 \text{ and}$$

$$\lambda_i(t) = \sum_{k=1}^{n} b_k t^k = \lambda_i^0 + b_k(-t_0 + t) + b_{k+1}(-t_0^2 + t^2) \text{ for } \lambda_i(t_0) = \lambda_i^0$$

$$\text{where } w_{x_i}^1 = w_{\lambda_i}^1 = (-t_0 + t) \text{ and } w_{\lambda_i}^1 = w_{x_i}^1 = (-t_0^2 + t^2) \quad (2.8)$$

Total decretization of (2.6) and (2.7) and the residual functions for both the state and the co-state equations yields

$$\begin{gathered} R_x = a_k + 2a_{k+1}t + \alpha_0 a_k t_0 + \alpha_0 a_{k+1} t_0^2 + \alpha_1 b_k t_0 + \alpha_1 b_{k+1} t_0^2 - \alpha_0 a_k t \\ - \alpha_0 a_{k+1} t^2 - \alpha_1 b_k t - \alpha_1 b_{k+1} t^2 - \alpha_0 x_i^0 - \alpha_1 \lambda_i^0 = 0 \\ \text{similarly} \\ R_\lambda = b_k + 2b_{k+1}t + \beta_0 a_k t_0 + \beta_0 a_{k+1} t_0^2 + \beta_1 b_k t_0 + \beta_1 b_{k+1} t_0^2 - \beta_0 a_k t \\ - \beta_0 a_{k+1} t^2 - \beta_1 b_k t - \beta_1 b_{k+1} t^2 - \beta_0 x_i^0 - \beta_1 \lambda_i^0 = 0 \end{gathered} \quad (2.9)$$

where R_x and R_λ are the residual functions for the state and the co-state respectively.
The weighted Residual integral for both yields

$$\int_{t_0}^{t_f} [R_x(t) w_{x_i}^j] dt = 0 \quad (2.10)$$

$$\int_{t_0}^{t_f} [R_\lambda(t) w_{\lambda_i}^j] dt = 0 \quad (2.11)$$

Solving (2.10) and (2.10) yielded system of linear equations with solutions:
$a_k = v_1, a_{k+1} = v_2, b_k = v_3$ and $b_{k+1} = v_4$
Substituting these values in into the assumed solutions in (2.8) to get

$$\begin{aligned} x_i(t) &= x_i^0 + v_1(-t_0 + t) + v_2(-t_0^2 + t^2) \\ \lambda_i(t) &= \lambda_i^0 + v_3(-t_0 + t) + v_4(-t_0^2 + t^2) \end{aligned} \quad (2.12)$$

3. Result

In this section, three optimal control problems labelled problem 3.1, 3.2 and 3.3 with an imposed initial conditions, solutions in tabular form to the itemized problems labelled table 3.1, 3.2 and 3.3 which were compared with their semi analytic solutions (generated using Mathematical program) were presentated

Presentation of Problem 3.1

$\underset{(x,u,\lambda)}{\text{minimize}} J = 0.5\int_0^1 [x^T(t)px(t) + u^T(t)Ru(t)]dt$

subject to $\dot{x_1}(t) = x_2(t)$

$\dot{x_2}(t) = -x_1(t) + 2xx_2(t) + u(t)$

$x_1(0) = 0.2, x_2(0) = 0.1, \lambda_1(1) = 0, \lambda_2(1) = 0.01,$

$R = 1, (\mathbf{P}) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, 0 \le t \le 1$

Analytical solutions to problem 3.1

$x_1(t) = -0.0963707e^{-3.38587t}(e^{0.628052t} - 2.92548e^{2.75782t)} - 0.138998e^{4.01392t} - 0.0108414e^{6.14368t})$

$x_2(t) = 0.265773e^{-3.38587t}(e^{0.628052t} - 0.666235e^{2.75782t} + 0.0316547e^{4.01392t} + 0.0108414e^{6.14368t})$

$\lambda_1(t) = 0.900102e^{-3.38587t}(e^{0.628052t} + 0.719547e^{2.75782t} - 0.212772e^{4.01392t} - 0.00308761e^{6.14368t})$

$\lambda_2(t) = 2.33625e^{-3.38587t}(e^{0.628052t} - 0.157012e^{2.75782t} + 0.0213483e^{4.01392t} - 0.000974846e^{6.14368t}),$

Galerkin's solutions to problem 3.1

$x_1(t) = 0.2 - \frac{17}{560}t$

$x_2(t) = 0.1 - \frac{99}{496}t$

$\lambda_1(t) = -\frac{1413}{1200}(-1 + t)$ $\lambda_2(t) = 0.01 - \frac{1061}{1200}(-1 + t)$

Table 3.1 : Numerical solution of problem 3.1

t-values	Methods	$x_1^*(t)$	$x_2^*(t)$	$\lambda_1^*(t)$	$\lambda_2^*(t)$	$u^*(t)$	$\lvert H(x^*, u^*, \lambda^*)\rvert$
$t = 0.1$	Num	0,196964	0.0400202	1.05975	0.80575	-0.402875	0.095213
	Exact	0.207265	0.0481822	1.08381	1.47878	-0.739391	0.0650569
	Error						0.0301561
$t = 0.2$	Num	0.193929	0.0300403	0.94200	0.717333	-0.358667	0.093991
	Exact	0.210139	0.00114732	0.867743	1.07487	-0.537436	0.0650569
	Error						0.0289341
$t = 0.3$	Num	0.190893	0.0200605	0.824250	0.628917	-0.314458	0.071864
	Exact	0.209943	-0.0137136	0.692394	0.77261	-0.386305	0.0650569
	Error						0.0068071
$t = 0.4$	Num	0.187857	0.0100806	0.706500	0.540500	-0.270250	0.488339
	Exact	207691	-0.030041	0.547882	0.547167	-0.273584	0.0650569
	Error						0.016223
$t = 0.5$	Num	0.184821	0.000100806	0.58875	0.452083	-0.226041	0.00490008
	Exact	0.204165	-0.0394544	0.426607	0.379663	-0.189831	0.0650569
	Error						0.0549416
$t = 0.6$	Num	0.181786	-0.00987903	0.471000	0.363667	-0.181833	0.0690589
	Exact	0.199984	-0.0433339	0.322671	0.255713	-0.127857	0.0650569
	Error						0.004002
$t = 0.7$	Num	0.178750	-0.0198589	0.353250	0.275250	-0.137625	0.0472245
	Exact	0.195651	-0.0426022	0.231442	0.164326	-0.082163	0.0650569
	Error						0.0078324
$t = 0.8$	Num	0.175714	-0.0298387	0.235500	0.186833	-0.0934167	0.0356594
	Exact	0.191599	-0.0377978	0.149215	0.0970525	-0.0485263	0.0650569
	Error						0.00306231
$t = 0.9$	Num	0.172679	-0.0398186	0.117750	0.0984167	-0.0492083	0.03013131
	Exact	0.188220	-0.0291186	0.0729405	0.0473352	-0.0236676	0.0650569
	Error						0.0348893
$t = 1.0$	Num	0.169643	-0.0497984	0.00	0.01	-0.005	0.0616792
	Exact	0.185908	-0.0164391	1.678^{-15}	0.01	-0.005	0.0650569
	Error						0.0033777

Presentation of Problem 3.2

$$\underset{(x,u,\lambda)}{\text{minimize}} J = \int_0^1 [x^T(t)px(t) + u^T(t)Ru(t)]dt$$

$$\text{subject to} \qquad \dot{x_1}(t) = 2x_2(t)$$

$$\dot{x_2}(t) = -x_1(t) - 3x_2(t) + u(t)$$

$$x_1(0) = 0.2, x_2(0) = 0.1, \lambda_1(1) = 0, \lambda_2(1) = 0.01,$$

$$R = 1, (\mathbf{P}) = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}, 0 \leq t \leq 1$$

Analytical solutions to problem 3.2

$x_1(t) = -0.44566e^{-2.41421t}(e^t - 1.33098e^{1.41421t} - 0.0614185e^{3.41421t} - 0.0563781e^{3.82843t})$
$x_2(t) = 0.630258e^{-2.41421t}(e^t - 0.941142e^{1.41421t} + 0.0434294e^{3.41421t} + 0.0563781e^{3.82843t}$
$\lambda_1(t) = 0.44566e^{-2.41421t}(1.e^t + 1.33098e^{1.41421t} - 0.184255e^{3.41421t} - 0.0563781e^{3.82843t})$
$\lambda_2(t) = 0.891319e^{-2.41421t}(e^t + 1.15417 * 10^-15e^{1.41421t} + 4.39259 * 10^-17e^{3.41421t} - 0.0563781e^{3.82843t})$,

Galerkin's solutions to problem 3.2

$x_1(t) = 0.2 + \frac{1353}{19880}t$
$x_2(t) = 0.1 + \frac{1929}{7100}t$
$\lambda_1(t) = -\frac{2178}{17750}(-1 + t)$
$\lambda_2(t) = 0.01 - \frac{1983}{23075}(-1 + t)$

Table 3.2 : Numerical solution of problem 3.2

t-values	Methods	$x_1^*(t)$	$x_2^*(t)$	$\lambda_1^*(t)$	$\lambda_2^*(t)$	$u^*(t)$	$\lvert H(x^*, u^*, \lambda^*)\rvert$
$t = 0.1$	Num	0.206805	0.102717	0.690211	0.783435	-0.391717	0.154034
	Exact	0.209021	0.0816071	0.803910	0.715841	-0.357945	0.154523
	Error						0.000489
$t = 0.2$	Num	0.213611	0.105434	0.613521	0.697497	-0.348749	0.168257
	Exact	0.216545	0.0699264	0.687872	0.605055	-0.302527	0.154523
	Error						0.007734
$t = 0.3$	Num	0.220418	0.108151	0.536831	0.611560	-0.305780	0.173475
	Exact	0.223204	0.06418	0.581752	0.50634	-0,25317	0.154523
	Error						0.008952
$t = 0.4$	Num	0.227223	0.110868	0.460141	0.525623	-0.262812	0.490492
	Exact	0.229558	0.0637541	0.483992	0.417769	-0.208884	0.154523
	Error						0.005969
$t = 0.5$	Num	0.234029	0.113585	0.383451	0.439686	-0.219843	0.499238
	Exact	0.236116	0.0681822	0.39317	0.337567	-0.168784	0.154523
	Error						0.007853
$t = 0.6$	Num	0.240835	0.116301	0.306761	0.353749	-0.176874	0.472077
	Exact	0.243345	0.0771304	0.307975	0.264128	-0.132064	0.154523
	Error						0.007554
$t = 0.7$	Num	0.247641	0.119018	0.230070	0.267812	-0.133906	0.480738
	Exact	0.251686	0.0903873	0.227186	0.19598	-0.0979900	0.154523
	Error						0.006215
$t = 0.8$	Num	0.254447	0.121735	0.153380	0.181874	-0.0909372	0.467217
	Exact	0.261563	0.107855	0.149654	0.131758	-0.0658792	0.154523
	Error						0.012694
$t = 0.9$	Num	0.261253	0.124452	0.0766901	0.0959372	-0.0479686	0.448562
	Exact	0.273397	0.129545	0.0742792	0.0701762	-0.0350881	0.154523
	Error						0.005961
$t = 1.0$	Num	0.268058	0.127168	0.000000	0.010000	-0.005000	0.424771
	Exact	0.287616	0.155573	$7.08 * 10^{-17}$	0.010000	-0.005000	0.154523
	Error						0.009752

Presentation of Problem 3.3

$\underset{(x,u,\lambda)}{\text{Minimize}}\ J = \int_0^1 [x^T(t)px(t) + u^T(t)Ru(t)]dt$
subject to $\dot{x_1}(t) = 2x_2(t)$
$\dot{x_2}(t) = -x_1(t) - 3x_2(t) + u(t)$
$x_1(0) = 0.2, x_2(0) = 0.1, \lambda_1(1) = 0, \lambda_2(1) = 0.01,$
$R = 1, (\mathbf{P}) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, 0 \le t \le 2$

Analytical solutions to problem 3.3
$x_1(t) = -0.376314e^{-3.55765t}(e^{1.19935t} - 1.52722e^{2.35829t} - 0.0044612e^{4.757t} + 0.000211265e^{5.91594t})$
$x_2(t) = 0.44373e^{-3.55765t}(e^{1.19935t} - 0.776695e^{2.35829t} + 0.00226882e^{4.757t} - 0.000211265e^{5.91594t})$
$\lambda_1(t) = -0.396799e^{-3.55765t}(e^{1.19935t} - 2.22251e^{2.35829t} + 0.0318776e^{4.757t} - 0.00141341e^{5.91594t})$
$\lambda_2(t) = 0.183141e^{-3.55765t}(e^{1.19935t} + 0.500877e^{2.35829t} - 0.0645023e^{4.757t} + 0.00635373e^{5.91594t})$

Galerkin's solutions to problem 3.3
$x_1(t) = 0.2 - \frac{3612}{77375}t$
$x_2(t) = 0.1 - \frac{57261}{619000}t$
$\lambda_1(t) = -\frac{69939}{309500}(-1 + t)$
$\lambda_2(t) = 0.01 - \frac{18357}{309500}(-1 + t)$

Table 3.3 : Numerical solutions of problem 3.3

t-values	Methods	$x_1^*(t)$	$x_2^*(t)$	$\lambda_1^*(t)$	$\lambda_2^*(t)$	$u^*(t)$	$\|H(x^*, u^*, \lambda^*)\|$
$t = 0.25$	Num	0.183328	0.0768923	0.329546	0.113796	-0.0568978	0.0420971
	Exact	0.219260	-0.0008093	0.417316	0.155685	-0.0778426	0.0050363
	Error						0.0370608
$t = 0.5$	Num	0.166656	0.0537845	0.282468	0.0989677	-0.0494838	0.0290336
	Exact	0.20258	-0.0512112	0.3409	0.0889485	-0.444743	0.0050363
	Error						0.0239973
$t = 0.75$	Num	0.148884	0.0306768	0.235389	0.0841387	-0.0420699	0.0166864
	Exact	0.173255	-0.0625855	-0.0231648	-0.0231648	0.0050363	0.0116501
	Error						0.0050363
$t = 1.0$	Num	0.133312	0.00756801	0.188312	0.0693118	-0.03465589	0.00872235
	Exact	0.14235	-0.0599554	0.192223	0.0180762	-0.00090381	0.00503631
	Error						0.003686
$t = 1.25$	Num	0.116639	-0.0155387	0.141234	0.0544838	-0.0272419	0.00514114
	Exact	0.114604	-0.0509672	0.130173	-0.0006239	0.00031199	0.0050363
	Error						0.00010484
$t = 1.5$	Num	0.0999677	-0.0386465	0.0941559	0.0396559	-0.0198279	0.00594326
	Exact	0.0915591	-0.0412557	0.0772087	-0.010886	0.00544302	0.0050363
	Error						0.00090696
$t = 1.75$	Num	0.0832956	-0.0617542	0.0470779	0.0248279	-0.0124139	0.00111283
	Exact	0.0731524	-0.0326933	0.033306-	-0.0100214	0.00501071	0.0050363
	Error						0.003908
$t = 2.0$	Num	0.0666236	-0.0848619	0.00000	0.01000	0.0206964	0.0080029
	Exact	0.0584328	-0.0267325	$-2.8 * 10^{-15}$	0.01	-0.005	0.0050363
	Error						0.00296666

4. Discussion

The tables show that the values of states and costates for both numerical and analytical solutions are generally similar. However, there are slight variations in the values of x_2 for Problems 2 and 3, indicating that the Galerkin method results in x_2 changing more rapidly than the solution generated by Wolfram Mathematica. Additionally, the objective function values for the solutions produced by Wolfram Mathematica were consistent across all three tested problems. This supports the notion that while optimal paths may vary, optimal values remain constant. As t (time) changes, the performance measures from the Galerkin method exhibit slight discrepancies compared to those from the analytical solutions, but with minimal error. The costate values at the terminal points for both the Galerkin method and the analytical solutions provided by Wolfram Mathematica were identical, suggesting that at the terminal points, the Galerkin method approximated the exact solution (produced global solutions) and provided local solutions along other paths during the search procesdure.

5. Conclusion

The paper proprosed an Embedding Galerkin method for numerical solution of optimal control problems. Basically, control problems can be solved by different approches which include: the direct method, indirect method and the operator based methods. The three methods have their strengths and weaknesses. The direct mehods is faster but may not be suitable for problems with random conditions or uncertainties, as it assumes deterministic inputs; The indirect method find an optimal solution by satisfying optimality conditions and provide results with high accuracy but the boundary value problems is often extremely difficult to solve for problems that span large time intervals or problem with interrior point constraints; Lastly, the operator based method overcame these two disadvantages of both direct and indirect method of solution but very difficult to develop and execute as it requires many inter-related theorems and also selective in its convergernce.

This paper focuses on combinations of the direct and the indirect method of solution to develop a better numerical method that will overcome the disadvantages of both methods. concentration was on the discretization of boundary conditions from the indirect method (i.e The state and the co-state equation) using the Galerkin method of solution with the objective of increasing the applicability and competitivenesss of the Galerkin method in comparison with other numerical methods. The contribution of the paper is to reduce the dominant limitations of the indirect method by employ a self starting method which is stable and easy to implement. Although, Galerkin's method is has computational cost and memory requirement due to its higher-order polynomials and numerical fluxes but demonstrate high level of robustness and reliability as it showed $x^*(t)$ as a through trajectory whose curve with respect to t can be sketched but exhibited the disadvantage in that it requires more computer time or many computations before solution is achieved, while the Galerkin method is not only simple but also popular in the finite element method since it offers ease of implementation due to some weight and trial functions.

Abbreviations:

$H(x^*, u^*, \lambda^*(t))$:	The value of the modified Objective functional or Hamiltonian at optimal point
$u^*(t)$:	The optimal value of the control variable at optimal point.
$x^*(t)$:	The optimal value of the state variable at optimal point.
$\lambda^*(t)$:	The optimal value of the co-state variable at optimal point.

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Matthew Folorunsho Akinmuyise*
Department of Mathematics, Adeyemi University of Education, Ondo, Ondo State, Nigeria.
E-mail address: `akinkwam33@gmail.com`

Sharimakin
Department of Economics, Adeyemi University of Education, Ondo, Ondo State, Nigeria.
E-mail address: `Shakwumi@gmail.com`

I. Fakunle*
Department of Mathematics, Adeyemi University of Education, Ondo, Ondo State, Nigeria.
E-mail address: `fakunlei@aceondo.edu.ng`