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# A SEMI-ANALYTICAL METHOD FOR SOLVING FRACTIONAL ORDER GENERALIZED BURGERS- HUXLEY EQUATION WITH A REFINED INITIAL GUESS.

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ABSTRACT: In this paper, a refined initial guess was incorporated into the Adomian Decomposition Method (ADM) in order to obtain an approximate solution to the classical order and fractional order Generalized Burgers-Huxley equations (GBHE). The first iteration was obtained using the refined initial guess and all other iterations are obtained using the ADM. The solutions obtained were computed as an infinite series with a fast convergence to the exact solution at classical order. The dependability and efficacy of the procedure are demonstrated by the supplied graph and the results.

## 1. INTRODUCTION

Fractional calculus is a crucial and useful extension of the ordinary derivatives of integer orders and integrals, because the current solution is not only influenced by the preceding one but also by the entire history of the solution. Numerous man-made and natural phenomena have characteristics that classical order calculus is unable to sufficiently explain. A more realistic foundation for simulating these intricate phenomena, like fractal geometry, anomalous diffusion, and viscoelasticity is offered by fractional calculus. Fractional calculus is important in memory effect, signal processing, control systems, fractal phenomenon, optimization and finance. It is essential to solve fractional calculus because it provides a more precise and adaptable mathematical framework for explaining complex systems and phenomena that display long-range interactions, memory effects, and non-local behavior. It is an important instrument for scientific study and engineering practice because of its wide range of applications.

**Literature Review.** A significant obstacle with this problem type is the difficulty in finding a solution, particularly when it is nonlinear. Several numerical methods have been developed to solve this set of equations, including the homotopy analysis technique by [14], the homotopy perturbation method by [11], the ADM by [9], the Aboodh transformation-based homotopy

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pertubation method by [6], kamal Adomian decomposition method by [8], laplace Adomian decomposition method by [10], Mohand variational transformation method by [5].

Alaje *et al.* [2] developed the concept of modified initial guess in the Homotopy perturbation method solely to solve the generalized fractional order korteweg- de-Vries (kdV) problems. This study aims to incorporate the initial guess proposed by [2] into the Adomian decomposition method to address fractional order Generalized Burgers-Huxley equations.

## 2. MATERIALS AND METHODS

**Definition 2.1.** A real function $\phi(t)$, $t>0$ is said to be in space $C_\mu, \mu \in R$ if there exist a real number $m>\mu$ such that $\phi(t)=t^m\phi(t)$ where $\phi(t)\in C(0,\infty)$ and it is said to be in the space $C_\mu^n$ if and only if $\phi^n \in C_\mu$, $n\in N$.

**Definition 2.2.** The Reimann-Liouville fractional Integration of order $\alpha \geq 0$ for a real positive function $\phi^n \in C_\mu$, $\mu \geq -1$, $t>0$ is defined by [3] as

$$I^\alpha \phi(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-x)^{\alpha-1}\phi(x)dx. \tag{1}$$

Where $\phi(t)$ is a function of t, $\phi(x)$ is a function of $x$, $\alpha$ is the order of the fractional integral, $\Gamma(\alpha)$ is a Gamma function of $\alpha$, $I^\alpha$ is called the fractional integral operator.
The following properties hold for fractional integral operator:

$$I^\alpha I^\beta \phi(t) = I^{\alpha+\beta}\phi(t) \tag{2}$$

$$I^\alpha I^\beta \phi(t) = I^\beta I^\alpha \phi(t) \tag{3}$$

$$I^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}t^{\alpha+\beta} \tag{4}$$

Let $\alpha$ be any positive real number and n be a natural number such that $n-1<\alpha<n$. Let $f(t)$ be a continuous function in the interval $[a,T]$, $T>a$. Then the Reimann-Liouville derivative of order $\alpha$ is given by [6]

$${}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)d\tau, \tag{5}$$

Here nth derivative is operated outside the integral sign.

**Definition 2.3.** The Caputo fractional derivative of a positive real function $\phi^n \in C_\mu$, is expressed as

$$D^\alpha \phi(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-x)^{n-\alpha-1}\phi^n(x)dx, \tag{6}$$

$$n-1 \leq \alpha \leq n, n\in N. \tag{7}$$

Where $D^\alpha$ is the Caputo derivative, $n$ is a natural number, $\phi^n(x)$ is a real positive function of order $n$, $\alpha$ is the order of Caputo derivative. The fractional integration of Caputo derivative for $n-1\leq\alpha\leq n$, $n\in N$, $\phi^n \in C_{-1}^n$, $\mu \leq -1$ is

$$I^{\alpha} D^{\alpha} \phi(t) = \phi(t) - \sum_{k=0}^{n-1} \phi^{k}(0) \frac{t^{k}}{k!}, \tag{8}$$

$\phi(t)$ is a function of t, $\phi^{k}(x)$ is a real positive function of order $k$, $\alpha$ is the order of the fractional integral, $I^{\alpha}$ is called the fractional integral operator, $D^{\alpha}$ is the Caputo derivative[3].

### 2.4. Refined Initial Guess Adomian Decomposition Method (RIG-ADM)

The general form of ADM [12] is given as:

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + N(u(x,t)) + R(u(x,t)) = g(x), \tag{9}$$

Where $N(u(x,t))$ is a Nonlinear function, $R(u(x,t))$ is the remaining linear function, $g(x)$ is the source term

$${}^{c}D_{t}^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}} \text{ represents the Caputo derivative }, \; 0 < \alpha \leq 1 \tag{10}$$

Applying the integral operator of $I_{t}^{\alpha}$ to both sides of Eq. (9)

$$u(x,t) = \phi(x) + \mathrm{I}_{t}^{\alpha} g(x) - \mathrm{I}_{t}^{\alpha} N(u) - \mathrm{I}_{t}^{\alpha} R(u), \tag{11}$$

Where the integral constant $\phi(x) = \sum_{k=0}^{n-1} \frac{\partial^{k}}{\partial t^{k}} u(x,0) \frac{t^{k}}{k!} = u(x,0)$ for $k = 0$ (12)

Substituting Eq. (12) into Eq. (11) gives;

$$u(x,t) = u(x,0) + \mathrm{I}_{t}^{\alpha} g(x) - \mathrm{I}_{t}^{\alpha} N(u) - \mathrm{I}_{t}^{\alpha} R(u), \tag{13}$$

But $u(x,t) = \sum_{n=0}^{\infty} u_{n}(x,t)$, (14)

The nonlinear term $Nu = \sum_{n=0}^{\infty} A_{n}$, (15)

Where $\mathrm{A}_{n}$ is the Adomian polynomial with the general formula:

$$\mathrm{A}_{n} = \frac{1}{n!} \frac{d^{n}}{d\lambda^{n}} N\left( \sum_{i=0}^{\infty} \lambda^{i} U_{i} \right)_{\lambda=0}. \tag{16}$$

The first few terms are:

When n=0

$$\mathrm{A}_{0} = N(U_{0}). \tag{17}$$

When n=1

$$\mathrm{A}_{1} = u_{1} N'(u_{0}). \tag{18}$$

When n=2

$$\mathrm{A}_{2} = u_{2} N'(u_{0}) + \frac{1}{2!} u_{1}^{2} N''(u_{0}). \tag{19}$$

When n=3

$$\mathrm{A}_{3} = u_{3} N'(u_{0}) + u_{1} u_{2} N''(u_{0}) + \frac{1}{3!} u_{1}^{3} N'''(u_{0}). \tag{20}$$

Substituting Eq. (14) and (15) into Eq. (13) gives:

$$\begin{aligned} u(x,t) &= u(x,0) + I_{t}^{\alpha} g(x) - I_{t}^{\alpha} (A_{0} + A_{1} + A_{2} + \ldots) \\ &- \mathrm{I}_{t}^{\alpha} R(u_{0}(x,t) + u_{1}(x,t) + u_{2}(x,t) + u_{3}(x,t) + ..), \end{aligned} \tag{21}$$

Comparing Eq. (21) and Eq. (14)

$$
\begin{aligned}
&u_0(x,t) = u(x,0) + I_t^{\alpha} g(x) \\
&u_1(x,t) = -I_t^{\alpha}\left(A_0 + R(u_0)\right) \\
&u_2(x,t) = -I_t^{\alpha}\left(A_1 + R(u_1)\right) \\
&. \qquad\qquad\qquad\qquad . \\
&. \\
&. \\
&u_n(x,t) = -I_t^{\alpha}\left(A_{n-1} + R(u_{n=1})\right)
\end{aligned} \tag{22}
$$

The modified initial guess proposed by [2] is given as;

$$u(x,t) = u(x,0) + \sum_{n=1}^{\infty} \lambda_n t^{n\alpha} \ ,0 < \alpha \le 1. \tag{23}$$

With the initial condition $u(x,0) = \beta(x)$. (24)

By comparing Eq. (24) to Eq. (14),

$$
\begin{aligned}
&u_0(x,t) = u(x,0) \\
&u_1(x,t) = \lambda_1 t^{\alpha} \\
&u_2(x,t) = \lambda_2 t^{2\alpha} \\
&. \qquad\qquad . \\
&. \\
&. \\
&u_n(x,t) = \lambda_n t^{n\alpha}
\end{aligned} \tag{25}
$$

These $u_i's$, $i = 1,2,3,...n$ are then calculated depending on the problem under consideration which are then sum up as the solution of the given problem.

## 3. RESULTS

Considering the generalized form of fractional order Burger-Huxley equation given as:

$$\frac{\partial^{\xi} u(x,t)}{\partial t} + \alpha u(x,t)^{\delta} \frac{\partial u(x,t)}{\partial x} - \frac{\partial^2 u(x,t)}{\partial x^2} = \beta u(x,t)(1 - u(x,t)^{\delta})(u(x,t)^{\delta} - \gamma)\,, \tag{26}$$

$\alpha, \delta, \beta \ge 0$ and $\gamma \in (0,1)$ with the initial condition

$$u(x,0) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}. \tag{27a}$$

The exact solution of the classical order when $\xi = 1$ of equation (26) was obtained by [15] as

$$u(x,t) = \left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh\left[\sigma\gamma\left(x - \left\{\frac{\gamma\alpha}{1+\delta} - \frac{(1+\delta-\gamma)(\rho-\alpha)}{2(1+\delta)}\right\}t\right)\right]\right]^{\frac{1}{\delta}}, \tag{27b}$$

where $\sigma = \dfrac{\delta(\rho - \alpha)}{4(1+\delta)}$ and $\rho = \sqrt{\alpha^2 + 4\beta(1+\delta)}$ . (27c)

The classical order of equation (26) has been solved by [4] using the Adomian decomposition method. The RIG-ADM will now be used to solve the fractional order of the same equation due to the non-local importance of fractional differential equations. Burger-Huxley models the interaction between mechanisms, convection effects and diffusion transport [13]. Where $u(x,t)$ is the population density, $\gamma$ is the species carrying capacity, $\alpha$ is the speed of advection and $\beta$ is the nonlinear source, $\delta$ is a constant parameter.
By expanding the right hand side of equation (26),

$$\frac{\partial^{\xi} u(x,t)}{\partial t^{\xi}} + \alpha u(x,t)^{\delta} \frac{\partial u(x,t)}{\partial x} - \beta(1+\gamma)u(x,t)^{\delta+1} + \beta u(x,t)^{2\delta+1} - \frac{\partial^2 u(x,t)}{\partial x^2} + \beta\gamma u(x,t) = 0 \tag{28}$$

From Eq. (25) we have

$$u_1(x,t) = \lambda_1 t^{\xi},\ u_2(x,t) = \lambda_2 t^{2\xi}, u_3(x,t) = \lambda_3 t^{3\xi} \tag{29}$$

$$\frac{\partial^{\xi} u_1(x,t)}{\partial t^{\xi}} = \lambda_1 \Gamma(\xi+1), \tag{30a}$$

$$\frac{\partial^{\xi} u_2(x,t)}{\partial t^{\xi}} = \frac{\lambda_2 \Gamma(2\xi+1)}{\Gamma(\xi+1)} t^{\xi}, \tag{30b}$$

$$\frac{\partial^{\xi} u_3(x,t)}{\partial t^{\xi}} = \frac{\lambda_3 \Gamma(2\xi+1)}{\Gamma(2\xi+1)} t^{2\xi} \tag{30c}$$

$$\frac{\partial^{\xi} u(x,t)}{\partial t^{\xi}} = \lambda_1 \Gamma(\xi+1) + \frac{\lambda_2 \Gamma(2\xi+1)}{\Gamma(\xi+1)} t^{\xi} + \frac{\lambda_3 \Gamma(3\xi+1)}{\Gamma(2\xi+1)} t^{2\xi} + \ldots \tag{31a}$$

$$u(x,0)^{\delta} \frac{\partial u(x,0)}{\partial x} = \frac{\frac{1}{2}\left[\left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{\delta} \left(\left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right) \gamma^2 \sigma \left(1 - \tanh^2(\sigma\gamma x)\right)}{\delta\left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]} \tag{31b}$$

$$u(x,0)^{\delta+1} = \left[\left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{\delta+1} \tag{31c}$$

$$u(x,0)^{2\delta+1} = \left[\left[\frac{\gamma}{2} + \frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{2\delta+1} \tag{31d}$$

$$\frac{\partial^2 u(x,0)}{\partial x^2} = \frac{1}{4}\frac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\gamma^4\sigma^2\left(1-\tanh^2(\sigma\gamma x)\right)^2}{\delta^2\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^2}$$

$$-\frac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\left(1-\tanh^2(\sigma\gamma x)\right)\gamma^3\sigma^2\tanh(\sigma\gamma x)}{\delta\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]} \tag{31e}$$

$$-\frac{1}{4}\frac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\gamma^4\sigma^2\left(1-\tanh^2(\sigma\gamma x)\right)^2}{\delta\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^2}$$

Substituting Eq. (31a-31e) into Eq. (28) when $t=0$ gives,

$$\lambda_1 = \frac{1}{\Gamma(\xi+1)}\left( \begin{array}{l} -\alpha \dfrac{\frac{1}{2}\left[\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{\delta}\left(\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right)\gamma^2\sigma\left(1-\tanh^2(\sigma\gamma x)\right)}{\delta\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]} \\ +\beta(1+\gamma)\left[\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{\delta+1} - \beta\left[\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{2\delta+1} \\ +\left( \begin{array}{l} \dfrac{1}{4}\dfrac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\gamma^4\sigma^2\left(1-\tanh^2(\sigma\gamma x)\right)^2}{\delta^2\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^2} \\ -\dfrac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\left(1-\tanh^2(\sigma\gamma x)\right)\gamma^3\sigma^2\tanh(\sigma\gamma x)}{\delta\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]} \\ -\dfrac{1}{4}\dfrac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\gamma^4\sigma^2\left(1-\tanh^2(\sigma\gamma x)\right)^2}{\delta\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^2} \end{array} \right) - \beta\gamma\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}} \end{array} \right) \tag{32}$$

From Eq. (12)

$u_1(x,t) = \lambda_1 t^{\xi}$,

Substituting Eq. (32) into Eq. (12) gives,

$$u_1(x,t)=\frac{1}{\Gamma(\xi+1)}\left(\begin{array}{l} -\alpha\dfrac{\frac{1}{2}\left[\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{\delta}\left(\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right)\gamma^2\sigma\left(1-\tanh^2(\sigma\gamma x)\right)}{\delta\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]} \\ +\beta(1+\gamma)\left[\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{\delta+1}-\beta\left[\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\right]^{2\delta+1} \\ +\left(\begin{array}{l} \frac{1}{4}\dfrac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\gamma^4\sigma^2\left(1-\tanh^2(\sigma\gamma x)\right)^2}{\delta^2\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^2} \\ -\dfrac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\left(1-\tanh^2(\sigma\gamma x)\right)\gamma^3\sigma^2\tanh(\sigma\gamma x)}{\delta\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]} \\ -\frac{1}{4}\dfrac{\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}}\gamma^4\sigma^2\left(1-\tanh^2(\sigma\gamma x)\right)^2}{\delta\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^2} \end{array}\right)-\beta\gamma\left[\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(\sigma\gamma x)\right]^{\frac{1}{\delta}} \end{array}\right)t^{\xi} \tag{33}$$

Other iterations can now be obtained using Adomian decomposition method
The nonlinear terms of equation (28) are

$$\mathrm{A_n}=u_n(x,t)^{\delta}\,\frac{\partial u_n(x,t)}{\partial x}, \tag{34a}$$

$$\mathrm{B_n}=u_n(x,t)^{\delta+1}, \tag{34b}$$

$$\mathrm{C_n}=u_n(x,t)^{2\delta+1} \tag{34c}$$

$$\begin{aligned}
\mathrm{A}_0 &= u_0(x,t)^{\delta}\,\frac{\partial u_0(x,t)}{\partial x}\\
\mathrm{A}_1 &= u_0(x,t)^{\delta}\,\frac{\partial u_1(x,t)}{\partial x}+u_1(x,t)^{\delta}\,\frac{\partial u_0(x,t)}{\partial x}\\
\mathrm{A}_2 &= u_0(x,t)^{\delta}\,\frac{\partial u_2(x,t)}{\partial x}+u_1(x,t)^{\delta}\,\frac{\partial u_1(x,t)}{\partial x}+u_2(x,t)^{\delta}\,\frac{\partial u_0(x,t)}{\partial x}\\
\mathrm{A}_3 &= u_0(x,t)^{\delta}\,\frac{\partial u_3(x,t)}{\partial x}+u_1(x,t)^{\delta}\,\frac{\partial u_2(x,t)}{\partial x}+u_2(x,t)^{\delta}\,\frac{\partial u_1(x,t)}{\partial x}+u_3(x,t)^{\delta}\,\frac{\partial u_0(x,t)}{\partial x}
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\mathrm{B}_0 = u_0(x,t)^{\delta+1}\\
&\mathrm{B}_1 = (\delta+1)u_1(x,t)u_0(x,t)^{\delta}\\
&\mathrm{B}_2 = (\delta+1)u_2(x,t)u_0(x,t)^{\delta}+\frac{1}{2}(\delta+1)(\delta)u_1(x,t)^2u_0(x,t)^{\delta-1}\\
&\mathrm{B}_3 = (\delta+1)u_3(x,t)u_0(x,t)^{\delta}+(\delta+1)(\delta)u_1(x,t)u_2(x,t)u_0(x,t)^{\delta-1}\\
&+\frac{1}{6}(\delta+1)(\delta)(\delta-1)u_1(x,t)^3u_0(x,t)^{\delta-2}
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\mathrm{C}_0 = u_0(x,t)^{2\delta+1}\\
&\mathrm{C}_1 = (2\delta+1)u_1(x,t)u_0(x,t)^{2\delta}\\
&\mathrm{C}_2 = (2\delta+1)u_2(x,t)u_0(x,t)^{2\delta}+\frac{1}{2}(2\delta+1)(2\delta)u_1(x,t)^2u_0(x,t)^{2\delta-1}\\
&\mathrm{C}_3 = (2\delta+1)u_3(x,t)u_0(x,t)^{2\delta}+(2\delta+1)(2\delta)u_1(x,t)u_2(x,t)u_0(x,t)^{2\delta-1}\\
&+\frac{1}{6}(2\delta+1)(2\delta)(2\delta-1)u_1(x,t)^3u_0(x,t)^{2\delta-2}
\end{aligned} \tag{37}$$

$$u_2(x,t) = -I_t^{\xi}\left(\alpha A_1-\beta(1+\gamma)B_1+\beta C_1-\frac{\partial^2 u_1(x,t)}{\partial x^2}+\beta\gamma u_1\right) \tag{38}$$

$$u_3(x,t) = -I_t^{\xi}\left(\alpha A_2-\beta(1+\gamma)B_2+\beta C_2-\frac{\partial^2 u_2(x,t)}{\partial x^2}+\beta\gamma u_2\right) \tag{39}$$

$$u_4(x,t) = -I_t^{\xi}\left(\alpha A_3-\beta(1+\gamma)B_3+\beta C_3-\frac{\partial^2 u_3(x,t)}{\partial x^2}+\beta\gamma u_3\right) \tag{40}$$

$$u(x,t) = u_0(x,t)+u_1(x,t)+u_2(x,t)+u_3(x,t)+u_4(x,t)+\ldots \tag{41}$$

## 4. Discussion

The refined initial guess was incorporated into the Adomian decomposition scheme to solve the generalized Burger-Huxley equation using Maple software. The first iteration was obtained using the refined initial guess proposed by [2] and other iterations were obtained using the Adomian decomposition method. The iterative solutions were in a series form whose closed-form converges to the exact solution. Table I shows the errors obtained when the approximate solutions of the first three iterations of the Generalized Burger Huxley Equation (GBHE) obtained by RIG-ADM when $\alpha=\beta=1, \delta=2, \gamma=0.01$ was compared with the exact solutions. The Mean Absolute Error (MAE) calculated was $1.65\times10^{-4}$ which was found to be the same as [4]. Table II shows the approximate solution of the fractional-order GBHE at different fractional order when $\alpha=\beta=1, \delta=2, \gamma=0.01$ and it was observed that as the fractional order increases the approximate solution increases and tends to the classical solution. Table III shows the absolute errors obtained when the approximate solutions of the second, third and fourth iterations of the GBHE obtained by RIG-ADM when $\alpha=0, \beta=2, \delta=1, \gamma=0.7$ were compared with the exact solution given by [1]. It was observed that the MAE of the first four iterations calculated is $3.3\times10^{-5}$ which is the same as [1]. When $\alpha=0, \beta=2$ this reduces the GBHE to Generalized Huxley Equation (GHE) and it was observed that the MAE of $u_2(x,t) > u_3(x,t) > u_4(x,t)$, this then implies that the accuracy of the method depends on the number of iterations. Table IV shows the approximate solution of the first four iteration of different fractional-order Huxley equation when $\alpha=0, \beta=2, \delta=1, \gamma=0.7$ and it was observed that as the fractional order increases the value obtained tends to the classical solution at different values of $t$.

Figure 1 shows the agreement between the 3D plot of RIG-ADM and exact of GHE when $\alpha=0, \beta=2, \delta=1, \gamma=0.7$ the figure shows the interaction between the propagation of neural pulses, the motion of liquid crystal walls and the dynamic of nerve fibres. Figure 2 shows the 3D plot of RIG-ADM of the GHE when $\alpha=0, \beta=2, \delta=1, \gamma=0.7$ the figure shows the interaction between the propagation of neural pulses, the motion of liquid crystal walls and the dynamic of nerve fibres.

Table I. Results of the first three iterations $u(x,t)$ of the classical Generalized Burger-Huxley equation when $\alpha=\beta=1, \delta=2, \gamma=0.01, \xi=1$.

| x | t | Exact | Numerical | Absolute error |
|---|---|---|---|---|
| 0.1 | 0.1 | 0.0707459068192873 | 0.0706907399638226 | 5.516685E-05 |
| | 0.2 | 0.0707657782238068 | 0.0706554324552584 | 1.1034577E-04 |
| | 0.3 | 0.0707856440150640 | 0.0706201073273971 | 1.6553669E-04 |
| | 0.4 | 0.0708055041852216 | 0.0705847646239787 | 2.2073956E-04 |
| | 0.5 | 0.0708253587264504 | 0.0705494043887428 | 2.7595434E-04 |
| 0.3 | 0.1 | 0.0707765915255728 | 0.0707214486683211 | 5.514286E-05 |
| | 0.2 | 0.0707964542583346 | 0.0706861564601093 | 1.1029779E-04 |
| | 0.3 | 0.0708163113657345 | 0.0706508465943210 | 1.6546477E-04 |
| | 0.4 | 0.0708361628399472 | 0.0706155191146305 | 2.2064372E-04 |
| | 0.5 | 0.0708560086731549 | 0.0705801740647122 | 2.7583461E-04 |
| MAE | | | | 1.65-04 |

Table II. Results of the first three iterations $u(x,t)$ of the Generalized Burger-Huxley equation when $\alpha = \beta = 1, \delta = 2, \gamma = 0.01$ at different fractional order.

| x | t | $\xi = 0.25$ | $\xi = 0.5$ | $\xi = 0.75$ |
|---|---|---|---|---|
| 0.1 | 0.1 | 0.0705065148685067 | 0.0705999619653633 | 0.0706577231172378 |
| | 0.2 | 0.0704648407844771 | 0.0705476400617371 | 0.0706111040273507 |
| | 0.3 | 0.0704368727317888 | 0.0705074514210270 | 0.0705702020566559 |
| | 0.4 | 0.0704152240549829 | 0.0704735435115939 | 0.0705326128971685 |
| | 0.5 | 0.0703973174192068 | 0.0704436493930238 | 0.0704973049742888 |
| 0.3 | 0.1 | 0.0705373021430817 | 0.0706307097483701 | 0.0706884461075625 |
| | 0.2 | 0.0704956455157502 | 0.0705784101596525 | 0.0706418471109080 |
| | 0.3 | 0.0704676891120356 | 0.0705382385517081 | 0.0706009626856422 |
| | 0.4 | 0.0704460494154736 | 0.0705043449405901 | 0.0705633895815824 |
| | 0.5 | 0.0704281501835144 | 0.0704744633727386 | 0.0705280966796442 |

Table III**.** Results of the absolute error of Generalized Huxley equation at second, third and fourth iterations when $\alpha = 0, \beta = 2, \delta = 1, \gamma = 0.7, \xi = 1$.

| x | t | $u_2(x,t)$ | $u_3(x,t)$ | $u_4(x,t)$ |
|---|---|---|---|---|
| -5 | 0.3 | 0.0000532641 | 0.000002746 | 0.000000065 |
| | 0.7 | 0.0006315957 | 0.000079942 | 0.000004767 |
| | 1 | 0.0017507672 | 0.000323676 | 0.000029134 |
| -3 | 0.3 | 0.0000975884 | 0.000001419 | 0.000000629 |
| | 0.7 | 0.0012427897 | 0.000021079 | 0.000039619 |
| | 1 | 0.0035952650 | 0.000033425 | 0.000219377 |
| -1 | 0.3 | 0. 000153432 | 0.000020390 | 0.000000327 |
| | 0.7 | 0.0016042828 | 0.000603898 | 0.000009177 |
| | 1 | 0.0039590272 | 0.002478818 | 0.000001848 |
| MAE | | 0.001454 | 0.000396 | 0.000033 |

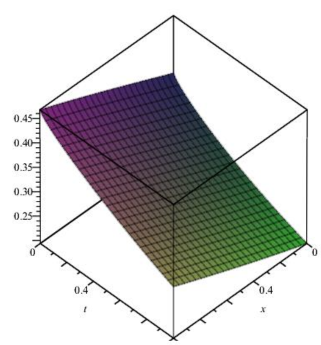
Table IV**.** Results of the first four iterations of Generalized Huxley equation at different fractional order when $\alpha = 0, \beta = 2, \delta = 1, \gamma = 0.7$.

| x | t | $\xi = 0.25$ | $\xi = 0.5$ | $\xi = 0.75$ |
|---|---|---|---|---|
| -5 | 0.3 | 0.01283922699 | 0.01283117497 | 0.01415141133 |
| | 0.7 | 0.01374792465 | 0.01112431781 | 0.01056777381 |
| | 1 | 0.01492244025 | 0.01110400506 | 0.00907629687 |
| -3 | 0.3 | 0.04600734938 | 0.04844961611 | 0.05785780517 |
| | 0.7 | 0.04530583812 | 0.04000883032 | 0.04008262693 |
| | 1 | 0.04646287702 | 0.03705522959 | 0.03322474263 |
| -1 | 0.3 | 0.14037733048 | 0.15804066800 | 0.17570580636 |
| | 0.7 | 0.12252861274 | 0.12746556140 | 0.13523389124 |
| | 1 | 0.11358418601 | 0.11119979799 | 0.11295581138 |

**(a)** **(b)**

**Fig 1:** exact solution (a) and numerical solution (b) of equation (26) respectively when $\alpha = 0, \beta = 2, \delta = 1, \gamma = 0.7, \xi = 1$.

**(c)** **(d)**



**(e)**

**Fig 2:** Numerical solution of equation (26) when $\alpha = 0, \beta = 2, \delta = 1, \gamma = 0.7$ (c) plot of $\xi = 0.25$ (d) plot of $\xi = 0.5$, (e) plot of $\xi = 0.75$.

## 5. CONCLUSION

Refined Initial Guess Adomian Decomposition Method (RIG-ADM) was used to solve the Generalized Burger-Huxley Equation at classical order and fractional order. The results obtained were presented in tabular and graphical formats and shows more efficient and accurate solutions when compared with other semi-analytical methods, making them a valuable tool for researchers.

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**Authors Contribution.** Alaje *et al*. developed the concept of refined initial guess in homotopy perturbation method to solve korteweg-de-Vries equation at classical order. This manuscript was incorporated the initial guess into the Adomian decomposition method to solve Generalized Burgers-Huxley Equation at classical and fractional order.

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