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BRIEF STUDY ON SOME PROPERTIES OF SYMMETRIC CARDIO-BAZILEVIC FUNCTIONS

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Abstract. Bazilevic functions consist of functions defined by certain integral I say, which are entirely univalent in the unit disk. They contain some other class of functions as special cases. In the recent time, the study of Bazilevic functions became so popular that researchers (especially in Geometric Function Theory, GFT) have had to study different subclasses of Bazilevic functions as related to various domains. However, their study seem to lack full vigour addressing relevant connections of Bazilevic functions to certain interesting domain called the symmetric cardioid domain. In characterization of these Bazilevic functions, the geometry of the image domains is very critical. Consequently, in this article, with the aid of Salagean derivative operator, the author derived a new Bazilevic class $B_n^\alpha(A, B, \sigma)$, type α, associated with symmetric cardioid domain. This was achieved via the Hadamard product of certain fractional analytic function $g(z)^\alpha$ and the normalized univalent function $f(z)$ using subordination principle. In the sequel, a new geometrical formation regarding the said class of Bazilevic functions was obtained. Additionally, sharp bounds on the first three Taylor-Maclaurin coefficients for functions belonging to the aforementioned class were obtained while the relationship of these bounds to the classical Fekete-Szego inequality was established using a very lucid Mathematical approach.

1. Introduction

The interaction of geometry and analysis is possibly the most fascinating aspect of complex function theory. Geometric function theory is the branch of complex analysis that deals with the study of geometric properties of analytic functions. These functions are pivotal in the analysis of practical problems such as image processing and signal processing among others. Furthermore, in 1955, a Russian Mathematician (Bazilevic) started the study of the Bazilevic functions I say, and since then, a rather fast flood of follow-up to his work have resulted on the study of various subclasses of Bazilevic functions. Since the configuration of the image

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domain is very germane in the enactment of Bazilevic functions, therefore, for a panoramic and effective study of these functions, a careful analysis of the geometrical properties of their domains should be of priority. In the recent time, the study of Bazilevic function became so popular that researchers (especially in Geometric Function Theory, GFT) have had to study different subclasses of Bazilevic functions as related to various domains. However, their studies seem to lack full vigour addressing relevant connections of Bazilevic functions to the symmetric cardioid domain. In [16], Malik et al. studied a domain called symmetric cardioid domain. Motivated in this direction, the present author aim at contributing to the existing literatures by obtaining the estimates of the first few Taylor-Maclaurin coefficients for functions belonging to the new class of Bazilevic function associated with Symmetric Cardioid Domain (otherwise known as cardio-Bazilevic function) while relevant connections of the coefficients so obtained to the popular Fekete-Szego inequality were investigated using lucid mathematical technique. This study is unique and became so necessary owned to the way the author defined the Bazilevic class $B_n^\alpha(A, B, \sigma)$ via convolution and the application of cardioid domain (which is very useful in radiography, physics, and other fields of science and engineering) in Geometric Function Theory.

Let E denote the unit disk, that is, $E = \{z : |z| < 1; z \in \mathbb{C}\}$. Also let Ω denotes the class of analytic functions $f(z)$ having the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad z \in \mathbb{C} \tag{1.1}$$

gratifying the normalization conditions:

$$f(0) = 0 \;\; and \;\; f'(0) = 1$$

in the unit disk E.

In [7], Hamzat and Oladipo considered certain fractional analytic function $g(z)^\alpha$ of the form:

$$g(z)^\alpha = \frac{z^\alpha}{1-z} = z^\alpha + \sum_{k=2}^{\infty} z^{\alpha+k-1},$$

for real number $\alpha(\alpha > 0)$ in E, see [8]. Using the concept of convolution and applying Salagean differential operator respectively, then

$$f(z)^\alpha = f(z) * g(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k z^{\alpha+k-1} \tag{1.2}$$

and

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty} (\alpha + k - 1)^n a_k z^{\alpha+k-1}. \tag{1.3}$$

It is observed that

$$\Re\left\{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha}\right\} > 0, \qquad (\alpha > 0,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ z \in \mathbb{C}). \tag{1.4}$$

Interestingly, (1.4) coincides with the famous class of Bazilevic functions studied by different authors, for details (see [5] and [19]) among others.

Furthermore, an analytic function $f(z)$ is said to be subordinate to another analytic function $g(z)$ (expressed as f $\prec$ g), if there exist a unit bound function $r(z)$ satisfying the conditions:

$$r(0) = 0 \quad and \quad |r(z)| < 1,$$

such that

$$f(z) = g(r(z)), \quad z \in E.$$

At this juncture, using the principle of subordination, the class $B^{\alpha}(n)$ of Bazilevic functions can be written using the principle of subordination as

$$\left\{ \frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}} \right\} \prec p(z), \tag{1.5}$$

where $p(z)$ belongs to the famous class of Caratheodory functions P (that is, functions with positive real parts) satisfying the conditions:

$$p(0) = 0 \quad and \quad \Re \{p(z)\} > 0 \quad z \in E.$$

It is quite interesting to note that the geometric formations of the image domain of $p(E)$ depend on the definition given to P. For instance, the class P of Caratheodory functions can be defined via subordination principle such that

$$P = \left\{ p(z) \prec \frac{1+z}{1-z}, \;\; p(0) = 1; \;\; p'(0) = 1; \;\; z \in E \right\}. \tag{1.6}$$

Then it is obvious from (1.6) that the geometry of the image domain of $p(z)$ is $\frac{1+z}{1-z}$, which is the right half of the complex plane. In addition, several subclasses of P are obtained for various choices of function $p(z)$. Below are the few cases (among others) with their respective symmetric domains, which were studied by different authors.

1. The authors in [9] and [26], studied the circular domain centered at $\frac{1-rs}{1-s^2}$ and radius $\frac{r-s}{1-s^2}$ for which

$$p_1(z) = \frac{1+rs}{1-sz}, \;\; -1 \leq s < r \leq 1.$$

2. Also the author in [25] studied the right half of the lemniscate of Bernoulli $|w^2 - 1| = 1$ for

$$p_2(z) = (1+z)^{1/2}.$$

3. The author in [4], examined the plane to the right of the vertical line $u = \beta$, for which

$$p_3(z) = \frac{1 + (1-2\beta)z}{1-z}, \;\; 0 \leq \beta < 1.$$

4. In [11] and [28], the authors investigated the parabolic domain, for which

$$p_4(z) = 1 + \big(\frac{2}{\pi^2}\big)^2 \Big(log \frac{1+\sqrt{z}}{1-\sqrt{z}} \Big)^2.$$

5. In [20], Paprocki and Sokol considered the leaf-like domain such that

$$p_5(z) = \Big(\frac{1+z}{1+\frac{1-b}{b}z}\Big)^{1/a}, \quad a \geq 1,\ b \geq 1/2.$$

6. The author in [27], worked on the nephroid domain whereby

$$p_6(z) = 1 + z - \frac{z^3}{3}.$$

7. Raina and Sokol in [23] investigated the crescent-shaped domain for which

$$p_7(z) = z + (1+z^2)^{1/2}.$$

8. The authors in [14], [18] and [24], examined the oval and petal type domain, where

$$p_8(z) = \frac{(r+1)p_j(z) - (r-1)}{(s+1)p_j(z) - (s-1)}, \quad -1 \leq s <\leq 1, \quad j = 3, 4, 5.$$

9. Kanas and Masih in [10], studied the pascal snail domain for which

$$p_9(z) = \frac{2(1-t)z}{(1-\beta z)^2}, \ \ 0 \leq \beta < 1,\ t \in (-1, 1) - \{0\}.$$

10. Cho et al. in [3], studied the eight-shaped domain, where

$$p_{10}(z) = 1 + sinz.$$

11. Piejko and Sokol in [21] examined the booth lemniscate for which

$$p_{11}(z) = 1 + \frac{z}{1-\beta z^2}, \ \ 0 \leq \beta < 1.$$

12. Masih and Kanas in [17], investigated the limacon-shaped domain, where

$$p_{12}(z) = (1+dz)^2, \quad 0 < d < \frac{1}{\sqrt{2}}.$$

13. Hamzat and Oladipo in [6], studied the shell-like curve for which

$$p_{13}(z) = \frac{1+\sigma^2 z^2}{1-\sigma z - \sigma^2 z^2}, \ \ \sigma = \frac{1-\sqrt{5}}{2} = -0.618.$$

14. Also, Hamzat and Oladipo in [6], considered S-shaped region, where

$$p_{14} = \frac{2}{1+e^{-z}}.$$

In addition, the mapping of the unit disk via the function $p_{13}(z)$ given above generates the conchoid of Maclaurin such that

$$p(e^{i\theta}) = \frac{\sqrt{5}}{2(3-2cos\theta)} + i\frac{sin\theta(4cos\theta-1)}{2(3-2cos\theta)(1+cos\theta)}, \quad 0 \leq \theta < 2\pi.$$

Moreover, $p_{13}(z)$ has the series form:

$$p_{13}(z) = \frac{1+\sigma^2 z^2}{1-\sigma z-\sigma^2 z^2} = 1 + \sum_{k=1}^{\infty} \big(u_{k-1}+u_{k+1}\big)\sigma^k z^k,$$

where $u_k = \frac{(1-\sigma)^k - \sigma^k}{\sqrt{5}}$, $k = 1, 2, 3, ...$ and $\sigma = -0.618$.
The above series representation in (1.7) has a close link with the sequence of Fibonacci number given that

$$u_0 = 0, \quad u_1 = 1, \quad u_{k+2} = u_k + u_{k+1}, \quad k = 0, 1, 2, 3, ...$$

Therefore, $p_{13}(z)$ can be expressed as

$$p_{13}(z) = 1 + \sigma z + 3\sigma^2 z^2 + 4\sigma^3 z^3 + 7\sigma^4 z^4 + 11\sigma^5 z^5 + ...$$

Here, we consider a case whereby

$$\bar{p}(A, B, \sigma; z) = \frac{2+(A-1)\sigma z+2A\sigma^2 z^2}{2+(B-1)\sigma z+2B\sigma^2 z^2}, \tag{1.7}$$

where $-1 \leq B < A \leq 1$, $\sigma = -0.618$ and $z \in E$. By letting

$$u = \Re\left\{\bar{p}(A, B, \sigma; e^{i\theta})\right\} \ \ and \ \ v = Im\left\{\bar{p}(A, B, \sigma; e^{i\theta})\right\},$$

then the image $\bar{p}(A, B, \sigma; e^{i\theta})$ of the unit disk generates a cardioid-like curve given by the parametric equations:

$$u = \frac{4+(A-1)(B-1)\sigma^2+4AB\sigma^4+2\psi cos\theta+(A+B)\sigma^2 cos\theta}{4+(s-1)^2\sigma^2+4B^2\sigma^4+4\sigma(B-1)(1+s\sigma^2)cos\theta+8B\sigma^2 cos2\theta}$$

and

$$\frac{2\sigma(A-B)(1-\sigma^2)sin\theta+2\sigma^2 sin2\theta}{4+(B-1)^2\sigma^2+4B^2\sigma^4+4\sigma(B-1)(1+s\sigma^2)cos\theta+8B\sigma^2 cos2\theta},$$

where $\psi = (A+B-2)\sigma+(2AB-A-B)\sigma^3$, $-1 \leq B < A \leq 1$ and $(0 \leq \theta < 2\pi)$, see [15].

Remark A

It is easily seen from (1.8) that
1. $\bar{p}(A, B, \sigma; 0) = 1$
2. $\bar{p}(A, B, \sigma; 1) = \frac{AB+9(A+B)+4(B-A)\sqrt{5}+1}{B^2+18B+1}$.

Definition 1:
Let $B_n^{\alpha}(A,B,\sigma)$ denote the class of cardio-Bazilevic functions consisting of the functions $f(z)^{\alpha}$ of the form (1.2), satisfying the geometric condition:

$$\frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}} \prec \bar{p}(A,B,\sigma;z), \tag{1.8}$$

where $\alpha > 0$, $-1 \leq B < A \leq 1$, $\sigma = -0.618$ and $\bar{p}(A,B,\sigma;z)$ is as defined in (1.8).

Remark B
Suppose that $A = 1$ and $B = -1$ in (1.9), then the class $B_n^{\alpha}(A,B,\sigma)$ crack down to the class of Bazilevic functions associated with Fibonacci number.

2. Results

Before proceeding into the results, the following lemmas which shall be useful in the course of this study shall be stated.

Lemma 2.1 [22]**:** Let the function $p \in P$ be of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \quad z \in E,$$

the inequality $|c_k| \leq 2$ holds true for each $k \geq 1$. Equality is attained for function $p(z)$ given by $p(z) = \frac{1+z}{1-z}$.

Lemma 2.2 [12]**,** [13]**:** Let $p \in P$. Then for complex number μ

$$\left|c_2 - \mu c_1^2\right| \leq 2max\left\{1, |2\mu - 1|\right\}.$$

Lemma 2.3 [1]**,** [2]**:** Let $p \in P$. Then for $\alpha, \beta, \gamma \in R$ and $z \in E$

$$\left|ac_1^3 - bc_1c_2 + xc_3\right| \leq 2|a| + 2|b - 2a| + 2|a - b + x|.$$

This first result is a motivation from the work of [15].

Theorem 2.4: Let $f(z)^{\alpha}$ be of the form (1.2). If $f(z)^{\alpha} \in B_n^{\alpha}(A,B,\sigma)$, then for $\alpha > 0$, $-1 \leq B < A \leq 1$, $\sigma = -0.618$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $m \geq 2$

$$\left|a_m\right| \leq \left(\frac{\alpha}{\alpha+m-1}\right)^n \left(\Gamma + \sum_{k=1}^{m-2}\left[1+|\sigma|^2\left(\left|\frac{B-1}{2}\right| + |B||\sigma|\right)^2\right]\left(\frac{\alpha+k-1}{\alpha}\right)^{2n}\left|a_k\right|^2\right)^{\frac{1}{2}}, \tag{2.1}$$

where

$$\Gamma = \left[1 + |\sigma|^2\left(\left|\frac{A-1}{2}\right|^2 + A|\sigma|^2\right)\right] + \left[1 + |\sigma|^2\left|\frac{B-1}{2}\right|^2\right]\left(\frac{\alpha+m-2}{\alpha}\right)^{2n}\left|a_{m-1}\right|^2.$$

Proof: Since $f(z)^{\alpha}$ belong to the class $B_n^{\alpha}(A,B,\sigma)$, then from (2.1), we have

$$\frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}} = \bar{p}(A,B,\sigma;\phi(z)) = \frac{2 + (A-1)\sigma\phi(z) + 2A\sigma^2(\phi(z))^2}{2 + (B-1)\sigma\phi(z) + 2B\sigma^2(\phi(z))^2}, \quad z \in E. \tag{2.2}$$

It is easily seen from (2.2) that

$$\sum_{k=2}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k-1}=\sigma(\phi(z))\left[\left(\frac{A-1}{2}\right)-\left(\frac{B-1}{2}\right)\left(1+\sum_{k=2}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k-1}\right)\right]$$
$$+\sigma^2(\phi(z))^2\left[A-B\left(1+\sum_{k=2}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k-1}\right)\right]. \tag{2.3}$$

Simplifying further, we obtain

$$\sum_{k=1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k-1}=1+\sigma(\phi(z))\left[\left(\frac{A-1}{2}\right)-\left(\frac{B-1}{2}\right)\sum_{k=1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k-1}\right]$$
$$+\sigma^2(\phi(z))^2\left[A-B\sum_{k=1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k-1}\right]\ (a_1=1).$$

That is,

$$\sum_{k=1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}=z+\sigma(\phi(z))\left[\left(\frac{A-1}{2}\right)z-\left(\frac{B-1}{2}\right)\sum_{k=1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}\right]$$
$$+\sigma^2(\phi(z))^2\left[Az-B\sum_{k=1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}\right]. \tag{2.4}$$

(2.4) can be expressed as

$$\sum_{k=1}^{m}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}+\sum_{k=m+1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n \delta_k z^{k}$$
$$=z+\sigma(\phi(z))\left[\left(\frac{A-1}{2}\right)z-\left(\frac{B-1}{2}\right)\sum_{k=1}^{m-1}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}\right]$$
$$-\sigma^2(\phi(z))^2\left[Az-B\sum_{k=1}^{m-2}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}\right],$$

where

$$\sum_{k=m+1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n \delta_k z^{k}=\sum_{k=m+1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}$$
$$+\sigma(\phi(z))\left(\frac{B-1}{2}\right)\sum_{k=m}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}$$
$$+\sigma^2(\phi(z))^2 B\sum_{k=m-1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}.$$

Also from (2.4), one can say that

$$\left|\sum_{k=1}^{m}\left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^{k}+\sum_{k=m+1}^{\infty}\left(\frac{\alpha+k-1}{\alpha}\right)^n \delta_k z^{k}\right|^2$$

$$= \left| z + \sigma(\phi(z)) \left[\left(\frac{A-1}{2}\right) z - \left(\frac{B-1}{2}\right) \sum_{k=1}^{m-1} \left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^k \right] \right.$$

$$\left. -\sigma^2 (r(z))^2 \left[Az - B \sum_{k=1}^{m-2} \left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^k \right] \right|^2.$$

Now

$$\left| \sum_{k=1}^{m} \left(\frac{\alpha+k-1}{\alpha}\right)^n \gamma_k z^k \right|^2 = \left| z + \sigma(r(z)) \left(\frac{A-1}{2}\right) z - \sigma(r(z)) \left(\frac{B-1}{2}\right) \left(\frac{\alpha+m-2}{\alpha}\right)^n a_{m-1} z^{m-1} - H \right|^2,$$

where

$$H = \sigma^2(\phi(z))^2 Az + \sigma(\phi(z)) \left(\frac{B-1}{2}\right) \sum_{k=1}^{m-2} \left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^k + \sigma^2(\phi(z))^2 \sum_{k=1}^{m-2} B \left(\frac{\alpha+k-1}{\alpha}\right)^n a_k z^k$$

and

$$\left(\frac{\alpha+k-1}{\alpha}\right)^n \gamma_k = \begin{cases} \left(\frac{\alpha+k-1}{\alpha}\right)^n a_k, \ for \quad 1 \le k \le m \\ \left(\frac{\alpha+k-1}{\alpha}\right)^n \delta_k, \ for \quad k > m \end{cases} \tag{2.5}$$

Recall that

$$\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{k=1}^{\infty} \left(\frac{\alpha+k-1}{\alpha}\right)^n \gamma_k \left(re^{i\theta}\right)^k \right|^2 d\theta = \sum_{k=1}^{\infty} \left| \left(\frac{\alpha+k-1}{\alpha}\right)^n \gamma_k \right|^2 r^{2k}$$

$$\left(since \ \left| e^{i2\theta} \right| = 1, \ \ z = re^{i\theta}, \ \ 0 < r < 1, \ \ 0 \le \theta < 2\pi \right).$$

It implies that

$$\sum_{k=1}^{\infty} \left| \left(\frac{\alpha+k-1}{\alpha}\right)^n \gamma_k \right|^2 r^{2k} < \frac{1}{2\pi} \int_0^{2\pi} \left| re^{i\theta} - \sigma(\phi(re^{i\theta})) \left(\frac{B-1}{2}\right) \left(\frac{\alpha+m-2}{\alpha}\right)^n a_{m-1} (re^{i\theta})^{m-1} - H \right|^2 d\theta,$$

where

$$H = \sigma^2(\phi(re^{i\theta}))^2 A(re^{i\theta}) + \sigma(\phi(re^{i\theta})) \left(\frac{B-1}{2}\right) \sum_{k=1}^{m-2} \left(\frac{\alpha+k-1}{\alpha}\right)^n a_k (re^{i\theta})^k$$

$$+ \sigma^2(\phi(re^{i\theta}))^2 \sum_{k=1}^{m-2} B \left(\frac{\alpha+k-1}{\alpha}\right)^n a_k (re^{i\theta})^k - \sigma(\phi(re^{i\theta})) \left(\frac{A-1}{2}\right) re^{i\theta}.$$

That is,

$$\sum_{k=1}^{\infty} \left| \left(\frac{\alpha+k-1}{\alpha}\right)^n \gamma_k \right|^2 r^{2k} < \frac{1}{2\pi} \int_0^{2\pi} \left(re^{i\theta} - \sigma(r(z)) \left(\frac{B-1}{2}\right) \left(\frac{\alpha+m-2}{\alpha}\right)^n a_{m-1} r^{m-1} (e^{i\theta})^{m-1} - H \right)$$

$$\mathbf{x} \left[re^{-i\theta} - \sigma(\phi(re^{-i\theta})) \left(\frac{B-1}{2}\right) \left(\frac{\alpha+m-2}{\alpha}\right)^n a_{m-1} (re^{-i\theta})^{m-1} - H^* \right] d\theta,$$

where

$$H=\sigma^2(\phi(re^{i\theta}))^2A(re^{i\theta})+\sigma(\phi(re^{i\theta}))\Big(\frac{B-1}{2}\Big)\sum_{k=1}^{m-2}\Big(\frac{\alpha+k-1}{\alpha}\Big)^n a_k(re^{i\theta})^k$$

$$+\sigma^2(\phi(re^{i\theta}))^2\sum_{k=1}^{m-2}B\Big(\frac{\alpha+k-1}{\alpha}\Big)^n a_k(re^{i\theta})^k-\sigma(\phi(re^{i\theta}))\Big(\frac{A-1}{2}\Big)(re^{i\theta}).$$

and

$$H^*=\sigma^2(\phi(re^{-i\theta}))^2A(re^{-i\theta})+\sigma(\phi(re^{-i\theta}))\Big(\frac{B-1}{2}\Big)\sum_{l=1}^{m-2}\Big(\frac{\alpha+l-1}{\alpha}\Big)^n \overline{a_l}(re^{-i\theta})^l$$

$$+\sigma^2(\phi(re^{-i\theta}))^2\sum_{l=1}^{m-2}B\Big(\frac{\alpha+l-1}{\alpha}\Big)^n \overline{a_l}(re^{-i\theta})^l-\sigma(\phi(re^{-i\theta}))\Big(\frac{A-1}{2}\Big)re^{-i\theta}.$$

Since the product of two integrals with $k\neq l$ yields zero, therefore using triangle inequality

$$\sum_{k=1}^{m}\Big(\frac{\alpha+k-1}{\alpha}\Big)^{2n}\big|a_k\big|^2r^{2k}+\sum_{k=m+1}^{\infty}\Big(\frac{\alpha+k-1}{\alpha}\Big)^{2n}\big|\delta_k\big|^2r^{2k}\leq r^2\Big[1+|\sigma|^2\Big(\Big|\frac{A-1}{2}\Big|^2+\big|\sigma\big|^2A\Big)\Big]$$

$$+\big|\sigma\big|^2\Big|\frac{B-1}{2}\Big|^2\Big(\frac{\alpha+m-2}{\alpha}\Big)^{2n}\big|a_{m-1}\big|^2r^{2(m-1)}+\big|\sigma\big|^2\sum_{k=1}^{m-2}\Big(\Big|\frac{B-1}{2}\Big|+\big|\sigma B\big|\Big)^2\Big(\frac{\alpha+k-1}{\alpha}\Big)^{2n}\big|a_k\big|^2r^{2k}.$$

Now, letting $r\to 1$ and using the fact that $a+b\leq c+d \implies a\leq c+d$, then

$$\sum_{k=1}^{m}\Big(\frac{\alpha+k-1}{\alpha}\Big)^{2n}\big|a_k\big|^2\leq\Big[1+|\sigma|^2\Big(\Big|\frac{A-1}{2}\Big|^2+\big|\sigma\big|^2A\Big)\Big]$$

$$+\big|\sigma\big|^2\Big|\frac{B-1}{2}\Big|^2\Big(\frac{\alpha+m-2}{\alpha}\Big)^{2n}\big|a_{m-1}\big|^2+\big|\sigma\big|^2\sum_{k=1}^{m-2}\Big(\Big|\frac{B-1}{2}\Big|+\big|\sigma B\big|\Big)^2\Big(\frac{\alpha+k-1}{\alpha}\Big)^{2n}\big|a_k\big|^2.$$

Furthermore,

$$\Big(\frac{\alpha+m-1}{\alpha}\Big)^{2n}\big|a_m\big|^2+\Big(\frac{\alpha+m-2}{\alpha}\Big)^{2n}\big|a_{m-1}\big|^2+\sum_{k=1}^{m-2}\Big(\frac{\alpha+k-1}{\alpha}\Big)^{2n}\big|a_k\big|^2$$

$$\leq\Big[1+|\sigma|^2\Big(\Big|\frac{A-1}{2}\Big|^2+\big|\sigma\big|^2A\Big)\Big]+\big|\sigma\big|^2\Big|\frac{B-1}{2}\Big|^2\Big(\frac{\alpha+m-2}{\alpha}\Big)^{2n}\big|a_{m-1}\big|^2$$

$$+\big|\sigma\big|^2\sum_{k=1}^{m-2}\Big(\Big|\frac{B-1}{2}\Big|+\big|\sigma B\big|\Big)^2\Big(\frac{\alpha+k-1}{\alpha}\Big)^{2n}\big|a_k\big|^2,$$

so that

$$\Big(\frac{\alpha+m-1}{\alpha}\Big)^{2n}\big|a_m\big|^2\leq\Big[1+|\sigma|^2\Big(\Big|\frac{A-1}{2}\Big|^2+\big|\sigma\big|^2A\Big)\Big]$$

$$+\Big[1+\big|\sigma\big|^2\Big|\frac{B-1}{2}\Big|^2\Big]\Big(\frac{\alpha+m-2}{\alpha}\Big)^{2n}\big|a_{m-1}\big|^2+\sum_{k=1}^{m-2}\Big[1+\big|\sigma\big|^2\Big(\Big|\frac{B-1}{2}\Big|+\big|\sigma B\big|\Big)^2\Big]\Big(\frac{\alpha+k-1}{\alpha}\Big)^{2n}\big|a_k\big|^2.$$

This obviously completes the proof of theorem 2.4.
Setting $\alpha = 1$ in theorem 2.4, then the following corollary is obtained.

Corollary 2.5: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_n^1(A,B,\sigma)$, then for $-1 \le B < A \le 1$, $\sigma = -0.618$, $n \in \mathbb{N}_0$ and $m \ge 2$

$$|a_m| \le \left(\frac{1}{m}\right)^n \left(\Gamma + \sum_{k=1}^{m-2}\left[1 + |\sigma|^2\left(\left|\frac{B-1}{2}\right| + |B||\sigma|\right)^2\right]k^{2n}|a_k|^2\right)^{\frac{1}{2}},$$

where

$$\Gamma = \left[1 + |\sigma|^2\left(\left|\frac{A-1}{2}\right|^2 + A|\sigma|^2\right)\right] + \left[1 + |\sigma|^2\left|\frac{B-1}{2}\right|^2\right](m-1)^{2n}|a_{m-1}|^2.$$

Corollary 2.6: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_n^1(1,-1,\sigma)$, then for $\sigma = -0.618$, $n \in \mathbb{N}_0$ and $m \ge 2$

$$|a_m| \le \left(\frac{1}{m}\right)^n \left(\Gamma + \sum_{k=1}^{m-2}\left[1 + |\sigma|^2\left(1 + |\sigma|\right)^2\right]k^{2n}|a_k|^2\right)^{\frac{1}{2}},$$

where

$$\Gamma = \left[1 + |\sigma|^4\right] + \left[1 + |\sigma|^2\right](m-1)^{2n}|a_{m-1}|^2.$$

Corollary 2.7: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_1^1(1,-1,\sigma)$, then for $\sigma = -0.618$ and $m \ge 2$

$$|a_m| \le \left(\frac{1}{m}\right) \left(\Gamma + \sum_{k=1}^{m-2}\left[1 + |\sigma|^2\left(1 + |\sigma|\right)^2\right]k^{2}|a_k|^2\right)^{\frac{1}{2}},$$

where

$$\Gamma = \left[1 + |\sigma|^4\right] + \left[1 + |\sigma|^2\right](m-1)^{2}|a_{m-1}|^2.$$

Corollary 2.8: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_0^1(1,-1,\sigma)$, then for $\sigma = -0.618$ and $m \ge 2$

$$|a_m| \le \left(\left[1 + |\sigma|^4\right] + \left[1 + |\sigma|^2\right]|a_{m-1}|^2 + \sum_{k=1}^{m-2}\left[1 + |\sigma|^2\left(1 + |\sigma|\right)^2\right]k^{2}|a_k|^2\right)^{\frac{1}{2}}.$$

Corollary 2.9: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_0^1(1,-1,-0.618)$, then for $m \ge 2$

$$|a_m| \le \left(1 + 1459 + 1.3819|a_{m-1}|^2 + \sum_{k=1}^{m-2}\left[1.7293\right]k^{2}|a_k|^2\right)^{\frac{1}{2}}.$$

Theorem 2.10: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_n^\alpha(A,B,\sigma)$, then for

$\alpha > 0$, $-1 \le B < A \le 1$, $\sigma = -0.618$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$

$$\big|a_2\big| \le \frac{\big|\sigma\big|}{2}\big(A-B\big)\Big(\frac{\alpha}{\alpha+1}\Big)^n, \tag{2.6}$$

$$\big|a_3\big| \le \frac{\big|\sigma\big|}{2}\big(A-B\big)\Big(\frac{\alpha}{\alpha+2}\Big)^n max\left\{1, \left|\frac{1}{2}\big[\sigma\big(5-B\big)\big]\right|\right\} \tag{2.7}$$

and

$$\big|a_4\big| \le \frac{\big|\sigma\big|}{2}\big(A-B\big)\Big(\frac{\alpha}{\alpha+3}\Big)^n. \tag{2.8}$$

The inequality in (2.6) is the best possible for the function $f_0(z)$ given by

$$f_0(z)^\alpha = \alpha^n \int_0^z t^\alpha.\bar{p}\big(A, B, \sigma; t\big)dt = z^\alpha + \frac{1}{2}\sigma(A-B)\Big(\frac{\alpha}{\alpha+1}\Big)^n z^{\alpha+1}$$

$$+\frac{1}{4}\sigma^2(A-B)(5-B)\Big(\frac{\alpha}{\alpha+2}\Big)^n z^{\alpha+2} + \frac{1}{8}\sigma^3(A-B)\big(5-10B+B^2\big)\Big(\frac{\alpha}{\alpha+3}\Big)^n z^{\alpha+3} + \dots . \tag{2.9}$$

Also the inequality in (2.7) is the best possible for the function $f_1(z)$ given by

$$f_1(z)^\alpha = \alpha^n \int_0^z t^\alpha.\bar{p}\big(A, B, \sigma; t^2\big)dt = z^\alpha + \frac{1}{2}\sigma(A-B)\Big(\frac{\alpha}{\alpha+2}\Big)^n z^{\alpha+2}$$

$$+\frac{1}{4}\sigma^2(A-B)(5-B)\Big(\frac{\alpha}{\alpha+4}\Big)^n z^{\alpha+4} + \frac{1}{8}\sigma^3(A-B)\big(5-10B+B^2\big)\Big(\frac{\alpha}{\alpha+6}\Big)^n z^{\alpha+6} + \dots . \tag{2.10}$$

Similarly, the inequality in (2.8) is the best possible for the function $f_2(z)$ given by

$$f_2(z)^\alpha = \alpha^n \int_0^z t^\alpha.\bar{p}\big(A, B, \sigma; t^3\big)dt = z^\alpha + \frac{1}{2}\sigma(A-B)\Big(\frac{\alpha}{\alpha+3}\Big)^n z^{\alpha+3}$$

$$+\frac{1}{4}\sigma^2(A-B)(5-B)\Big(\frac{\alpha}{\alpha+6}\Big)^n z^{\alpha+6} + \frac{1}{8}\sigma^3(A-B)\big(5-10B+B^2\big)\Big(\frac{\alpha}{\alpha+9}\Big)^n z^{\alpha+9} + \dots . \tag{2.11}$$

Proof: Let $f(z)^\alpha \in B_n^\alpha(A, B, \sigma)$, then by the principle of subordination, it follows that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = \bar{p}\big(A, B, \sigma; r(z)\big), \tag{2.12}$$

Suppose that $p(z)$ is defined such that

$$p(z) = \frac{1+r(z)}{1-r(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots , \tag{2.13}$$

where $r(z)$ has the properties that $r(0) = 0$ and $\big|r(z)\big| < 1$ for $z \in E$ and $p \in P$ (class of Caratheodory functions or functions with positive real part). It is easily verified from (2.13) that

$$r(z) = \frac{p(z)-1}{p(z)+1} = \frac{(1)}{2}c_1(z) + \frac{1}{4}\big(2c_2 - c_1^2\big)z^2$$

$$+\frac{1}{8}\big(c_1^3 - 4c_1c_2 + 4c_3\big)z^3 + \frac{1}{16}\big(6c_1^2c_2 - c_1^4 - 8c_1c_3 - 4c_2^2 + 8c_4\big)z^4 + \dots . \quad (2.14)$$

Now one can say that

$$\bar{p}\big(A,B,\sigma;r(z)\big) = 1 + \bar{p}_1 r(z) + \bar{p}_2\big(r(z)\big)^2 + \bar{p}_3\big(r(z)\big)^3 + \bar{p}_4\big(r(z)\big)^4 + \dots$$

$$= 1 + \sum_{k=0}^{\infty} \bar{p}_k\big(r(z)\big)^k, \quad (2.15)$$

where

$\bar{p}_1 = \frac{1}{2}\big(A-B\big)\sigma,\;\; \bar{p}_2 = \frac{1}{4}\big(A-B\big)\big(5-B\big)\sigma^2, \bar{p}_3 = \frac{1}{8}\big(A-B\big)\big(B^2-10B+5\big)\sigma^3,$

$\bar{p}_4 = \frac{1}{16}\big(A-B\big)\big(5-35B+15B^2-B^3\big)\sigma^4, \bar{p}_5 = \frac{1}{32}\big(A-B\big)\big(B^4-20B^3+90B^2-60B+5\big)\sigma^5 \dots .$

Therefore, in view of (2.14) and (2.15), we obtain

$$\bar{p}\big(A,B,\sigma;r(z)\big) = 1 + \frac{1}{2}\bar{p}_1c_1z + \Big[\bar{p}_1\Big(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\Big) + \frac{1}{4}\bar{p}_2c_1^2\Big]z^2$$

$$+\Big[\bar{p}_1\Big(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\Big) + \bar{p}_2\Big(c_1c_2 - \frac{1}{4}c_1^3\Big) + \frac{1}{8}\bar{p}_3c_1^3\Big]z^3 + \dots . \quad (2.16)$$

Also

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} = 1 + \Big(\frac{\alpha+1}{\alpha}\Big)^n a_2z + \Big(\frac{\alpha+2}{\alpha}\Big)^n a_3z^2 + \Big(\frac{\alpha+3}{\alpha}\Big)^n a_3z^3 + \dots . \quad (2.17)$$

By comparing the coefficients of $z, z^2 and z^3$ in (2.16) and (2.17), we obtain

$$\Big(\frac{\alpha+1}{\alpha}\Big)^n a_2 = \frac{1}{2}\bar{p}_1c_1, \quad (2.18)$$

$$\Big(\frac{\alpha+2}{\alpha}\Big)^n a_3 = \bar{p}_1\Big(\frac{1}{2}c_2 - \frac{1}{4}c_1^2\Big) + \frac{1}{4}\bar{p}_2c_1^2 \quad (2.19)$$

and

$$\Big(\frac{\alpha+3}{\alpha}\Big)^n a_4 = \bar{p}_1\Big(\frac{1}{8}c_1^3 - \frac{1}{2}c_1c_2 + \frac{1}{2}c_3\Big) + \bar{p}_2\Big(c_1c_2 - \frac{1}{4}c_1^3\Big) + \frac{1}{8}\bar{p}_3c_1^3. \quad (2.20)$$

Applying Lemma 2.1, Lemma 2.2 and Lemma 2.3 respectively, in (2.18), (2.19) and (2.20), we obtain the inequalities in (2.6), (2.7) and (2.8). This completes the proof of Theorem 2.10.

Corollary 2.11: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_n^1(A,B,\sigma)$, then for $-1 \le B < A \le 1$, $\sigma = -0.618$ and $n \in \mathbb{N}_0$

$$\big|a_2\big| \le \frac{|\sigma|}{2^{n+1}}\big(A-B\big),$$

$$\big|a_3\big| \le \frac{|\sigma|}{2}\big(A-B\big)\Big(\frac{1}{3}\Big)^n max\left\{1, \Big|\frac{1}{2}\big[\sigma\big(5-B\big)\big]\Big|\right\}$$

and

$$\left|a_4\right| \leq \frac{\left|\sigma\right|}{2^{2n+1}}\big(A-B\big).$$

Corollary 2.12: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_n^1(1,-1,\sigma)$, then for $-1 \leq B < A \leq 1$, $\sigma = -0.618$ and $n \in \mathbb{N}_0$

$$\left|a_2\right| \leq \frac{\left|\sigma\right|}{2^n}, \quad \left|a_3\right| \leq \frac{\left|\sigma\right|}{3^n} max\left\{1, \left|3\sigma\right|\right\} \ \ and \ \ \left|a_4\right| \leq \frac{\left|\sigma\right|}{2^{2n}}.$$

Corollary 2.13: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_1^1(1,-1,\sigma)$, then for $-1 \leq B < A \leq 1$, $\sigma = -0.618$ and $n \in \mathbb{N}_0$

$$\left|a_2\right| \leq \frac{\left|\sigma\right|}{2}, \quad \left|a_3\right| \leq \frac{\left|\sigma\right|}{3} max\left\{1, \left|3\sigma\right|\right\} \ \ and \ \ \left|a_4\right| \leq \frac{\left|\sigma\right|}{4}.$$

Corollary 2.14: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_0^1(1,-1,\sigma)$, then for $-1 \leq B < A \leq 1$, $\sigma = -0.618$ and $n \in \mathbb{N}_0$

$$\left|a_2\right| \leq \left|\sigma\right|, \quad \left|a_3\right| \leq \left|\sigma\right| max\left\{1, \left|3\sigma\right|\right\} \ \ and \ \ \left|a_4\right| \leq \left|\sigma\right|.$$

Corollary 2.15: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_1^1(1,-1,-0.618)$, then for $-1 \leq B < A \leq 1$, $\sigma = -0.618$ and $n \in \mathbb{N}_0$

$$\left|a_2\right| \leq 0.309, \quad \left|a_3\right| \leq 0.3819 \ \ and \ \ \left|a_4\right| \leq 0.155.$$

Corollary 2.16: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_0^1(1,-1,-0.618)$, then for $-1 \leq B < A \leq 1$, $\sigma = -0.618$ and $n \in \mathbb{N}_0$

$$\left|a_2\right| \leq 0.618, \quad \left|a_3\right| \leq 0.1458 \ \ and \ \ \left|a_4\right| \leq 0.618.$$

Theorem 2.17: Let $f(z)^\alpha$ be of the form (1.2). If $f(z)^\alpha \in B_n^\alpha(A,B,\sigma)$, then for $\alpha > 0$, $-1 \leq B < A \leq 1$, $\sigma = -0.618$, $n \in \mathbb{N}_0$ and for complex number μ

$$\left|a_3 - a_2^2\right| \leq \frac{\left|\sigma\right|}{2}(A-B)\Big(\frac{\alpha}{\alpha+2}\Big)^n max\left\{1, \frac{\left|\sigma\right|}{2}\left|(A-B)\Big(\frac{\alpha+2}{\alpha}\Big)^n\Big(\frac{\alpha}{\alpha+1}\Big)^{2n} - (5-B)\right|\right\} \tag{2.21}$$

Proof: $f(z)^\alpha \in B_n^\alpha(A,B,\sigma)$, then from (2.18) and (2.19), we obtain

$$\begin{aligned} a_3 - a_2^2 &= \frac{1}{4}\sigma\Big(\frac{\alpha}{\alpha+2}\Big)^n\left[c_2 - \frac{1}{4}(2-(5-B)\sigma)c_1^2\right] - \frac{1}{16}\sigma^2(A-B)^2\Big(\frac{\alpha}{\alpha+1}\Big)^{2n} \\ &= \frac{1}{4}\sigma(A-B)\Big(\frac{\alpha}{\alpha+2}\Big)^n\left\{c_2 - \mu c_1^2\right\}, \end{aligned} \tag{2.22}$$

where

$$\mu = \frac{1}{4}\Big(\frac{\alpha+2}{\alpha}\Big)^n\left[\Big(\frac{\alpha}{\alpha+2}\Big)^n\big(2-(5-B)\sigma\big) + \sigma(A-B)\Big(\frac{\alpha}{\alpha+1}\Big)^{2n}\right].$$

Applying Lemma 2.2 on (2.22), then the desired inequality is obtained as seen in (2.21).

Corollary 2.18: Let $f(z)^\alpha \in B_n^1(A, B, \sigma)$, then

$$\left|a_3 - a_2^2\right| \leq \frac{|\sigma|}{2(3)^n}(A-B)max\left\{1, \frac{|\sigma|}{2}\left|\frac{3^n}{2^{2n}}(A-B) - (5-B)\right|\right\}$$

Corollary 2.19: Let $f(z)^\alpha \in B_n^1(1, -1, \sigma)$, then

$$\left|a_3 - a_2^2\right| \leq \frac{|\sigma|}{(3)^n}max\left\{1, |\sigma|\left|\frac{3^n}{2^{2n}} - 3\right|\right\}.$$

Corollary 2.20: Let $f(z)^\alpha \in B_1^1(1, -1, \sigma)$, then

$$\left|a_3 - a_2^2\right| \leq \frac{|\sigma|}{3}max\left\{1, \frac{9|\sigma|}{4}\right\}.$$

Corollary 2.21: Let $f(z)^\alpha \in B_0^1(1, -1, \sigma)$, then

$$\left|a_3 - a_2^2\right| \leq \left|\sigma\right|max\left\{1, 2|\sigma|\right\}.$$

Corollary 2.22: Let $f(z)^\alpha \in B_1^1(1, -1, -0, 618)$, then

$$\left|a_3 - a_2^2\right| \leq 0.2864.$$

Corollary 2.23: Let $f(z)^\alpha \in B_0^1(1, -1, -0, 618)$, then

$$\left|a_3 - a_2^2\right| \leq 0.7638 .$$

Conclusion:

Finally, in this paper, anew class of Cardio-Bazilevic functions $B_n^\alpha(A, B, \sigma)$, type α, associated with symmetric cardioid domain in the open unit disk E is derived with the aid of Salagean derivative operator. This was achieved via the subordination principle and Hadamard product of certain fractional analytic function $g(z)^\alpha$ the normalized univalent function $f(z)$. Additionally, sharp bounds on the first three Taylor-Maclaurin coefficients for functions belonging to the aforementioned class were obtained while the relationship of these bounds to the classical Fekete-Szego inequality was established using a very lucid Mathematical approach while some of the consequences of the results obtained were discussed as corollaries.

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