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ERGODIC THEOREM FOR QUANTUM DYNAMICAL SEMIGROUP ON DECOHERENCE-FREE SUBALGEBRA OF A QUANTUM MARKOV SEMIGROUP

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Abstract. In this work, we study ergodic theorem for quantum dynamical semigroup on decoherence-free subalgebra of quantum Markov semigroup. A quantum dynamical semigroup of non-expansive maps which has at least one stationary point was established on decoherence-free subalgebra.The measurability of the semigroup was then used to ensure the weak convergence of the Cesaro means to a stationary point.

1. Introduction

A decoherence-free subspace(DFs) is a subspace of a quantum systems Hilbert space that is invariant to non-unitary dynamics. The study of decoherence-free subspace began with a search for structured methods to avoid decoherence in the subject of quantum information processing (QIP). The methods involved attempts to identify particular states which have the potential of being unchanged by certain decohering processes (i.e certain interaction with the environment). These studies started with observations made by [25] who studied the consequences of pure dephasing on two qubits that have the same interaction with the environment. They found that two such qubits do not decohere. They also characterized decoherence- free subspace (DFs) as a special class of quantum error correcting codes.

In the approach of [4], if the space is an algebraic space (i.e a von Neumann or a C*-algebra), the decoherence- free subspace becomes a decoherence-free subalgebra. Therefore, if a von Neumann algebra is a quantum system (say open), there is a part of the algebra that behaves like a closed system (i.e a part that is decoherence-free). This can be used to study the asymptotic behavior

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of the quantum system.The evolution of a closed quantum system which does not interact with the environment, can be described by a one-parameter group of automorphism $(a_t)_{t\geq 0}$, with $a_t(x) = e^{itH}xe^{-itH}$ and H self-adjoint. Inside an open Quantum system, sometimes, one can have a subsystem evolving like a closed quantum system where the typical effects of the interaction with the environment do not appear and the typical quantum features of the system, like quantum coherence and entanglement of quantum states are preserved [11].

In recent years, there has been a growing interest in the use of Quantum Markov Semigroup to model open quantum systems having subsystems which are not affected by decoherence. In these applications, the Quantum Markov Semigroup (in the Heisenberg picture) acts as a semigroup of automorphisms of a von Neumann subalgebra $N(T)$ of $B(h)$, called the Decoherence-free subalgebra. This subalgebra allows identification of noise protected subsystems where states evolve unitarily, moreover its structure and relationship with the set of fixed points also has important consequences on the asymptotic behavior of the Quantum Markov Semigroup. Decoherence-free subalgebra allows us to gain insight into the structure of a Quantum Markov Semigroup. Decoherence-free (DF) subalgebra of T is therefore defined as follows:

$$N(T) = \{x \in B(h) : T_t(x^*x) = T_t(x)^*T_t(x), T_t(xx^*) = T_t(x)T_t(x)^* \; \forall \; t \geq 0\}. \tag{1.1}$$

In [12], the quantum Markov semigroup and their stationary states were studied. The convergence of the Cesaro means of quantum dynamical semigroup was established. The aims of this work is to establish a quantum dynamical semigroup of non-expansive maps which has at least one stationary point on decoherence-free subalgebra and to find the weak convergence of the Cesaro means to a stationary point.

2. Preliminaries

The following definitions, lemmas and propositions are necessary for our results

Definition 2.1. [24] A point x_α is a stationary point of a semigroup of operators $\{T_t\}$ if and only if

$$T_t x_\alpha = x_\alpha \; \forall \; t \geq 0. \tag{2.1}$$

Definition 2.2. [24] Let K be a closed convex subset of x_α and $D \subseteq R_+$, a family of maps T_t from K to K is called a semigroup of non-expansive maps if

1 $T_0 = 1, \; \forall \; t, s \in D$;
2 $T_{t+s} = T_t T_s \; \forall \; t, s \in D$; and
3 $\forall \; t \in D, x, y \in K, \| T_t x - T_t y \| \leq \| x - y \|$.

Definition 2.3. [24] If x_α is a stationary point of the semigroup, the function $t \to \| T_t x - x_\alpha \|$ is non-increasing. That is $t \geq s$,

$$\| T_t x - x_\alpha \| = \| T_{t-s} T_s x - T_{t-s} x_\alpha \| \leq \| T_s x - x_\alpha \| . \tag{2.2}$$

Therefore, if there exists at least one stationary point x_α, the function $t \to \| T_t x - x_\alpha \|^2$ has a limit as $t \to \infty$; hence $t \to T_t x$ is bounded and thus it has an asymptotic centre.

Definition 2.4. [23] Let $D = [0, +\infty)$, consider a function $Q : D \times D \to D$ satisfying the properties

1 $\forall\, N \in D, s \to Q(N, s)$ is μ-measurable
2 $\forall\, N \in D, \int_D Q(N, s) d\mu(s) = 1$.

If we set

$$Q(N,t) = \frac{1}{N};\ \text{when}\ 0 \leq t \leq N,\ 0;\ \text{when}\ t > N. \tag{2.3}$$

Assuming $t \to T_t x$ is strongly μ-measurable, we define the Cesaro means $\sigma_Q(N)x$ by

$$\sigma_Q(N)x = \int_D Q(N, s) T_s x d\mu(s). \tag{2.4}$$

Proposition 2.5. [23] *The asymptotic centre of a bounded sequence of elements $a_n \in A$ belongs to the closed convex hull of its weak cluster points. Moreover, if the bounded sequence of elements a_n converges weakly, then the limit coincides with the asymptotic centre.*

Proof. Let a_n be the asymptotic centre of the bounded sequence of elements a_n and b_∞ be the projection of a_∞ onto the closed convex hull C of the weak cluster points of the sequence $\{a_n\}_n$. It is non-empty because any bounded sequence is weakly relatively compact. There exists a subsequence of elements a_n,such that $\phi(b_\infty) = \lim_{n' \to \infty} \| a_{n'} - b_\infty \|^2$. Hence

$$\lim_{n' \to \infty} \sup \| a_{n'} - a_\infty \|^2 \leq \phi(x_\infty). \tag{2.5}$$

Also there exists a subsequence of elements $a_{n''}$ that converges to some $z \in C$. Therefore

$$\lim_{n'' \to \infty} \langle b_\infty - a_\infty, b_\infty - a_{n''} \rangle = \langle b_\infty - a_\infty, b_\infty - z \rangle \leq 0, \tag{2.6}$$

because b_∞ is the projection of a_∞ to C. From the identity

$$\| a_n - a_\infty \|^2 = \| a_n - b_\infty \|^2 + \| b_\infty - a_\infty \|^2 + 2\langle a_n - b_\infty, b_\infty - a_\infty \rangle. \tag{2.7}$$

We deduce that

$$\phi(a_\infty) \geq \phi(b_\infty) + \| a_\infty - b_\infty \|^2 \geq \phi(b_\infty), \tag{2.8}$$

from the uniqueness of the minimum of ϕ, it follows that $a_\infty = b_\infty \in C$. The second part of the proposition is an immediate consequence of the first. □

Proposition 2.6. *If all the weak cluster points of $T_t x$ are stationary points, then $T_t x$ converges weakly to its asymptotic centre as $t \to \infty$.*

Proof.

$$M = \{y \in X : w^* - \lim_{t \to \infty} \| T_t x - y \| \ \text{exist}\}, \tag{2.9}$$

and N denotes the set of stationary points x_α of the semigroup; then

$$\| T_t x - x_\alpha \| = \| T_{t-s} T_s x - T_{t-s} x_\alpha \| \leq \| T_s x - x_\alpha \|, t \geq s. \tag{2.10}$$

Implies that $N \subset M$. The result therefore follows from the above proposition. □

Proposition 2.7. *Let T be a non-expansive map from a closed convex subset $K \subset A$ to itself. Assume that a sequence of elements $x_N \in K$ satisfies the condition*

$$\lim_{N\to\infty} \| x_N - T_{x_N} \| = 0. \tag{2.11}$$

Then any weak cluster point of this sequence is a fixed point of T.

Proof. Let x_α be the weak limit of a generalized subsequence $\{x_{N'}\}_{N'}$ of the sequence $\{x_N\}_N$ and let us set $x_\lambda = (1-\lambda)x_\alpha + \lambda T x_\alpha$ where $\lambda \in (0,1]$. Since T is non-expansive, we deduce that

$$\langle x_\lambda - Tx_\lambda - (x_{N'} - Tx_{N'}), x_\lambda - x_{N'}\rangle \geq \| x_\lambda - x_{N'} \|^2 - \| Tx_\lambda - Tx_{N'} \| \\ \| x_\lambda - x_{N'} \| \geq 0.$$

By letting $x_{N'} \to x_\alpha$, we deduce from the inequality

$$\langle x_\lambda - Tx_\lambda, x_\lambda - x_\alpha\rangle \geq 0. \tag{2.12}$$

that $x_{N'} \to Tx_{N'}$ strongly. This latter inequality can be written, after division by $\lambda > 0$,

$$\langle (1-\lambda)x_\alpha + \lambda Tx_\alpha - T[(1-\lambda)x_\alpha + \lambda Tx_\alpha], Tx_\alpha - x_\alpha\rangle \geq 0. \tag{2.13}$$

By letting $\lambda \to 0$, we deduce that $\| Tx_\alpha - x_\alpha \|^2 \leq 0$,which implies that $x_\alpha = Tx_\alpha$. □

Lemma 2.8. *Assume that there exists at least one stationary point of the QDS $\{T_t\}$ and that*

$$t \geq 0, \lim_{N\to\infty} \| \sigma_Q(N)x - T_t\sigma_Q(N)x \| = 0. \tag{2.14}$$

when $N \to \infty$,$\sigma_Q(N)x$ converges weakly to the asymptotic centre of $t \to T_t x$, which is a stationary point of the QDS.

Proof. Set $y_N = \sigma_Q(N)x$, proposition 2.6 implies that all the weak cluster points of the sequence $\{y_N\}$ are stationary points of the semigroup; these weak cluster points belong to the subset

$$K = \{y \in A : \lim_{t\to\infty} \| T_t x - y \| \text{ exist}\}. \tag{2.15}$$

□

Lemma 2.9. [24] *Set $y_N = \sigma_Q(N)x$. Then*

$$\| T_t y_N - y_N \|^2 \leq \int_{D\cap[0,t]} Q(N,s) \| T_s x - T_t y_N \|^2 \, d\mu(s) \\ + \int_D |Q(N,s+t) - Q(N,s)| \, \| T_s x - y_N \|^2 \, d\mu(s).$$

Proof. Since $T_{t+s}x - T_t y \| \leq \| T_s x - y \|$,we deduce that

$$\begin{aligned} 0 \leq & \| T_s x - T_t y + T_t y - y \|^2 - \| T_{t+s}x - T_t y \|^2 \\ & + \| T_s x - T_t y \|^2 + \| T_t y - y \|^2 - \| T_{t+s}x - T_t y \|^2 \\ & + 2\langle T_s x - T_t y, T_t y - y \rangle. \end{aligned}$$

If we take $y = y_N$, the above inequality becomes

$$\| T_t y_N - y_N \|^2 \leq \int_D Q(N,s)(\| T_s x - T_t y_N \|^2 - \| T_{t+s}x - T_t y_N \|^2) d\mu(s) \quad (2.16)$$

Since,

$$\begin{aligned} \int_D Q(N,s) \| T_s x - T_t y_N \|^2 \, d\mu(s) = & \int_{D\cap[0,t]} Q(N,s) \| T_s x - T_t y_N \|^2 \, d\mu(s) \\ & + \int_D Q(N,s+t) \| T_{t+s}x - T_t y_N \|^2 \, d\mu(s). \end{aligned}$$

The above inequality can be rewritten as

$$\begin{aligned} \| T_t y_N - y_N \|^2 \leq & \int_{D\cap[0,t]} \| T_s x - T_t y_N \|^2 \, d\mu(s) \\ & + \int_D |Q(N,s) - Q(N,s+t)| \, \| T_{t+s}x - T_t y_N \|^2 \, d\mu(s). \end{aligned}$$

We estimate the last integral by using

$$\| T_{t+s}x - T_t y \|^2 \leq \| T_s x - y_N \|^2 . \quad (2.17)$$

□

Lemma 2.10. [14] *If $S : X \to X$ is affine and nonexpansive in the sense that $\| Sx - Sy \| \leq \| x - y \|$ such that X is a Banach space, and if*

$$A_n x = \frac{1}{n} S^n x \quad (2.18)$$

then $\{A_n\}$ converges pointwise on X.

3. Main Results

Here, we want to establish the weak convergence of Cesaro means

$$\frac{1}{N}\int_0^N T_t x^* x dt. \quad (3.1)$$

Definition 3.1. Let $N(T)$ be a closed convex subset of x_α and $D \subseteq R_+$, a family of maps T_t from $N(T)$ to $N(T)$ is called a semigroup of non-expansive maps if

1 $T_0 = 1, \ \forall \, t, s \in D$;
2 $T_{t+s} = T_t T_s \ \forall \, t, s \in D$, and
3 $\forall \ t \in D, x, y \in N(T), \| T_t x - T_t y \| \leq \| x - y \|$ and $x^*, y^* \in N(T)$,$\| T_t x^* - T_t y^* \| \leq \| x^* - y^* \|$.

Lemma 3.2. *Set* $y_N^* y_N = \sigma_Q(N) x^* x$. *Then*

$$\| T_t y_N^* y_N - y_N^* y_N \|^2 \leq \int_{D\cap[0,t]} \| y_N^* \|^2 \, Q(N,s) \| T_s x - T_t y_N \|^2 \, d\mu(s) + \int_D |Q(N,s+t) - Q(N,s)| \, \| y_N^* \|^2 \| T_s x - y_N \|^2 \, d\mu(s).$$

Proof.

$$\| T_t y_N^* y_N - y_N^* y_N \|^2 \leq \| y_N^* \|^2 \| T_t y_N - y_N \|^2 . \tag{3.2}$$

Since,

$$\| T_t y_N - y_N \|^2 \leq \int_{D\cap[0,t]} Q(N,s) \| T_s x - T_t y_N \|^2 \, d\mu(s) + \int_D |Q(N,s+t) - Q(N,s)| \, \| T_s x - y_N \|^2 \, d\mu(s).$$

So, we have

$$\| T_t y_N^* y_N - y_N^* y_N \|^2 \leq \int_{D\cap[0,t]} \| y_N^* \|^2 \, Q(N,s) \| T_s x - T_t y_N \|^2 \, d\mu(s) + \int_D |Q(N,s+t) - Q(N,s)| \, \| y_N^* \|^2 \| T_s x - y_N \|^2 \, d\mu(s)$$

□

Theorem 3.3. *Let* $T_t : N(T) \to N(T)$ *be* μ*-measurable Quantum dynamical semigroup of non-expansive maps which has at least one stationary point. Let* Q *be any function* $D \times D$ *to* R_+, *measurable with respect to the second variable of bounded variation, satisfying*

1 $\forall\, N > 0, \int_D Q(N,s) d\mu(s) = 1$;
2 $\forall\, t \in D, \lim_{N\to\infty} \int_{N\cap[0,t]} Q(N,s) d\mu(s) = 0$; *and*
3 $\forall\, t \in D, \lim_{N\to D} \int_D |Q(N,s+t) - Q(N,s)| d\mu(s) = 0$,

Then the averages $\sigma_Q x^* x$ *converges weakly to the asymptotic centre of* $t \to T_t x^* x$ *which is a stationary point of the semigroup, as* $N \to \infty$.

Proof. We have to check that

$$\forall\, t \geq 0, \| T_t y_N^* y_N - y_N^* y_N \|^2 = 0 \tag{3.3}$$

We will use the above lemma (1) to prove this. Observe that if $x_\alpha \in K$ is a stationary point of the semigroup, we can estimate

$$\| T_s y - z \| = \| T_s y - T_s x_\alpha + x_\alpha - z \| . \tag{3.4}$$

It can be solved further by

$$\| T_s y - T_s x_\alpha + x_\alpha - z \| \leq \| T_s y - T_s x_\alpha \| + \| x_\alpha - z \| = \| T_s x_\alpha - T_s y \| + \| x_\alpha - z \| \leq \| x_\alpha - y \| + \| x_\alpha - z \| .$$

We also observe that

$$\| x_\alpha - y_N \| = \| \int_D Q(N,s)(T_s x_\alpha - T_s x) d\mu(s) \| \leq \| x_\alpha - x \| . \tag{3.5}$$

By taking $y = x$ and $z = T_t y_N$, we deduce that

$$\begin{aligned} \| T_s x - T_t y_N \| &\leq \| x_\alpha - x \| + \| x_\alpha - T_t y_N \| \\ &\leq \| x_\alpha - x \| + \| x_\alpha - y_N \| \\ &\leq \| x_\alpha - x \| + \| x_\alpha - x \| \\ &\leq 2 \| x_\alpha - x \| . \end{aligned}$$

And by taking $y = x$ and $z = y_N$, we also deduce that

$$\begin{aligned} \| T_s x - y_N \| &\leq \| x_\alpha - x \| + \| x_\alpha - y_N \| \\ &\leq \| x_\alpha - x \| + \| x_\alpha - x \| \\ &\leq 2 \| x_\alpha - x \| . \end{aligned}$$

Hence, by Lemma 2.8,

$$\begin{aligned} \| T_t y_N^* y_N - y_N^* y_N \|^2 \leq & \int_{D\cap[0,t]} \| y_N^* \|^2 Q(N,s) \| T_s x - T_t y_N \|^2 \, d\mu(s) \\ & + \int_D |Q(N,s+t) - Q(N,s)| \, \| y_N^* \|^2 \| T_s x - y_N \|^2 \, d\mu(s). \end{aligned}$$

By substituting the above inequalities,we have

$$\begin{aligned} \| T_t y_N^* y_N - y_N^* y_N \|^2 \leq & \int_{D\cap[0,t]} \| y_N^* \|^2 Q(N,s) 4 \| x_\alpha - x \|^2 \, d\mu(s) \\ & + \int_D |Q(N,s+t) - Q(N,s)| \, \| y_N^* \|^2 \, 4 \| x_\alpha - x \|^2 \, d\mu(s). \end{aligned}$$

By taking $y_N^* = y^*$ and $y^* = x^*$, we have

$$\begin{aligned} \| T_t y_N^* y_N - y_N^* y_N \|^2 \leq & \int_{D\cap[0,t]} \| x^* \|^2 Q(N,s) 4 \| x_\alpha - x \|^2 \, d\mu(s) \\ & + \int_D |Q(N,s+t) - Q(N,s)| \, \| x^* \|^2 \, 4 \| x_\alpha - x \|^2 \, d\mu(s) \\ & \leq \int_{D\cap[0,t]} Q(N,s) 4 \| x^* x_\alpha - x^* x \|^2 \, d\mu(s) \\ & + \int_D |Q(N,s+t) - Q(N,s)| 4 \| x^* x_\alpha - x^* x \|^2 \, d\mu(s). \end{aligned}$$

□

The right-hand side of the inequality converges as $N \to \infty$ by assumptions (2) and (3) of Theorem 3.3

Corollary 3.4. *Let $T_t : N(T) \to N(T)$ be a quantum dynamical semigroup of non-expansive maps which has at least one stationary point. If the semigroup T is measurable, then*

$$x \in N(T), \frac{1}{N} \int_0^N T_\tau x^* x d\tau$$

converges weakly to a stationary point.

Example 3.5. Gievn $\mathcal{B}$ to be a closed convex subset of a decoherence-free subalgebra $N(T)$ whose dual has Frechet differentiable norm, and let $S : \mathcal{B} \to \mathcal{B}$ be nonexpansive. Then $\lim T_t x * x$ exists for each $x \in \mathcal{B}$. If we let $0 \in \mathcal{B}$, without loss of generality. Since $0 \in \mathcal{B}$ is a contraction of $\mathcal{B}$, the map $x^*x \to (1-r)^{-1}Sx^*x$ has a fixed point $(y^*y)_r \in \mathcal{B}$, for $r > 0$. By the triangle inequality,

$$\| (y^*y)_r - x^*x \| - \| (y^*y)_r - Sx^*x \| \geq d - 2r \| Sx^*x \| \quad (x \in \mathcal{B}) \tag{3.6}$$

where $d = \inf \| Sx^*x - x^*x \|$. If we apply this repeatedly,we have

$$\| (y^*y)_r \| - \| (y^*y)_r - S^n 0 \| \geq nd - 2r(\| S0 \| + - - - + \| S^n 0 \|) \tag{3.7}$$

Pick $(y^*y)_r^* \in N(T)^*$ with $\| (y^*y)_r^* \| = 1$ and $\langle (y^*y)_r, (y^*y)_r^* \rangle = \| (y^*y)_r \|$. Then $\langle S^n 0, (y^*y)_r^* \rangle$ dominates the left side of 2.18. Letting $r \to 0$, the Banach-Alaoglu theorem furnishes a $(y^*y)^* \in N(T)^*$ with $\| (y^*y)^* \| = 1$ and $\langle T_t 0, (y^*y)^* \rangle \geq d$. It is easy to verify that $\varsigma = \limsup \| T_t x^*x \|$ is independent of x^*x and that $\varsigma \leq d$. Thus, we conclude that for all $x \in \mathcal{B}$,

$$\lim \langle T_t x^*x, (y^*y)^* \rangle = \lim \| T_t x^*x \| . \tag{3.8}$$

Hence, $T_t x^*x$ converges.

4. Conclusion

From the results, we were able to conclude that the measurability of a quantum dynamical semigroup of non-expansive maps which has at least one stationary point will ensure the weak convergence of the Cesaro means to a stationary point.

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