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THE GAMMA AND BETA MATRIX FUNCTION AND OTHER APPLICATIONS

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ABSTRACT. The paper presents Gamma and Beta functions and their applications to real life problems. After relating the gamma function with Euler's infinite products, the Hermite series and Hypergeometric series, then the computation of perimeter (p_n) of a polygon involving gamma function by which a factor $\frac{p_n}{2\pi}$ exceeds the perimeter of a unit circle is presented in the sense of Jorda and Cortise. Application to Hermite-Laguerre polynomial and multivariate calculus Hypergeometric matrix functions are presented with convergence of gamma matrix using the Ratio Test. The procedure for detecting nearness to singularity of the gamma matrix is described in terms of condition number where eigenvalues are ordered according to their magnitudes. In addition, the Numerical radius of the Gamma matrix is introduced which helps in the computation of a bound for the condition number of the matrix. In particular, we paid special attention to the analysis of the pendulum problem as a second order differential initial value problem wherein, Jacobi elliptic integrals of first and second kinds play major roles. The bounds for these elliptic integrals are discussed in details using some ideas in the existing literatures. It is established in this paper that, there exists no universally most acceptable bound for these Jacobi elliptic integrals as attested to by various authors. It is therefore suggested in this paper that these bounds may be subjected to probabilistic analysis in our future work. It is also hoped to link these bounds for the Jacobi elliptic integrals with the Weierstrass elliptic functions as well.

1. INTRODUCTION

The first aim of this article is to answer in the affirmative that there exists [3].

1.1. The Gamma and Beta functions have several applications in scientific and engineering practices other than statistical density functions in which the later were originally defined and intended for use Kargin and Kurt (2013), Bao (2021). Gamma function has several uses in the

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representation of hypergeometric function and infinite products, Huang et al (2017). This paper aims at a general framework for computing Gamma and Beta functions as well as conformal radius applied on multivariate functional equations found in optimization theory and complex integral calculus. We use the complex contour integrals as deriving basic properties for the gamma function and relate this to inverse Laplace transform using ideas due to Berg (2004), taking the orientation counterclockwise, the deformed Bromwich contour. With deep knowledge of Laurent series, various residue theorems could be obtained which are useful in the applications of gamma function and its allied functions.

Gamma, Beta, hypergeometric functions have major roles they play in the treatments and analyses of swinging pendulum differential equation problems. For instance, Jacobi (or Legendre) elliptic integrals of both first and second kinds owe much of treatise to the gamma and hypergeometric functions. Jacobi elliptic integrals of both first and second kinds have various uses in engineering designs of rotating rods and a great impetus for expressing strong solutions to mathematical differential equations.

Definition 1.1, Berg (2004): (Contour or path of complex integral). Let $\eta:[a,b]\to\mathbb{C}$ be an oriented differentiable $C^{/}$-curve and $f:\eta^{*}\to\mathbb{C}$ be continuous. By the contour (or path) integral of f along η we mean the complex number

$$\int_{\eta} f = \int_{\eta} f(t)dt = \int_{a}^{b} f(\eta(t))\eta^{/}(t)dt$$

for which the orientation over Jordan arc is rectifiable.

In a simple language, the word contour mean a continuous parameterization $\eta:[a,b]\to\mathbb{C}$ which is piecewise $C^{/}$ continuous for which exists a partition

$a=t_0<t_1<\ldots<t_{n-1}=b$, such that, $\eta_k=\eta\left|t_{k-1},t_k\right|,(k=1,2,\ldots,n)$ are $C^{/}$ parameterizations

$$\int_{\eta} f = \int_{\eta_1\cup..\cup\eta_n} f = \sum_{k=1}^{n}\int_{\eta_k} . = \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} f(\eta(t))\eta^{/}(t)dt .$$

The Cauchy's integral theorem states that if $f:D\subset\mathbb{C}\to\mathbb{C}$ be holomorphic in a simply connected domain D and given that η be a closed path in D, then

$$f(z_0)=\frac{1}{2\pi i}\oint_{\eta}\frac{f(z)}{z-z_0}dz .$$

Cauchy's integral theorem is the basis upon which many formulae for complex functions are derived and their convergences analyzed. The Borel's covering theorem for instance, asserts that if K be a closed and bounded subset of$\mathbb{C}$, and for every family $\{F_i\}_{i\in I}$ of open sets in $\mathbb{C}$ covering K such that $K \subseteq \bigcup_{i\in I} F_i$, there are finitely many indices $i_1, i_2, ..., i_n \in I$ for which holds $K \subseteq F_{i_1} \cup F_{i_2} \cup ... \cup F_{i_n}$.

We introduce the gamma function by the equation

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \qquad (\text{Re } z>0) \qquad (1.1.1)$$

where, $t^{z-1} = e^{(z-1)\log t}$, and $\log t \in R$ so that $\Gamma(z+1) = z\Gamma z,\ \Gamma(1) = 1,\ \Gamma(n+1) = n!$.

The limiting Stirling's series is defined by the equation

$$n! = \sqrt{\pi n}\left(n^n e^{-n}\right), \text{ (for } n \to \infty) \qquad (1.1.2)$$

The Spouge's formula for gamma function Pugh (2004) is defined to be

$$\Gamma(z+1) = (z+a)^{z+\frac{1}{2}} e^{-(z+a)} (2\pi)^{\frac{1}{2}} \left[c_0 + \sum_{k=1}^{N} \frac{c_k}{z+k} + \varepsilon(k) \right], \qquad (1.1.3)$$

where $N = [a]-1, c_0 = 1, c_k$ is the residue of $\Gamma(z+1)(z+a)^{-\left(z+\frac{1}{2}\right)} e^{z+a} (2\pi)^{-\frac{1}{2}}$ at $z = -k$ .The a is a free parameter suitable for proper adjustment in achieving good accuracy to the approximation. More definitive in our presentation Pugh(2004) for the Stirling series is the equation:

$$\log\left[\Gamma(z+1)\prod_{k=1}^{N}(z+k)\right] = \left(z+N+\frac{1}{2}\right)\log(z+N) - (z+N) + \frac{1}{2}\log 2\pi + \sum_{j=1}^{n} \frac{B_{2j}}{2j(2j-1)(z+N)^{2j-1}} - \int_0^\infty \frac{B_{2n}(x)}{2n(z+N+x)^{2n}} dx \qquad (1.1.4)$$

where, B_{2j} are the Bernouli numbers obtained from the Maclaurin series for $\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_{2j}}{j!} t^j$ To define the inverse gamma function Temme (1996), the method of Euler was initiated. Firstly define

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)(z+2)...(z+n)} \qquad (1.1.5)$$

Then the inverse gamma function in view of Equation (1.1.5) is given in the form:

$$\frac{1}{\Gamma(z)} = \lim_{n\to\infty} \frac{n^{-z}}{n!} inv[\, z(z+1)(z+2)...(z+n)] \ , \ (\forall z) \tag{1.1.6}$$

where $inv(.)$ denotes inverse function, and the poles are at $z = 0,-1,-2,...,$ such that the reflection formula is therefore expressed in the form:

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z} \quad \text{with} \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

For $n \in N$, the first three rational (fractional) form of gamma functions are well known, Rainville (1960) in the form:

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1.3.5...(2n-1)}{2^n}\sqrt{\pi}, \qquad (n = 1,2,...)$$
$$\Gamma\left(n+\frac{1}{3}\right) = \frac{1.4.7.(3n-2)}{3^n}\Gamma\left(\frac{1}{3}\right), \quad (n = 1,2,...),$$
$$\Gamma\left(n+\frac{1}{4}\right) = \frac{1.5.9...(4n-3)}{4^n}\Gamma\left(\frac{1}{4}\right), \quad (n = 1,2,...).$$

Conversely, for negative integers their gamma functions Hannah (2013) are defined in the form:

$$\Gamma(k) = \frac{\Gamma(k+n)}{k(k+1)(k+2)...(k+n-1)}, \text{ where }, \ -n < k < -n+1, \quad n \in N$$

Setting as $t = su$ into Equation (1.1.1) gives rise to the Laplace transform, Schmelzer and Trefethen (2007) in the form:

$$F(s) = \frac{\Gamma(z)}{s^z} = \int_0^\infty u^{z-1} e^{-su}\, du \tag{1.1.7}$$

The recoverable part of Equation (1.1.7) as the byproduct in the calculation yields for the inverse Laplace transform u^{z-1} . This is given by the equation

$$u^{z-1} = \Im^{-1}(F(s)) = \frac{1}{2\pi i}\int_\eta e^{ku} F(k)dk = \frac{1}{2\pi i}\int_\eta \frac{e^{ku}\Gamma(z)}{k^z} dk \tag{1.1.8}$$

The path η is the Jordan arc taken counterclockwise the deformed Bromwich contour.

Now, assuming instead, we set $s = ku$ in Equation (1.1.7) this gives

$$u^{z-1} = \frac{1}{2\pi i}\int_C e^s \frac{\Gamma(z)u^z}{s^z u} ds \ , \tag{1.1.9}$$

where

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_C s^{-z}e^s ds \qquad (1.1.10)$$

Equations (1.1.9) and (1.1.10) are often computed using suitable trapezoid quadrature rule along the contour, the path traced out by the complex integrals.

The link between gamma function and Euler infinite products is presented. Thus, the Euler's infinite product, for $z \neq 0,-1,-2,\ldots$, is defined in the form:

$$\Gamma(z) = \frac{1}{z}\prod_{1\le n<\infty}\frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}} \qquad (1.1.11)$$

Using Holder's inequality e.g., $(1 < p < \infty)$ and $\frac{1}{p}+\frac{1}{q}=1$, there follows:

$$\Gamma\left(\frac{z}{p}+\frac{y}{q}\right) = \int_0^\infty \left(t^{z-1}e^{-t}\right)^{\frac{1}{p}}\left(t^{y-1}e^{-t}\right)^{\frac{1}{q}}dt \le \left(\int_0^\infty t^{z-1}e^{-t}dt\right)^{\frac{1}{p}}\left(\int_0^\infty t^{y-1}e^{-t}dt\right)^{\frac{1}{q}} = \left(\Gamma(z)\right)^{\frac{1}{p}}\left(\Gamma(y)\right)^{\frac{1}{q}}.$$

If we set, instead that $\lambda = \frac{1}{p}$ and, $1-\lambda = \frac{1}{q}$, the convexity of $\log\Gamma(z)$ is expressed following from convexity of

$$\log\Gamma(\lambda z + (1-\lambda)y) \le \lambda\log\Gamma(z) + (1-\lambda)\log\Gamma(y). \qquad (1.1.12)$$

We give the respective Euler's and Gauss' limiting formula in their equivalent forms for gamma function similar to equation (1.1.5) in their forms:Euler's :

$$\Gamma(z+1) = \lim_{n\to\infty}\left[(n+1)^z\prod_{i=1}^{n}\frac{i}{i+z}\right] = \lim_{n\to\infty}\frac{n!(n+1)^z}{(z+1)(z+2)\ldots(z+n)} \qquad (1.1.13)$$

Gauss' :

$$\Gamma(z) = \lim_{n\to\infty}\frac{n!n^z}{z(z+1)(z+2)\ldots(z+n)} = \lim_{n\to\infty}\frac{n!n^z}{(z)_{n+1}} \qquad (1.1.14)$$

for $(z)_{n+1} = z(z+1)(z+2)\ldots(z+n)$.

The categorization of this paper is as follows: Section 2 discusses the Beta function and its associated conformal radius. We link the gamma function with regularity spaces of Hankel functions and the hypergeometric function. The matrix gamma and Beta functions are presented for the multivariate functional calculus. We give further information on the condition number of a gamma matrix which can be computed as a ratio of Numerical radius of a matrix to the spectral radius of the matrix. Condition for nearness to singularity is discussed in terms of ratio of largest eigenvalue to the smallest eigenvalue assuming the eigenvalues are ordered according to their magnitudes and counting their multiplicities (if any) so that the number of eigenvalues is exact. We give numerical example demonstrating the discussed formulae in section3. Useful bounds for the Jacobi elliptic integral are given for the pendulum problem as a second order differential equation Initial Value problem using gamma and hypergeometric functions. Section 4 gives the discussions aspect of results and analysis in the paper. In section 5, we give conclusion based on the findings of our numerical examples with these methods.

2. MATERIALS AND METHODS

The following methods and materials shall be adopted for our approach.

2.1 The Beta Function and Associated Conformal Radius

We define the Euler Beta function by the equation

$$B(\alpha,\beta)=\int_0^1 u^{\alpha}(1-u)^{\beta}\frac{du}{u(1-u)}=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad , \quad \left(B(\alpha,1)=\frac{1}{\alpha}\right) \tag{2.1.1}$$

Here $\alpha>0$, and $\beta>0$ with $B(\alpha,\beta)$ is a function of α and β. $\Gamma(\alpha)=\alpha\Gamma(\alpha-1)$, $\Gamma(\beta)=\beta\Gamma(\beta-1)$.

Integration by part applied on Equation (2.1.1) gives that

$$B(\alpha,\beta)=\frac{(\beta-1)}{\alpha}B(\alpha+1,\beta-1)\ , \tag{2.1.2}$$

$$B(\alpha+1,\beta+1)=\frac{\alpha!\beta!}{(\alpha+\beta)!}=\frac{1}{(\alpha+\beta+1)!}\binom{\alpha+\beta}{\alpha}^{-1} \tag{2.1.3}$$

We link the hyper geometric function with the gamma function. Firstly, we give the Hankel contour integral. As usual, consider the integral given by

$$I_{\eta}=\int_{\eta}(-t)^{z-1}e^{-t}dt\,, \tag{2.1.4}$$

We consider a positive oriented contour at the point $x+i0^{+}$, $(x>0)$ above the real x-axis which encircles the coordinate origin in counterclockwise manner Pugh (2004) that returns to the point $x-i0^{+}$ below the x-axis by fixing the branch of the multivalued function $(-t)^{z-1}$. Then define that

$(-t)^{z-1}=e((z-1)\ln(-t))$, where $\ln(-t)$ is purely real on the negative real axis and argument on η being $-\pi \leq \arg(-t) \leq \pi$.

The radius being denoted by ρ such that $\arg(-t)=\pm\pi$ and $(-t)^{z-1}=e^{\mp i(z-1)}t^{z-1}$ on each segment of the contour.

Using $(-t)=\rho e^{i\theta}$, and writing that

$$I_{\eta}(z)=\int_{x}^{\rho} e^{-i\pi(z-1)}t^{z-1}e^{-t}dt+\int_{-\pi}^{\pi}\rho e^{i\theta}i\left(\rho e^{i\theta}\right)^{z-1}e^{-\rho(\cos\theta+i\sin\theta)}d\theta+\int_{\rho}^{x} e^{i\pi(z-1)}t^{z-1}e^{-t}dt \text{ s}$$

$$=2i\sin(\pi z)\int_{\rho}^{x} t^{z-1}e^{-t}dt+i\rho^{2}\int_{-\pi}^{\pi} e^{iz\theta}-\rho(\cos\theta+i\sin\theta)dt \tag{2.1.5}$$

As $\rho\to 0$ for $\operatorname{Re} z>0$, then $I_{\eta}(z)=-2i\sin(\pi z)\int_{0}^{x} t^{z-1}e^{-t}dt$.

We take the limit for $x\to\infty$ and obtained

$$\Gamma(z)=-\frac{1}{2i\sin(\pi z)}\int_{\eta}(-t)^{z-1}e^{-t}dt \tag{2.1.6}$$

Equation (2.1.6) is the Hankel's representation of gamma function which is holomorphic except at the points $z=0,\pm1,\pm2,\ldots$

The hypergeometric series with variable z is defined to be the equation

$$F(a,b,c,z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n z^n}{(c)_n\, n!} \tag{2.1.7}$$

Then, $(a)_0=1,(a)_1=a,\ (a)_2=a(a+n)$,

$$(a)_n=a(a+1)\ldots(a+n-1)=\frac{\Gamma(a+n)}{\Gamma(a)},\ (1)_n=n!,(b)_n=\frac{\Gamma(b+n)}{\Gamma(b)},(c)_n=\frac{\Gamma(c+n)}{\Gamma(c)}.$$

The a,b,c are arbitrary complex numbers. Therefore, the hypergeometric series is in the form

$$F(a,b,c,z)=\sum_{n=0}^{\infty}A_n z^n \quad \text{(Where, } A_n=\frac{(a)_n(b)_n}{(c)_n\ n!}\text{)} \tag{2.1.8}$$

The convergence of hypergeometric series in Equation (2.1.8) is demonstrated by the ratio test

$\lim_{n\to\infty}\left|\frac{A_{n+1}z^{n+1}}{A_n z^n}\right|=\lim_{n\to\infty}\frac{(n+a)(n+b)}{(n+c)(n+1)}|z|=|z|$. This is absolutely convergent inside the unit circle $|z|<1$.

We give two commonly used cases as shown below:

Case 1: b=c , given by

$$F(a,b,b,z)=\sum_{n=0}^{\infty}\frac{(a)_n}{n!}z^n=(1-z)^{-a} \tag{2.1.9}$$

Setting $a=1$, gives $F(1,b,b,z)=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z}$.

Case 2: $b=a+\frac{1}{2},c=\frac{3}{2}$:

$$F(a,\frac{1}{2}+a,\frac{3}{2},z^2)=\frac{1}{2^z(1-2a)}\left[(1+z)^{1-2a}-(1-z)^{1-2a}\right] \tag{2.1.10}$$

From Equation (2.1.10), by letting $a=1,b=c$ would give that $F(1,\frac{3}{2},\frac{3}{2},z^2)=\left(1-z^2\right)^{-1}$.The duplicative formula is in the form:

$$A_n=\frac{\Gamma(a+n)\Gamma\left(a+\frac{1}{2}+n\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(a)\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(\frac{3}{2}+n\right)n!}=\frac{\Gamma(3a+2n)}{\Gamma(2a)(n+1)!} \tag{2.1.11}$$

It beholds that

$$F\left(a,\frac{1}{2}+a,\frac{3}{2},z^2\right)=\frac{1}{z(2a-1)}\sum_{n=0}^{\infty}\frac{\Gamma((2a-1+2n+1)}{\Gamma(2a-1)(3n+1)!}z^{2n+1}$$

2.2 Applications to Multivariate analysis: The role of Gamma and Beta matrix functions

We implement Hermite polynomials and Hypergeometric matrix function. Writing the Hermite polynomials in the sense of Kargin and Kurt (2013) we have that

$$H_n(x,A)=\sum_{k=0}^{\left(\frac{n}{2}\right)}\frac{(-1)^k n!\left(x\sqrt{2A}\right)^{n-2k}}{k!(n-2k)!},\ n\geq 0 \tag{2.2.1}$$

This satisfies the three term recurrence

$$H_n(x,A)=x\sqrt{2A}\ H_{n-1}(x,A)-2(n-1)H_{n-2}(x,A),\ n\geq 1, \tag{2.2.2}$$

$H_{-1}(x,A)=0,\ H_1(x,A)=I$, where A is a real or complex square matrix, I is an identity matrix.

To proceed further, we noted that for a real matrix A and for $\mathrm{Re}(z)>0$ where $z\in\sigma(A)$, and given that $n\geq 1$, Jodar and Cortes (1998) we have the equations

$$E(z)=\int_0^1(1-s)^n s^{z-1}ds=n!\left[z(z+1)\ldots(z+n)\right]^{-1} \tag{2.2.3}$$

$$g(z)=\int_0^n(1-\frac{s}{n})^n s^{A-I}ds=n!n^A\left[A(A+I)\ldots(A+nI)\right]^{-1} \tag{2.2.4}$$

To compute the inverse matrix for A, Golub and Van-Loan (1989), Horn and Johnson (1993) we adopt the Gaussian-LU factorization technique. Other possible methods for inversion of the matrix are the SDV Decomposition or QR Cholesky Factorization (cf Bjorck (2009)). The error estimates for the equations (2.2.3) and (2.2.4) are in the form:

$$\Gamma(A)-n!n^A\left[A(A+I)...(A+nI)\right]^{-1}=$$

$$\int_0^\infty e^{-t}t^{A-I}dt-\int_0^n\left(1-\frac{t}{n}\right)^n t^{A-I}dt=\int_0^n\left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{A-I}dt+\int_n^\infty e^{-t}t^{A-I}dt$$

The term $\int_n^\infty e^{-t}t^{A-I}dt\to 0$ as $n\to\infty$.

By signifying with notation

$$A(A+I)\ldots(A+(n-1)I)\Gamma^{-1}(A+nI)=\Gamma^{-1}(A),\quad n\geq 1 \tag{2.2.5}$$

$$(z)_n=z(z+1)(z+2)\ldots(z+n-1),\ n\geq 1,\ (z)_0=1, \tag{2.2.6}$$

$$(A)_n = A(A+I)(A+2I)\ldots(A+(n-1)I),\ n \geq 1,\ (A)_0 = I \tag{2.2.7}$$

using hypergeometric function, Kargin and Kurt (2013), the power matrix series is given by

$${}_0F_1(-, A, z) = \sum_{n \geq 0} \left[(A)_n\right]^{-1} \frac{z^n}{n!} \tag{2.2.8}$$

We thus give

$$(A+nI)^{-1} = \left[n\left(\frac{A}{n}+I\right)\right]^{-1} = \frac{1}{n}\left(\frac{A}{n}+I\right)^{-1} \tag{2.2.9}$$

By geometric series, we see that

$$\left\| \left(\frac{A}{n}+I\right)^{-1} \right\| \leq \frac{1}{1-\frac{\|A\|}{n}} = \frac{n}{n-\|A\|} \tag{2.2.10}$$

We compute the power ratio test on the matrix $(A)_n$ in the form:

$$\frac{\left[(A)_{n+1}\right]^{-1} z^{n+1} n!}{\left[(A)_n\right]^{-1} z^n (n+1)!} = \frac{z(A+nI)^{-1}}{n+1} = \frac{z\left(\frac{A}{n}+I\right)^{-1}}{n(n+1)} \tag{2.2.11}$$

By taking norm of both sides of equation (2.2.11), we see that

$$\left\| \frac{\left[(A)_{n+1}\right]^{-1} z^{n+1} n!}{\left[(A)_n\right]^{-1} z^n (n+1)} \right\| \leq \frac{|z| n}{n(n+1)(n-\|A\|)} \to 0 \text{ as } n \to \infty \tag{2.2.12}$$

The nearness to singularity of the matrix A is explained by the ratio of largest eigenvalue λ_1 of A to the smallest eigenvalue λ_k assuming they are ordered according to their magnitudes $\lambda_1 > \lambda_2 > \ldots > \lambda_k$ counting their multiplicities of occurrences. A very high condition number shows that the given matrix is highly ill-conditioned. However, the dominant eigenvalue of A can be computed by the Power method. In this way ill-conditioning occurring in the coefficients of the respective matrices can be detailed.

Furthermore, by computing the field of values of the matrix A as the set of all possible Rayleigh Quotients defined by

$F(A) = \{X^T Ax \ |x \in R^n \ \|x\| = 1\}$ we are able to compute the numerical radius in the form

$r(A) = \max\{|y| | y \in F(A)\}$. Following Bjorck (2009) the bound for the numerical radius of matrix A is in the form

$$\frac{1}{2}\|A\|_2 \le r(A) \le \|A\|_2 .$$

In any case, for such a diagonalizable matrix $A = UDV$ it follows that the bound $\frac{r(A)}{\rho(A)} \le \kappa(U)$ is best possible.

3. RESULT

3.1 **Numerical Examples**

Example 3.1.1

The hypergeometric matrix power series Jorda et al (1994) has applications in the treatments of Laguerre matrix polynomial, an important aspect in mathematical physics and is defined by the equation

$$L_A^{(\lambda,A)}(x) = \sum_{k=0}^{n} \frac{(-1)^k \lambda^k}{k!(n-k)!}[(A+I)]^{-1} x^k \tag{3.1.1}$$

where, λ is any complex number and in addition, $\mathrm{Re}(\lambda) > 0$.

Example 3.1.2:

Consider the Bessel function $I_v(z)$ in terms of hypergeometric matrix series taken from Jordar and Cortis (1998) assuming that one can find a D_0 in the complex domain in the region of the negative real axis such that for all $z \in D_0$, we have

$$I_v(z) = \sum_{k=0}^{n} \frac{\left(\frac{z}{2}\right)^{v+2k}}{\Gamma(k+1)\Gamma(k+v+1)}; \quad |z| < \infty, \ |\arg(z)| < \pi \tag{3.1.2}$$

Equation (3.1.2) is an entire function of v in complex plane for the set of integers Z .

Because of equation (3.1.2) the matrix Bessel function is given by the equation

$$I_A(z) = \sum_{k\ge 0} \frac{\Gamma^{-1}\left((k+1)I + A\right)\left(\frac{z}{2}\right)^{A+2kI}}{\Gamma(k+1)} = \left(\frac{z}{2}\right)^A \sum \frac{\left((k+1)I + A\right)\left(\frac{z}{2}\right)^{2k}}{\Gamma(k+1)} \tag{3.1.3}$$

To obtain the perimeter of a polygon Nunemacher(1986) involving gamma function, let

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right) = \frac{\pi}{\sin\left(\frac{\pi}{n}\right)} \tag{3.1.4}$$

and let p_n be the perimeter of a polygon, it can be derived in the sense of Jordar and Cortis (1998) that for $n \in N$,

$$p_n = 2n \sin\left(\frac{\pi}{n}\right) R_n \ , \quad (n \in N), \tag{3.1.5}$$

with

$$R_n = \frac{1}{n}\int_0^1 u^{-\frac{2}{k}} (1-u)^{\frac{1}{n-1}} du = \frac{\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(1-\frac{1}{n}\right)} \to 1 \text{ (as } n \to \infty) \tag{3.1.6}$$

It holds that $p_n \to 2\pi$ as $n \to \infty$ for $\Gamma(1) = 0! = 1$.

The factor $\frac{p_n}{2\pi}$ by which the perimeter of a polygon exceeds the perimeter of the unit circle could be obtained in the form: $(p_3, p_{4,} p_5, p_6) \approx (1.461, 1.180, 1.098, 1.043)$. Thus as $n \to \infty$, the size of $\frac{p_n}{2\pi}$ deceases in the unit circle.The conformal radius of the unit square (sides of length 2) is computed by the quantity

$$\frac{8}{p_4} = 4\frac{\Gamma\left(\frac{3}{4}\right)^2}{\pi^{\frac{3}{2}}} = 1.078705. \tag{3.1.7}$$

3.2. THE PENDULUM PROBLEM

Example3.2.1. The pendulum problem

The Jacobi elliptic function has two simple poles and two simple zeros per cell while the Weierstrass elliptic function is one with a second order pole and two zeros per cell.

By defining that $q = e^{\pi i \tau}$ for $sn(\tau) > 0$ the function

$$g(z,q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2inz} \tag{3.2.1}$$

gives the theta Jacobi elliptic function.

Because $|z| < M$ where M is a positive constant, it follows that $\left|q^{n^2} e^{2niz}\right| \le |q|^{n^2} e^{2nM}$, $n = 1,2,\ldots$

Thus the series expansion

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} e^{2inz} &= 1+\sum_{1}^{\infty}(-1)^n\left[q^{n^2}\left(e^{2inz}+e^{-2inz}\right)\right] \\ &= 1+2\sum_{1}^{\infty}(-1)^n q^{n^2}\cos 2nz\end{aligned} \tag{3.2.2}$$

defines the theta function of Jacobi elliptic integrals whose numerous applications range over mechanical engineering, Celestial mechanics and applied mathematics.

Equation (3.2.2) is a periodic function with $g(z,q)=g(z+\pi,q)$. It can be proved that $g(z+\tau\pi,q)=-q^{-1}e^{-2iz}g(z,q)$.

We set $g_4(z,q)=g(z,q)$. We obtain values for $g_3(z,q), g_1(z,q), g_2(z,q)$ as follows

$$g_3(z,q)=g(z+\frac{\pi}{2},q)=1+2\sum_{n=1}^{\infty}q^{n^2}\cos 2nz \tag{3.2.3}$$

$$g_1(z,q)=-ie^{\left(iz+\frac{i\tau\pi}{4}\right)}g\left(z+\frac{\pi\tau}{2},q\right)=2\sum_{n=0}^{\infty}(-1)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)z \tag{3.2.4}$$

$$g_2(z,q)=g_1\left(z+\frac{\pi}{2},q\right)=2\sum_{n=0}^{\infty}q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)z \tag{3.2.5}$$

The differential equation Whittaker and Watson (1958),Okeke (1990) satisfies by the Jacobi elliptic function is in the form:

$$\frac{d}{dz}\left[\frac{g_1(z,q)}{g_4(z,q)}\right]=\left[g_4(0,q)^2\right].\frac{g_2(z,q)g_3(z,q)}{g_4(z,q).g_4(z,q)}. \tag{3.2.6}$$

Setting as $\eta=\dfrac{g_1(z,q)}{g_4(z,q)}$, we obtain in the form the differential equation

$$\left(\frac{d\eta}{dz}\right)^2=\left[g_2(0,q)-\eta^2 g_3^2(0,q)\right].\left[g_3^2(0,q)-\eta^2 g_2^2(0,q)\right] \tag{3.2.7}$$

Next, by setting as

$y=\left[\dfrac{g_3(0,q)}{g_2(0,q)}\right].\eta$ and $x=z(g_3(0,q))^2$, then one obtains that

$$\left(\frac{dy}{dx}\right)^2=\left(1-y^2\right)\left(1-r^2y^2\right) \tag{3.2.8}$$

wherefrom,

$$r^{\frac{1}{2}} = \frac{g_2(0,q)}{g_3(0,q)} \quad (3.2.9)$$

The following identities Rainville(1960) hold true for the Jacobi elliptic integral:

$$Sn^2u + Cn^2u = 1,$$
$$dn^2u + k^2\, sn^2u = 1$$
$$Cn^2u + (1-k^2)Sn^2u = dn^2u$$

The Jacobi elliptic function becomes the trigonometric and hyperbolic functions for $k = 0$; $Snu = \sin u$

$Cnu = \cos u,\ dnu = 1$. For $k = 1 : Snu = \tanh u, Cnu = dnu = \sec hu$.

So that one has the Jacobi elliptic function of a second kind integral

$$x = \int_0^y \frac{du}{(1-u^2)^{\frac{1}{2}}(1-r^2u^2)^{\frac{1}{2}}} = s_n^{-1} y \quad (3.2.10)$$

$$\frac{dy}{dx} = \frac{d}{dx} Snx = \frac{1}{dx/dy} = \sqrt{(1-y^2)(1-r^2y^2)} = \sqrt{1-Sn^2x}\sqrt{1-r^2y^2} = Cnxdnx \quad (3.2.11)$$

$$\frac{d}{dx}(dnx) = \frac{d}{dx}\sqrt{1-r^2Sn^2x} = \frac{1}{2}(1-r^2Sn^2x)^{\frac{-1}{2}} . -2r^2(SnxCnx)dnx = n - r^2SnCx \quad (3.2.12)$$

The inverse sine integral defined by

$\sin^{-1} y = \int_0^y \frac{du}{\sqrt{1-u^2}}$ wherefrom, $x = -\int_0^y \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}$ for $u = -t$ in the given expression.

The period for the elliptic function is

$$K(r) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}, \text{(where } t = \sin\theta .) \quad (3.2.13)$$

To motivate our discussion in the right senses, consider the problem

$$L\overset{\circ\circ}{\theta} + g\sin\theta = 0, \theta(0) = \alpha,\quad \overset{\circ}{\theta}(0) = 0 \quad (3.2.14)$$

Multiplying both sides of equation by $\overset{\circ}{\theta}$ in the form:

$$L\overset{\circ\circ}{\theta}\overset{\circ}{\theta} + g\sin\theta\overset{\circ}{\theta} = 0 \text{ and then using } \frac{d}{dt} = \overset{\circ}{\theta} \text{ we have,}$$

$$L\frac{d}{dt}\frac{\overset{\circ}{\theta}^2}{2} - g\frac{d}{dt}\cos\theta = 0$$

Solving this by integrating with respect to t leads to the expression

$$\overset{\circ}{\theta} = -2\sqrt{\frac{g}{L}\left(-\sin^2\frac{\theta}{2} + \sin^2\frac{\alpha}{2}\right)} \tag{3.2.15}$$

The negative sign was taken since $\sin\theta$ was initially negative.

It is the aim of the paper to transform this into the Jacobi Elliptic function and obtain both lower and upper bounds for the problem.

By writing as

$$2\left(\frac{g}{L}\right)^{1/2} t = -\int_{\alpha}^{\theta} \frac{du}{\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{u}{2}\right)^{\frac{1}{2}}} \quad , \tag{3.2.16}$$

using substitution of variable $v = \sin\frac{\alpha}{2}$, $\sin\frac{u}{2} = rv$, then $dv = \frac{2rdu}{\left(1-rv^2\right)^{\frac{1}{2}}}$.

Therefore,

$$\left(\frac{g}{L}\right)^{\frac{1}{2}} t = -\int_{1}^{x} \frac{dv}{\left[\left(1-v^2\right)\left(1-r^2v^2\right)\right]^{\frac{1}{2}}} = \int_{0}^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}} = 2\pi\left(\frac{L}{g}\right)^{\frac{1}{2}} 2F_1\left(\frac{1}{2},\frac{1}{2},1,r^2\right) , \tag{3.2.17}$$

where, $r^2 = \sin^2\frac{\alpha}{2}$.

The modulus of the Jacobian are K and $4K$ and,

$$Sn^{-1}(x) = \left(-\left(\frac{g}{L}\right)^{\frac{1}{2}}\right)t + K . \tag{3.2.18}$$

Our contribution to the existing problem is now to give the bound using hyper geometric function for the Jacobi elliptic integral. Firstly, the expression $F(a,b;c;s)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)\,n!}s^n, |s|<1$ is well known in the field of complex analysis and applied mathematics where for instance, the

$$(a,n)=a(a+1)(a+2)\ldots(a+n-1)=\frac{\Gamma(n+a)}{\Gamma(a)}$$
$$(n=1,2,\ldots).$$

Thus the elliptic integral $\mathrm{K}(r)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}$ is bounded by

$$K(r)<\log\left(1+\frac{4}{\sqrt{1-r^2}}\right)-\left(\log 5-\frac{\pi}{2}\right)(1-r) \tag{3.2.19}$$

If we use Anderson et al (1992) it holds that the upper bound for this was given in Alzer and Richards (2004) see the discussion also in Bao (2021) in the form

$$K(r)<\frac{\pi}{2}\frac{16-5\log(1-r^2)}{16+(5\pi-16)r^2}\ (0<r<1). \tag{3.2.20}$$

However, recent findings due to Bao (2021) showed that the double inequality holds verbatim for the Jacobi elliptic integral

$$\max\left\{\frac{\pi[16-5\log(1-r^2)]-2[\alpha+(\beta-\alpha)r^2]r^4}{32+2(5\pi-16)r^2}-\theta r^2,\frac{\pi[16-5\log(1-r^2)]+2\delta(1-r)r^2}{32+2(5\pi-16)r^2}-\lambda r^2\right\}$$

$$\leq K(r)\leq$$
$$\min\left\{\frac{\pi[16-5\log(1-r^2)]-2\alpha r^4}{32+2(5\pi-16)r^2}-\theta r^2,\frac{\pi[16-5\log(1-r^2)+2\delta r^2]}{32+2(5\pi-16)r^2}-\lambda r^2\right\} \tag{3.2.21}$$

Note that the approximation $\ln\left(1+\frac{4}{r'}\right)$ is actually better than $\ln\frac{4}{r'}$. This investigation holds for the monotonicity of the ratio $\frac{K(r)}{\ln\left(1+\frac{4}{r'}\right)}$ wherefrom,

$$K(r)<\ln\left(1+\frac{4}{r'}\right)-\left(\ln 5-\frac{\pi}{2}\right)(1-r) \tag{3.2.22}$$

Following Yang and Tian (2017), Wank *et al*. (2020) the asymptotic formula for $K(r)$ is given in the form:

$K(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2},1,r^2\right)\approx \ln\frac{4}{r^/}$ as $r\to \bar{1}$ and $r^/=\sqrt{1-r^2}$. The ratio $\frac{K(r)}{\ln\frac{c}{r^/}}$ is a strictly decreasing function if and only if $0<c\le 4$ and $c\in(0,4]$. Now it can be derived in a similar manner that $\phi(r)=\frac{K(\sqrt{r})}{\ln\left(\frac{c}{\sqrt{1-r}}\right)}$ is also strictly convex on the interval (0,1) provided that $c=e^{\frac{4}{3}}$ .Also the $\frac{1}{\phi(r)}$ is a concave function on the interval $(0,1)$ .This means that the inverse of its second derivative is negative We seek further information Yang and Tian (2017), Bao (2021) on the functions

$f(r)=F\left(\frac{1}{2};\frac{1}{2};1;r\right)$ and $h(r)=\ln\left(1+\frac{4}{\sqrt{1-r}}\right)$ where, $r\in r^2\in(0,1)$. Differentiating we have

$$f^/(r)=\frac{1}{4}F\left(\frac{3}{2};\frac{3}{2},2,r\right)=\frac{1}{4}\left(\frac{1}{1-r}\right)F\left(\frac{1}{2};\frac{1}{2};2r\right),$$

$$h^/(r)=\frac{2}{4+\sqrt{1-r}\,.\,(1-r)}=\frac{2\left(4-\sqrt{1-r}\right)}{(15+r)(1-r)}$$

$$\text{The ratio } \frac{f^/(r)}{h^/(r)}=\frac{(15+r)F\left(\frac{1}{2};\frac{1}{2};2;r\right)}{8\left(4-\sqrt{1-r}\right)}\quad , \tag{3.2.23}$$

calls for further scrutiny in the analysis of results.

Power series expansion Qi et al (2004), Jorda *et a l*. (1994) of the ratio $\frac{f^/(r)}{h^/(r)}$ in the form due to equation (3.2.23) gives the following result:

$$\frac{f^/(r)}{h^/(r)}=\frac{15+\sum_{j=1}^{\infty}\frac{(16j^2-56j+15)}{(2j-1)^2(j+1)}W_j^2r^2}{24+\sum_{j-1}^{\infty}\frac{4\Gamma\left(j-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma(j+1)}r^2} \tag{3.2.24}$$

The W_j is the Wallis factor and is given by $W_j = \dfrac{\Gamma(j+\frac{1}{2})}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(j+\frac{1}{2}\right)}$; $W_{j+1} = \dfrac{\left(j+\frac{1}{2}\right)}{j+1}W_j$

Summing up these, it follows that

$$\sqrt{\frac{\ln\left(1+\frac{4}{r^{/}}\right)\ln\left(1+\frac{4}{r}\right)}{K(r)\ K(r^{/})}} \le \frac{1}{2}\left(\frac{1+\frac{4}{r^{/}}}{K(r)}\right)+\frac{\ln\left(1+\frac{4}{r}\right)}{K(r^{/})} \le \frac{\ln\left(1+4\sqrt{2}\right)}{K(\sqrt{\frac{1}{2}})}$$

Finally, we have that the inequality holds again and lends supports to the discussion.

$$1-\frac{2\ln 5}{\pi} < \frac{1}{r^2}\left(\frac{\ln\left(1+\frac{4}{r^{/}}\right)}{K(r)}\right)-\frac{2\ln 5}{\pi} < \frac{2}{\pi}\left(\frac{2}{5}-\frac{1}{4}\ln 5\right)$$

4. DISCUSSION

The paper reviewed the gamma function, the Beta function, hypergeometric series and their applications. We drew example of hypergeometric matrix power series for the Laguerre matrix polynomial. It was showed that the gamma matrix function is convergent with respect to power series ratio test. The nearness to singularity of the matrix A with respect to the ratio of largest eigenvalue to the smallest eigenvalue was discussed which helps in the analysis of nature of condition number of the gamma matrix function. We related this to the class of Laguerre matrix polynomials and can be extended to the class of Bessel matrix functions. Drawing example from Jordan and Cortis (1998), the computation of perimeter (p_n) of polygon involving gamma function by a factor $\frac{p_n}{2\pi}$ for which the perimeter exceeds the perimeter of a unit circle was also given. We discussed in details that the elliptic integral $\mathrm{K}(r) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}}$ for the pendulum problem is bounded by the inequality

$$K(r) < \log\left(1+\frac{4}{\sqrt{1-r^2}}\right)-\left(\log 5-\frac{\pi}{2}\right)(1-r)\ . \qquad (4.1)$$

This bound is convex and that the approximation $\ln\left(1+\frac{4}{r^{/}}\right)$ is actually better than $\ln\frac{4}{r^{/}}$ .it was pointed out that $\phi(r)=\frac{K\left(\sqrt{r}\right)}{\ln\left(\frac{c}{\sqrt{1-r}}\right)}$ is also strictly concave on the interval $(0,1)$ provided that $c=e^{\frac{4}{3}}$.

It should be noted that the following bounds mentioned in Wang et al (2020) and the references mentioned therein hold verbatim for the same bounds on $K(r)$ in the pendulum problem in the form:

$$K(r)\leq\frac{\frac{\pi}{2}\tan^{-1}\left(\frac{\sqrt{1-r^{/}}}{\sqrt{r^{/}}}\right)}{\sqrt{r^{2}+r^{/}-1}};\tag{4.2}$$

$$K(r)<\frac{\pi}{4r}\log\left(\frac{1+r}{1-r}\right);\tag{4.3}$$

$$K(r)\leq\frac{\pi\sqrt{r^{4}-32r^{2}+32}}{8\sqrt{2}\left(\sqrt[4]{(1-r^{2})^{3}}\right)}.\tag{4.4}$$

Thus, there exists yet no most universally acceptable bound for the Jacobi elliptic functions as attested to by the various authors in the Literatures. Probabilistic analysis may be necessary for these various bounds as given by various authors. This may put further insights in the solution to the shape of a ‘’ Hanging Rope’’ problem in geodesy.

It is left as an exercise for readers to choose most appropriate value of these bounds for $K(r)$.

5. CONCLUSION

The paper presented gamma, beta and hypergeometric functions for solving various mathematical and engineering problems. Various types of gamma functions have been given. The Stirlling series for the gamma function was discussed as a member of this class. The hypergeometric series for the matrix function was discussed and analyzed and its convergence in the terms of ratio test presented. We gave information on the condition number estimate for the matrix of gamma function in terms of ratio of largest eigenvalue to the smallest eigenvalue assuming their order of occurrence are in order of magnitudes where their multiplicities if any are counted so that their number is exact. We also discussed this in terms of ratio of radius of a matrix to the spectral radius of the matrix. The Bessel function in terms of hypergeometric function for approximating a polygon was given. The factor within which the factor $\frac{p_n}{2\pi}$ cannot exceed the perimeter of the unit circle was discussed in the form:

$(p_3, p_{4,} p_5, p_6) \approx (1.461, 1.180,\ 1.098,\ 1.043)$. Thus as $n \to \infty$, the size of $\frac{p_n}{2\pi}$ deceases in the unit circle.

Detailed analysis is given for the pendulum problem based on the Jacobi elliptic integrals. The convexity for continuity function of the Jacobi integral and concavity of the inverse function have been discussed.Our contribution to the existing problem was to give the bound using hyper geometric function for the Jacobi elliptic integral. Firstly, the expression $F(a,b;c;s) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)\, n!} s^n, |s| < 1$ is well known in the field of complex analysis and applied mathematics where for instance, the

$$(a,n) = a(a+1)(a+2)\ldots(a+n-1) = \frac{\Gamma(n+a)}{\Gamma(a)}.$$

With this, various theoretical analytical bounds were presented and analyzed using ideas on the existing literatures. It is established that there exists yet no universally acceptable bounds in the existing Literature for the Jacobi elliptic integrals.

The best choice for the optimal error bounds for the Jacobi elliptic integral with respect to the pendulum second order differential equation may be resolved through probabilistic

optimization approach. It is also hoped to link these theoretical bounds of Jacobi elliptic integrals with Weierstrass elliptic functions Whittaker and Watson (1963) relative to theta functions. It is hoped that this will form our next line of thought in the next focus of research in the coming months. In particular, the gamma matrix logarithm problems will form a beacon upon which so many formulae could be obtained in our work.

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