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COEFFICIENT INEQUALITIES OF BAZILEVIC FUNCTIONS COLLIGATED WITH CONIC DOMAIN

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Abstract. In this paper, the concept of Bazilevic function as well as Janowski function and the conic regions are combined effectively to define a new domain that exemplify the conic-type regions. The sub-classes of these types of functions which map the open unit disk U onto this changed Conic domain are defined. Also, the sub-classes of k -uniformly Janowski convex and k- uniformly Janowski starlike function involving Bazilevic functions are defined using Sălăgean derivative operator. New results were obtained along with some corollaries and the consequences of our results were pointed out.

1. Introduction

Let A be the usual class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$. The class $S^*(\alpha)$, $C(\alpha)$ are the well-known classes of starlike and convex univalent functions of order α $(0 \leq \alpha < 1)$ respectively, for details, see [2] and [19].

A function $h(z)$ is said to be in the class $P[A, B]$ if it is analytic in U with $h(0) = 1$ and $h(z) \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, where $\prec$ stands for subordination. Geometrically, a function $h(z) \in P[A, B]$ maps the open unit disk U onto the disk defined by the domain

$$\Omega[A, B] = \left\{ \omega : \left| \omega - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}.$$

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The class $[A, B]$ is connected with the class P of functions with positive real parts by the relation

$$h(z) \in P \iff \frac{(A+1)h(z)-(A-1)}{(B+1)h(z)-(B-1)} \in P[A, B].$$

This class was introduced by Janowski [10] and then studied by several authors, for example see [15], [18] and [27] among others.

Kanas and Wisniowska [13] and [12] introduced and studied the class $k-UCV$ of $k-$ uniformly convex functions and the corresponding class $k-ST$ of $k-$ starlike functions. These classes were defined based on the conic domain Ω_k, $k \geq 0$ which was defined by Kanas and Wisniowska [12] and [12] as

$$\Omega_k = u + iv : u > k\sqrt{(u-1)^2 + v^2}.$$

For further details see [17].

The function which play the role of extremal functions for these regions are given as

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, & \text{if } k = 0 \\ 1 + \frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & \text{if } k = 1, \\ 1 + \frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right) arc\tanh\sqrt{z}\right], & \text{if } 0 < k < 1, \\ 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{k^2-1}, & \text{if } k > 1, \end{cases}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0, 1)$, $z \in U$ and z is chosen so that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$; for more details, see [13] and [12]. If $p_k(z) = 1 + \delta_k z + ...$, then it was shown in [11] that from (1.2), one can have

$$\delta_k = \begin{cases} \frac{8(arccosk)^2}{\pi^2(1-k^2)} & 0 \leq k < 1, \\ \frac{8}{\pi^2} & k = 1, \\ \frac{\pi^2}{4(k^2-1)\sqrt{t}(1+t)R^2(t)} & k > 1, \end{cases} \tag{1.2}$$

These conic regions are being studied by several authors, see [1, 14, 16, 9, 8].

The classes $K-UCV$ and $k-ST$ are defined as follows.

A function $f(z) \in A$ is said to be in the class $k-UCV$, if and only if,

$$\frac{(zf'(z))'}{f'(z)} \prec p_k(z) \; z \in U, \; k \geq 0.$$

A function $f(z) \in A$ is said to be in the class $K-ST$, if and only if,

$$\frac{zf'(z)}{f(z)} \prec p_k(z) \; z \in U, \; k \geq 0.$$

These classes were then generalized to $KD(k, \alpha)$ and $SD(k, \alpha)$ respectively by Shams et al. [26] based on the conic domain $G(k, \alpha)$, $k \geq 0$, $0 \leq \alpha < 1$, which is

$$G(k, \alpha) = \{\omega : \mathcal{R}e\ \omega > k|\omega - 1| + \alpha\}.$$

Noor and Malik [17] investigated the concept of Janowski functions and the conic domain, by using the following definitions.

Definition 1.1 [17]: A function $p(z)$ is said to be in the class $K - P[A, B]$, if and only if ,

$$p(z) \prec \frac{(A+1)p_k(z) - (A-1)}{(B+1)p_k(z) - (B-1)},\ k \geq 0 \tag{1.3}$$

where $p_k(z)$ is defined by (1.2) and $-1 \leq B < A \leq 1$. Geometrically, the function $p(z) \in K - P[A, B]$ takes all values from the domain $\Omega_k[A, B]$, $-1 \leq B < A \leq 1$, $k \geq 0$ which is defined as

$$\Omega_k[A, B] = \Big\{\omega : \mathcal{R}e\Big(\frac{(B-1)\omega(z) - (A-1)}{(B+1)\omega(z) - (A+1)}\Big) > k\Big|\frac{(B-1)\omega(z) - (A-1)}{(B+1)\omega(z) - (A+1)} - 1\Big|\Big\} \tag{1.4}$$

Or equivalently,

$$\Omega_k[A, B] = \{u + iv : [(B^2 - 1)(u^2 + v^2) - 2(AB - 1)u + (A^2 - 1)]^2$$
$$> k^2[(-2(B+!)(u^2+v^2)+2(A+B+2)u-2(A+1))^2+4(A-B)^2v^2]\}.$$

See [13] and [17] for more details.

Remark 1.2. (1) $K - P[A, B] \subset p\Big(\frac{2k+1-A}{2k+1-B}\Big)$, the well-known class of functions with real part greater than $\frac{2k+1-A}{2k+1-B}$.
(2) $K - P[1. - 1] = P(p_k)$, the well-known class introduced by Kanas and Wisniowska [12].
(3) $0 - PA, B = p[A, B]$, the well-known class introduced by Janowski [10].

Definition 1.3[17]: A function $f(z) \in A$ is said to be in the class $K - UCV[A, B]$, $k \geq 0$,$-1 \leq B < A \leq 1$, if and only if,

$$\mathcal{R}e\left(\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{(zf'(z))'}{f'(z)} - (A-1)}{(B+1)\frac{(zf'(z))'}{f'(z)} - (A+1)} - 1\right|$$

Or equivalently,

$$\frac{(zf'(z))'}{f'(z)} \in K - P[A, B] \tag{1.5}$$

Definition 1.4[17]: A function $f(z) \in A$ is said to be in the class $k - ST[A, B]$, $k \geq 0$,$-1 \leq B < A \leq 1$, if and only if,

$$\mathcal{R}e\left(\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)}\right) > k\left|\frac{(B-1)\frac{zf'(z)}{f(z)} - (A-1)}{(B+1)\frac{zf'(z)}{f(z)} - (A+1)} - 1\right|$$

Or equivalently,

$$\frac{zf'(z)}{f(z)} \in K - P[A, B]. \tag{1.6}$$

It can be easily seen that

$$p(z) \in k - UCV[A, B \longleftrightarrow zf'(z) \in k - ST[A, B]]. \tag{1.7}$$

Special cases.
(i) $K - ST[1, -1] = K - ST, K - UCV[1, -1] = K - UCV$, the well-known classes of $K-$ uniformly convex and $K-$ starlike functions respectively, introduced by Kanas and Wisniowska [13] and [12] .
(ii) $K - ST[1 - 2\alpha, -1] = SD(k, \alpha), K - UCV[1 - 2\alpha, -1] = KD(k, \alpha)$, the classes, introduced by Shams et al. in [26].
(iii) $0 - ST[A, B] = S^*[A, B], 0 - UCV[A, B] = C[A, B]$, the well-known classes of Janowski starlike and Janowski convex functions respectively, introduced by Janowski [10].

Lemma 1.5[24]: Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be subordinate to $H(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$. If $H(z)$ is univalent in U and $H(U)$ is convex, then $|c_n| \leq |C_1|, n \geq 1$.

Sălăgean [21] introduced the following differential operator:

$$\begin{aligned} D_{\omega}^{0} f(z) &= f(z) \\ D^1 f(z) = D(D^0 f(z)) &= zf'(z) \\ &\vdots \\ D^m f(z) = D(D^{m-1} f(z)) &= z(D^{m-1} f(z))' \end{aligned} \tag{1.8}$$

The differential operator D^m is the one defined by Sălăgean.
observe that we can express equation (1.1) in the form

$$f(z)^{\tau} = \left(z + \sum_{k=2}^{\infty} a_k z^k\right)^{\tau} \tag{1.9}$$

Applying Binomial expansion and indices we have

$$f(z)^{\tau} = z^{\tau} + \sum_{n=2}^{\infty} a_k(\tau) z^{\tau+n-1} \tag{1.10}$$

where $\tau \geq 1$.
Oladipo and Breaz [20] investigated and study Bazilevic functions whose general

equation takes the form

$$f(z)=\left\{\frac{\alpha}{1+\epsilon^2}\int_0^z \frac{p(v)-i\epsilon}{v^{1+\frac{i\alpha\epsilon}{(1+\epsilon^2)}}}g(v)^{\frac{\alpha}{1+\epsilon^2}}dv\right\}^{\frac{1+i\epsilon}{\alpha}} \tag{1.11}$$

If $\epsilon=0$ equation (1.12) becomes

$$f(z)=\left\{\alpha\int_0^z \frac{p(v)}{v}g(v)^{\alpha}dv\right\}^{\frac{1}{\alpha}} \tag{1.12}$$

Differentiating (1.13) we have

$$\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}=p(z) \quad z\in U \tag{1.13}$$

Or equivalently,

$$\Re\left\{\frac{zf'(z)f(z)^{\alpha-1}}{g(z)^{\alpha}}\right\} \tag{1.14}$$

The subclass of functions satisfying (1.14) are called Bazilevic functions of type α and are denoted by $B(\alpha)$. For further details, see [3, 4, 7, 5, 8, 20, 21].
Let A^{τ} be the subclass of A consisting of analytic and $\tau-$ valent functions of the form

$$D^m f(z)^{\tau}=\tau^m z^{\tau}+\sum_{n=2}^{\infty}(\tau+n-1)a_n(\tau)z^{\tau+n-1} \tag{1.15}$$

where $m\in N_0, \tau\geq 1$ and D^m is the Sălăgean derivative operator.

$$\frac{\left(\frac{D^{m+1}f(z)^{\tau}}{\tau^{m+1}z^{\tau}}\right)'}{\frac{D^m f(z)^{\tau}}{\tau^m z^{\tau}}}\prec p_k(z), \quad z\in U, \quad k\geq 0 \tag{1.16}$$

where $m\in N_{|0}$ and D^m is the Sălăgean derivative operator.

Definition 1.6: A function $D^m f(z)^{\tau}\in A^{m,\tau}$ is said to be in the class $K-ST_{\tau}^{m}$, if and only if,

$$\frac{\frac{D^{m+1}f(z)^{\tau}}{\tau^{m+1}z^{\tau}}}{\frac{D^m f(z)^{\tau}}{\tau^m z^{\tau}}}\prec p_k(z), \quad z\in U, \quad k\geq 0 \tag{1.17}$$

where $m\in N_0$ and D^m is the Sălăgean derivative operator.

Definition 1.7: A function $D^m f(z)^{\tau}\in A^{m,\tau}$ is said to be in the class $K-$

$UCV_\tau^m[A,B], \quad k \geq 0, -1 \leq B < A \leq 1$ if and only if,

$$\mathcal{R}e\left\{\frac{(B-1)\frac{\left(\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}\right)'}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A-1)}{(B+1)\frac{\left(\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}\right)'}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A+1)}\right\} > k\left|\frac{(B-1)\frac{\left(\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}\right)'}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A-1)}{(B+1)\frac{\left(\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}\right)'}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A+1)}-1\right|$$

Or equivalently,

$$\frac{\left(\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}\right)'}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}} \in K - P_\tau^m[A,B] \tag{1.18}$$

where $\tau \geq 1, m \in N_0$ and D^m is the Sălăgean derivative operator.

Definition 1.8: A function $D^m f(z)^\tau \in A^{m,\tau}$ is said to be in the class $k - ST_\tau^m[A,B], \quad k \geq 0, -1 \leq B < A \leq 1$ if and only if,

$$\mathcal{R}e\left\{\frac{(B-1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A-1)}{(B+1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A+1)}\right\} > k\left|\frac{(B-1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A-1)}{(B+1)\frac{\left(\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}\right)'}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A+1)}-1\right|$$

Or equivalently,

$$\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}} \in K - P_\tau^m[A,B] \tag{1.19}$$

where $\tau \geq 1, m \in N_0$ and D^m is the Sălăgean derivative operator.
It can easily be seen that

2. Main Results

Theorem 2.1: A function $f(z)^\tau \in A^{m,\tau}$ is of the form (11) is in the class $K - ST_\tau^m[A,B]$ if it satisfies the condition

$$\sum_{n=2}^{\infty}\left\{2(k+1)\left(\frac{\tau+n-1}{\tau}\right)^m\left(\frac{n-1}{\tau}\right)+\left|(B+1)\left(\frac{\tau+n-1}{\tau}\right)^{m+1}-(A+1)\left(\frac{\tau+n-1}{\tau}\right)^m\right|\right\} \tag{2.1}$$

$$|a_n(\tau)| < |B-A|$$

where $-1 \leq B < A \leq 1, k \geq 0, m \in N_0, \tau \geq 1$ and D^m is the Sălăgean derivative operator.

Proof: Assuming that (2-1) holds, then it suffices to show that

$$k\left|\frac{(B-1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A-1)}{(B+1)\frac{\left(\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}\right)'}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A+1)}-1\right|-\Re\left\{\frac{(B-1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A-1)}{(B+1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A+1)}\right\}<1 \quad (2.2)$$

By considering the L.H.S. of (2.2) we have

$$k\left|\frac{(B-1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A-1)}{(B+1)\frac{\left(\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}\right)'}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A+1)}-1\right|-\Re\left\{\frac{(B-1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A-1)}{(B+1)\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-(A+1)}\right\}<1$$

We have

$$\leq (k+1)\left|\frac{(B-1)\frac{D^{m+!}f(z)^\tau}{\tau^{m+1}z^\tau}-(A-1)\frac{D^m f(z)^\tau}{\tau^m z^\tau}}{(B+1)\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}-(A+1)\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-1\right|$$

$$=(k+1)\left|\frac{(B-1)\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}-(A-1)\frac{D^m f(z)^\tau}{\tau^m z^\tau}-(B+1)\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}+(A+1)\frac{D^m f(z)^\tau}{\tau^m z^\tau}}{(B+1)\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}-(A+1)\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-1\right|$$

$$=2(k+1)\left|\frac{\frac{D^m f(z)^\tau}{\tau^m z^\tau}-\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{(B+1)\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}-(A+1)\frac{D^m f(z)^\tau}{\tau^m z^\tau}}-1\right|,$$

$$\leq \frac{2(k+1)\sum_{n=2}^{\infty}\left(\frac{\tau+n-1}{\tau}\right)^m\left[\frac{\tau+n-1}{\tau}\right]|a_n(\tau)|}{|B-A|-\sum_{n=2}^{\infty}\left|(B+1)\left(\frac{\tau+n-1}{\tau}\right)^{m+1}-(A+!)\left(\frac{\tau+n-1}{\tau}\right)^m\right||a_n(\tau)|}$$

The last expression (2.3) is bounded above by 1 if

$$\sum_{n=2}^{\infty}\left\{2(k+1)\Big(\frac{\tau+n-1}{\tau}\Big)^m\Big(\frac{n-1}{\tau}\Big)+\left|(B+1)\left(\frac{\tau+n-1}{\tau}\right)^{m+1}-(A+1)\left(\frac{\tau+n-1}{\tau}\right)^m\right|\right\}$$

$$|a_n(\tau)|<|B-A|$$

and this complete the proof.
By specializing some parameters, we have the following interesting results: when $m=0$ and $\tau=1$, then we have the following known result, proved by Noor and Malik [17].

Corollary 2.2: A function $f\in A$ and of the form (1.1) is in the class $K-ST[A,B]$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\Big\{2(k+1)(n-1)+|n(B+1)-(A+1)|\Big\}|a_n|<|B-A| \quad (2.3)$$

when $A = 1$, $B = -1$, $m \in N_0$ and $\tau \geq 1$, then we have the following new result

Corollary 2.3: A function $f^\tau \in A^{m,\tau}$ and of the form (1.11) is in the class $K - ST^{m,\tau}$ if it satisfies condition:

$$\sum_{n=2}^{\infty} \Big\{ \Big(\frac{\tau + n - 1}{\tau}\Big)^m \Big(\frac{(k+1)(n-1)+\tau}{\tau}\Big) \Big\} |a_n(\tau)| < 1 \quad k \geq 0 \qquad (2.4)$$

when $A = 1$, $B = -1$, $m = 0$ and $\tau = 1$, then we have the following result, proved by Kanas and Wisniowska [12].

Corollary 2.4: A function $f \in A$ and of the form (1.1) is in the class $K - ST[A, B]$, if it satisfies the condition

$$\sum_{n=2}^{\infty} \Big\{ n + k(n-1) \Big\} |a_n| < 1, \qquad k \geq 0 \qquad (2.5)$$

when $A = 1 - 2\alpha$, $B = -1$, with $0 \leq \alpha < 1$ $m \in N_0$ and $\tau \geq 1$. Then we have the following new result,

Corollary 2.5: A function $f^\tau \in A^{m,\tau}$ and of the form (1.11) is in the class $SD^{m,\tau}$ if it satisfies condition:

$$\sum_{n=2}^{\infty} \Big\{ \Big(\frac{\tau + n - 1}{\tau}\Big)^m \Big(\frac{(k+1)(n-1)+\tau(1-\alpha)}{\tau}\Big) \Big\} |a_n(\tau)| < 1 - \alpha \quad k \geq 0 \quad (2.6)$$

where $0 \leq \alpha < 1$.
when $A = 1 - 2\alpha$, $B = -1$, with $0 \leq \alpha < 1$ $m = 0$ and $\tau = 1$, then we have the following result, proved by Shams et al [26].

Corollary 2.6: A function $f \in A$ and of the form (1.1) is in the class $SD(k, \alpha)$, if it satisfies the condition

$$\sum_{n=2}^{\infty} \Big\{ n(k+1) - (k+\alpha) \Big\} |a_n| < 1 - \alpha \qquad (2.7)$$

where $0 \leq \alpha < 1$ and $k \geq 0$
when $A = 1 - 2\alpha$, $B = -1$, with $0 \leq \alpha < 1$, $k = 0$ $m \in N_0$ and $\tau \geq 1$, and then we have the following new result

Corollary 2.7: A function $f^\tau \in A^{m,\tau}$ and of the form (1.11) is in the class $S^{*,m,\tau}(\alpha)$ if it satisfies condition:

$$\sum_{n=2}^{\infty} \Big\{ \Big(\frac{\tau + n - 1}{\tau}\Big)^m \Big(\frac{n - 1 + \tau(1-\alpha)}{\tau}\Big) \Big\} |a_n(\tau)| < 1 \quad 0 < -\alpha < 1 \qquad (2.8)$$

when $A = 1 - 2\alpha$, $B = -1$, with $0 \leq \alpha < 1$, $k = 0$ $m = 0$ and $\tau = 1$, then we have the following result, proved by Selverman in [25].

Corollary 2.8: A function $f \in A$ and of the form (1.1) is in the class $S^*(\alpha)$, if it satisfies the condition

$$\sum_{n=2}^{\infty}\Big\{(n-\alpha)\}|a_n| < 1-\alpha, \qquad 0 \leq \alpha < 1. \tag{2.9}$$

Theorem 2.9: A function $f(z)^\tau \in A^{m,\tau}$ is of the form (1.11) is in the class $K-CV_\tau^m[A,B]$ if it satisfies the condition

$$\sum_{n=2}^{\infty}\left(\frac{\tau+n-1}{\tau}\right)^{m+1}\Bigg\{2(k+1)\Big(\frac{\tau+n-1}{\tau}\Big)^m\Big(\frac{n-1}{\tau}\Big)+$$

$$\left|(B+1)\left(\frac{\tau+n-1}{\tau}\right)^{m+1}-(A+1)\left(\frac{\tau+n-1}{\tau}\right)^m\right|\Bigg\}|a_n(\tau)| < |B-A| \tag{2.10}$$

where $-1 \leq B < A \leq 1, k \geq 0, m \in N_0, \tau \geq 1$ and D^m is the Sălăgean derivative operator.

The proof follows immediately by using Theorem 2.1 and (1.21).

Theorem 2.10: A function $f(z)^\tau \in K-ST_\tau^m[A,B]$ and is of the form (1.11), then, for $n \geq 2$,

$$|a_n| \leq \prod_{j=0}^{n-2}\frac{|\delta_k(A-B)-2jB|}{2}\Big(\frac{\tau}{\tau+j-1}\Big)^m\Big(\frac{\tau}{j+1}\Big) \tag{2.11}$$

where δ_k is defined by (1.3),$m \in N_0, \tau \geq 1$ and D^m is the Sălăgean derivative operator.
Proof: By definition (1.11), for $f(z)^\tau \in K-ST_\tau^m[A,B]$, we have

$$\frac{\frac{D^{m+1}f(z)^\tau}{\tau^{m+1}z^\tau}}{\frac{D^m f(z)^\tau}{\tau^m z^\tau}} = p(z) \tag{2.12}$$

where

$$p(z) \prec \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}$$

$$= ((A+1)p_k(z)-(A-1))((B+1)p_k(z)-(B-1))^{-1}$$

$$= \left(\left(\frac{A-1}{B-1}\right)-\left(\frac{A-1}{B-1}\right)\left(\frac{A+1}{A-1}\right)p_k(z)\right)\left(1+\sum_{n=1}^{\infty}\left(\frac{B+1}{B-1}p_k(z)\right)\right)$$

$$= \left(\left(\frac{A-1}{B-1}\right)-\left(\frac{A-1}{B-1}\right)\left(\frac{A+1}{A-1}\right)p_k(z)\right)\left(1+\frac{B+1}{B-1}p_k(z)+\left(\frac{B+1}{B-1}\right)^2(p_k(z))^2+\dots\right)$$

Clearly, we have

$$\begin{aligned} p(z) = {} & \frac{A-1}{B-1}-\frac{A+1}{B-1}p_k(z) \\ & + \frac{(B+1)(A-1)}{(B-1)^2}p_k(z) \\ & - \frac{(B+1)(A+1)}{(B-1)^2}(p_k(z))^2 \\ & + \frac{(B+1)^2(A-1)}{(B-1)^3}(p_k(z))^2 \\ & - \frac{(B+1)^2(A+1)}{(B-1)^3}(p_k(z))^3 \\ & + \frac{(B+1)^3(A-1)}{(B-1)^4}(p_k(z))^3 \\ & - \frac{(B+1)^3(A+1)}{(B-1)^4}(p_k(z))^4+\dots \end{aligned}$$

Notice if $p_k(z) = 1+\delta_k z+\dots$, then the following desirable computation readily comes handy:

$$p(z) \prec \sum_{n=1}^{\infty}\frac{-2(A-B)^{n-1}}{(B-1)^n}+\left\{\sum\frac{2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}}\right\}\delta_k z\dots$$

Now we observe that the series $\sum\frac{-2(B+1)^{n-1}}{(B-1)^n}$ and $\sum_{n=1}^{\infty}\frac{2n(A-B)(B+1)^{n-1}}{(B-1)^{n+1}}$ are convergent to 1 and $\frac{A-B}{2}$ respectively.
Consequently,

$$p(z) \prec 1+\frac{1}{2}(A-B)\delta_k z+\dots$$

Now if $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then by Lemma 1.5 [24] , we have

$$|c_n| \leq \frac{1}{2}(A - B)\delta_k, \quad n \geq 1. \tag{2.13}$$

Right away (34), we have

$$\frac{D^{m+1} f(z)^\tau}{\tau z^\tau} = \frac{D^m f(z)^\tau}{\tau z^\tau}$$

This implies that

$$z + \sum_{n=2}^{\infty} \Big(\frac{\tau + n - 1}{\tau}\Big)^{m+1} a_n(\tau) z^n = \Big(z + \sum_{n=2}^{\infty} \Big(\frac{\tau + n - 1}{\tau}\Big)^m a_n(\tau) z^n\Big)\Big(1 + \sum_{n=1}^{\infty} C_n z^n\Big)$$

$$z + \sum_{n=2}^{\infty} \Big(\frac{\tau + n - 1}{\tau}\Big)^{m+1} a_n(\tau) z^n = z + \sum_{n=2}^{\infty} \Big(\frac{\tau + n - 1}{\tau}\Big)^m a_n(\tau) z^n + \sum_{n=1}^{\infty} c_n z^{n+1} +$$

$$\sum_{n=1}^{\infty} \sum_{n=2}^{\infty} \Big(\frac{\tau + n - 1}{\tau}\Big)^m C_n a_n(\tau) z^{2n}$$

Equating coefficients of z^n on both sides, we have

$$\Big(\frac{\tau + n - 1}{\tau}\Big)^{m+1} a_n(\tau) - \Big(\frac{\tau + n - 1}{\tau}\Big)^m a_n(\tau) = \sum_{j=1}^{\infty} a_{n-j}(\tau) c_j,$$

$$\Big(\frac{\tau + n - 1}{\tau}\Big)^m \Big(\frac{\tau + n - 1}{\tau} - 1\Big) a_n(\tau) = \sum_{j=1}^{n-1} a_{n-j}(\tau) c_j, a_1 = 1,$$

$$\Big(\frac{\tau + n - 1}{\tau}\Big)^m \Big(\frac{n - 1}{\tau}\Big) a_n(\tau) = \sum_{j=1}^{n-1} a_{n-j}(\tau) c_j, a_1 = 1,$$

This equally implies that

$$|a_n(\tau)| \leq \Big(\frac{\tau}{\tau + n - 1}\Big)^m \Big(\frac{\tau}{n - 1}\Big) \sum_{j=1}^{n-1} |a_{n-j}||c_j|, a_1 = 1.$$

Using (2.14), we have

$$|a_n(\tau)| \leq \frac{|\delta|(A - B)}{2} \Big(\frac{\tau}{\tau + n - 1}\Big)^m \Big(\frac{\tau}{n - 1}\Big) \sum_{j=1}^{n-1} |a_j|, a_1 = 1. \tag{2.14}$$

Now we prove that

$$\frac{|\delta|(A-B)}{2}\Big(\frac{\tau}{\tau+n-1}\Big)^{m}\Big(\frac{\tau}{n-1}\Big)\sum_{j=1}^{n-1}|a_j| \leq \prod_{j=0}^{n-2}\frac{|\delta_k(A-B)+2jB|}{2}\Big(\frac{\tau}{\tau+n-1}\Big)^{m}\Big(\frac{\tau}{n-1}\Big) \tag{2.15}$$

For this, we use the famous mathematical induction method.
For $n=2$, from (2.15), we have

$$|a_2(\tau)| \leq \frac{|\delta_k(A-B)}{2}\Big(\frac{1}{\tau+1}\Big)^{m}\tau^{m+1}$$

From (2.12), we have

$$|a_2(\tau)| \leq \frac{|\delta_k|(A-B)}{2}\Big(\frac{1}{\tau+1}\Big)^{m}\tau^{m+1}$$

For $n=3$, from (2.15), we have

$$|a_3(\tau)| \leq \frac{|\delta_k|(A-B)}{4}\Big(\frac{1}{\tau+2}\Big)^{m}(\tau^{m+1})(1+|a_2|)$$

$$\leq \frac{|\delta_k|(A-B)}{4}\Big(\frac{1}{\tau+2}\Big)^{m}(\tau^{m+1})\Big(1+\frac{|\delta_k|(A-B)}{2}\Big(\frac{1}{\tau+1}\Big)^{m}(z^{n+1})\Big)$$

From (2.12), we have

$$|a_3(\tau)| \leq \frac{|\delta_k|(A-B)}{2}\Big(\frac{1}{\tau+1}\Big)^{m}(\tau^{m+1})\frac{|\delta_k|(A-B)}{4}\Big(\frac{1}{\tau+2}\Big)^{m}(\tau^{m+1})$$

$$\leq \frac{|\delta_k|(A-B)}{2}\Big(\frac{1}{\tau+1}\Big)^{m}(\tau^{m+1})\frac{|\delta_k(A-B)+2jB|}{4}\Big(\frac{1}{\tau+2}\Big)^{m}(\tau^{m+1})$$

$$\leq \frac{|\delta_k|(A-B)}{2}\Big(\frac{1}{\tau+1}\Big)^{m}(\tau^{m+1})\Big(\frac{|\delta_k|(A-B)}{2}\Big(\frac{1}{\tau+2}\Big)^{m}(\tau^{m+1})+1\Big)$$

Let the hypothesis be true $n=t$.
From (2.15), we have

$$a_t(\tau) \leq \frac{|\delta_k|(A-B)}{2}\Big(\frac{\tau}{\tau+t-1}\Big)^{m}\Big(\frac{\tau}{t-1}\Big)\sum_{j=1}^{t-1}|a_j|.$$

From (2.12), we have

$$|a_t(\tau)| \leq \prod_{j=0}^{t-2} \frac{|\delta_k(A-B)+2jB|}{2}\Big(\frac{\tau}{\tau+j-1}\Big)^m\Big(\frac{\tau}{j-1}\Big),$$

$$\leq \prod_{j=0}^{t-2} \frac{|\delta_k|(A-B)+2jB}{2(j+1)}$$

By the induction hypothesis, we have

$$\frac{|\delta_k|(A-B)}{2}\Big(\frac{\tau}{\tau+t-1}\Big)^m\Big(\frac{\tau}{t-1}\Big)\sum_{j=1}^{t-1}|a_j| \leq \prod_{j=0}^{t-2}\frac{|\delta_k|(A-B)+2j}{2}\Big(\frac{\tau}{\tau+j=1}\Big)^m\Big(\frac{\tau}{j+1}\Big)$$

Multiply both sides by $\frac{|\delta_k|(A-B)+2(t-1)}{2}\Big(\frac{\tau}{\tau+1}\Big)^m\Big(\frac{\tau}{t}\Big)$
We have

$$\prod_{j=0}^{t-2}\frac{|\delta_k|(A-B)+2j}{2}\Big(\frac{\tau}{\tau+j-1}\Big)^m\Big(\frac{\tau}{j+1}\Big) \geq$$

$$\frac{|\delta_k|(A-B)}{2}\Big(\frac{\tau}{\tau+j-1}\Big)^m\Big(\frac{\tau}{t-1}\Big)\frac{|\delta_k|(A-B)+2(t-1)}{2}\Big(\frac{\tau}{t-1}\Big)^m\Big(\frac{\tau}{t}\Big)\sum_{j=1}^{t-1}|a_j|$$

$$= \frac{|\delta_k|(A-B)}{2}\Big(\frac{\tau}{t}\Big)\Big(\frac{\tau}{\tau+t-1}\Big)^m\Big\{\frac{|\delta_k|(A-B)}{2}\Big(\frac{\tau}{\tau+t-1}\Big)^m\Big(\frac{\tau}{t-1}\Big)\sum_{j=1}^{t-1}|a_j| + \sum_{j=1}^{t-1}|a_j|\Big\}$$

$$\geq \frac{|\delta_k|(A-B)}{2}\Big(\frac{\tau}{\tau+t-1}\Big)^m\Big(\frac{\tau}{t}\Big)\Big\{|a_j| + \sum_{j=1}^{t-1}|a_j|\Big\}$$

$$= \frac{|\delta_k|(A-B)}{2}\Big(\frac{\tau}{\tau+t-1}\Big)^m\Big(\frac{\tau}{t}\Big)\sum_{j=1}^{t}|a_j|$$

That is,

$$\frac{|\delta_k|(A-B)}{2}\Big(\frac{\tau}{\tau+t-1}\Big)^m\Big(\frac{\tau}{t}\Big)\sum_{j=1}^{t}|a_j| \leq \prod_{j=0}^{t-2}\frac{|\delta_k|(A-B)+2j}{2}\Big(\frac{\tau}{\tau+j-1}\Big)^m\Big(\frac{\tau}{j+1}\Big)$$

and this shows that inequality (2.16) is true for $n = t+1$. This complete the proof.

Corollary 2.11:when $m = 0$ and $\tau = 1$ then Theorem 2.10 reduces to

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|\delta_k(A-B) - 2jB|}{2(j+1)} \quad n \ge 2 \tag{2.16}$$

. This result was obtained in [17].

Corollary 2.12:when $A = 1,\ B = -1, m \in N_0$ and $\tau \ge 1$ then (2.1) reduces to

$$|a_n| \le \prod_{j=0}^{n-2} |\delta_k + j| \Big(\frac{\tau}{\tau + j - 1}\Big)^m \Big(\frac{\tau}{j+1}\Big) \quad n \ge 2 \tag{2.17}$$

. A new result which involves the coefficient inequality of the class $k - ST^{m,\tau}$

Corollary 2.13:when $A = 1,\ B = -1, m = 0$ and $\tau = 1$ then (2.1) reduces to

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|\delta_k + j|}{j+1} \quad n \ge 2 \tag{2.18}$$

.
This is the coefficient inequality of the class $K - ST$, introduced by Kanas and Wisniowska [12]

Corollary 2.14:when $A = 1,\ B = -1, m = 0$ and $\tau = 1$ with $0 \le \alpha < 1$, then (2.1) reduces to

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|\delta_k(1-\alpha) + j|}{j+1} \quad n \ge 2 \tag{2.19}$$

.
This is the coefficient inequality of the class $SD(k, \alpha)$, introduced by Shams et al [26].
When $k = 0$, then $\delta_k = 2$, $m = 0$ and $\tau = 1$ and we obtain the following known result, prove in [10].

Corollary 2.15: Let $f(z) \in S^*[A, B]$ and is of the form (1.1) then for $n \ge 2$

$$|a_n| \le \prod_{j=0}^{n-2} \frac{|(A-B) - jB|}{j+1} \quad -1 \le B < A \le 1. \tag{2.20}$$

.

Conclusion: This study considered two subclasses of Bazilevic functions that were colligated with the conic domain. These classes were introduced in the definitions 1.7 and 1.8 by using the well-established Sălăgean derivative operator that was seen in equation (1.3) and it was used to remodify the Bazilevic function of type τ-valent function was seen in equation (1.16). This was then used in subordination relation to the function with positive real part that were introduced

in the definitions 1.1 - 1.4 which represented some known subclasses of analytic-univalent that motivated the interest and focus of this work.
The leading results are contained in two theorems. The first theorem involved the coefficient inequalities for the class $K - ST_{\tau}^{m}[A, B]$ along with some corollaries that were pointed out by specializing some parameters to obtain some new and existing results in this perspective.
The second theorem equally involved the coefficient inequalities for the class $K - UCV_{\tau}^{m}[A, B]$ along with some corollaries that were pointed out by specializing some parameters to obtain some new and existing results in this perspective as well.
Thus, the exciting imports of each theorem followed when employing the proved of each result. The new results presented in this paper are exciting for research benefits. In particular, the coefficient inequalities obtained in this work could be extended in order to invesigate some peculiar behaviours of some other subclasses of analytic-univalent functions.

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