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NUMERICAL APPROXIMATION OF SINGULAR MULTI-ORDER FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS BY LEAST SQUARES AND AKBARI-GANJI'S METHODS

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Abstract. This article is concerned with the numerical solution of singular multi-order fractional Volterra integro-differential equations. Two numerical methods are proposed; Least Squares and Akbari-Ganji's Methods using Legendre polynomials as basis functions. The proposed methods were demonstrated on some examples to verify their practicability and the results obtained were very close to the exact solution.
Key words: Singular Multi-order Fractional; Volterra Integro-differential Equations; Least Squares Method and Akbari-Ganji's Method

1. Introduction

There has been an increased interest in fractional calculus by researchers due to its applications in the fields of sciences and engineering. Fractional calculus (non-integer derivatives or integrals) possess a memory effect which are useful in several materials such as viscoelastic materials or polymers as well as principles of applications such as anomalous diffusion that enhances the transformation of physical problems. Weibeer (2005) These transformation are usually expressed either as fractional differential equations (ordinary and partial differential) or fractional integro-differential equations. However, most of these equations do not have solutions analytically and so it becomes important to use numerical methods as an alternative methed of solution. Hence, the need to provide good approximate solution schemes.
Numerical schemes or methods are constructed and their effectiveness in terms of accuracy and efficiency are verified by comparing the solutions of the methods with the exact solutions if they exist or that of the existing scheme results in the

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literatures.
Here, we place side by side the results of Least Squares and that of Akbari-Ganji's Methods using Legendre polynomials as basis functions with the exact solutions of some singular multi-order fractional Volterra integro-differential equations. The general form of the class of singular multi-order fractional integro-differential equation is given as:

$$D^{\alpha}y(x)+\sum_{i=0}^{n}p_i y^{(i)}(x)+\lambda\int_0^x \frac{y(t)}{\sqrt{x-t}}dt=f(x) \tag{1.1}$$

subject to the condition

$$y^{(k)}(0)=\alpha_k; \quad k=0,1,2,\cdots n-1; \tag{1.2}$$

Where $D^{\alpha}y(x)$ the α^{th} Caputo derivatives of $y(x)$, $y^{(i)}(x)$ is the i^{th} derivative of $y(x)$, p_i, $i=0,1,2,\cdots n$ are constants, x and t are given real variables in the interval [0,1], $y(x)$ is the unknown function to be determined and $\frac{1}{\sqrt{x-t}}$ is the kernel, the singular part

Some Relevant Terms Used

In this section, we defined some of the relevant terminologies that would aid the understanding of the work. They are:

Integro-Differential Equation: An integro-differential equation is a type of differential equation that involves both derivatives and integrals operators and the two operators are present in the same equation. It represents a mathematical relationship between a function and its derivatives as well as its integral over certain intervals. The general form of an integro-differential equation is given as:

$$y^{(n)}(x)=f(x)+\lambda\int_a^b k(x,t)y(t)dt \tag{1.3}$$

subject to conditions $y^{(k)}(0)=\phi_k$, a and b are the limits of integration and $k(x,t)$ is the kernel.

Fractional Order Differential Equations: Fractional order differentioal equations are generalized non-integer order differential equations. Equation (3) becomes a fractional differential equation if the differential n is replaced by a fractional operator α and it is now written as

$$D^{(\alpha)}y(x)=f(x)+\lambda\int_a^b k(x,t)y(t)dt \tag{1.4}$$

where $D^{(\alpha)}$ denote derivative of $y(x)$ and α is fractional number.

Integro-Differential Difference Equation: A differenntial equation is called integro-differential difference equation if the kernel say $k(x,t)$ under the integral sign depends on the difference $x-t$ so that equation (4) becomes

$$y^{(n)}(x)=f(x)+\lambda\int_a^b (x-t)y(t)dt \tag{1.5}$$

and the kernel is called difference kernel.

Fredholm Integro-differential equation: If the limit of integration are fixed then the Integro-differential equation is called Fredholm Integro-differential equation. This is of the form

$$y^{(n)}(x) = f(x) + \lambda \int_a^b K(x;t)y(t)dt, \quad y^{(n)} = \frac{d^n y}{dx^n} \tag{1.6}$$

where a and b are lower and upper limits of integration which are constants

Volterra Integro-differential Equations: If one of the limit is a variable then Integro-differential equation is called Volterra integral-differential equation. This is of the form

$$y^{(n)}(x) = f(x) + \lambda \int_a^x K(xt)y(t)dt, \qquad u^{(n)} = \frac{d^n y}{dx^n} \tag{1.7}$$

where a is a constant and x is a variable.

Equation (7) is called Integro-differential Difference Equation if the kernel $k(x,t)$ is given as $k(x-t)$ and the general form is

$$y^{(n)}(x) = f(x) + \lambda \int_0^x (x-t)y(t)dt \tag{1.8}$$

Caputo Fractional Derivative: Let $m-1 \leq \alpha \leq m$, and $\alpha \leq 0$. Caputo fractional derivative denoted by $D_x^\alpha f(x)$ is defined as follows:

$$D_x^\alpha f(x) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1}[D^m f(\tau)]d\tau; \ m-1 \leq \alpha \leq m \\ \frac{d^m f(x)}{dx^m}, \ \alpha = m; \ m\epsilon N \end{cases}$$

The Caputo fractional derivative has the following properties:

$$J^\alpha J^v f(x) = J^{\alpha+v} f(x), \alpha, v \geq 0$$

$$J^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} x^{\beta+\alpha}$$

$$D^\alpha x^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}$$

$$J^\alpha D^\alpha f(x) = f(x)$$

$$D_x^\alpha f(x) = D_x^{\alpha-n} f(x) = j^{m-\alpha}[D^m f(x)]; \quad m-1 \leq \alpha \leq m$$

Legendre Polynomials: Legendre polynomials belong to the class of classical orthogonal polynomials. In the Rodrigues formula

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n \tag{1.9}$$

$P_0(x) = 1$ and $P_1(x) = x$ and for $n \geq 1$, the recurrence

$$P_{n+1}(x) = \frac{2n+1}{n+1} x P_n(x) - \frac{n}{n+1} P_{n-1}(x); \tag{1.10}$$

becomes useful. Therefore, using (9) and (10) together, we can generate a few Legendre polynomial valid in [0, 1] as we replace every x obtained from equation (10) by $2x-1$. So we have the following shifted legendre polynomial

$\bar{P}_0(x) = 1$

$\bar{P}_1(x) = 2x - 1$

$\bar{P}_2(x) = 6x^2 - 6x + 1$
$\bar{P}_3(x) = 20x^3 - 30x^2 + 12x - 1$
$\vdots$

1.1. **Literature Review.** Recently, singular multi-order fractional differential equations and singular multi-order fractional integro-differential equations have attracted great attention from various researchers in the field of physics, visco-elasticity, fluid mechanism, signal processing, electric circuit etc. These researchers have tried to apply the concept fractional calculus to real life problems solving and the likes. Some of these researchers includes Beleanu (2022) who introduced a new technique of fractional model of COVID-19 pandemic including the effects of isolation and quarantine to determine the reproduction number of the pandemic. In the model, ordinary time-derivatives was developed and then modified by applying the general structure of fractional operators. After some numerical simulations, the results provided a better fit to the real data which were compared to other classical and fractional models. Abu-arqab (2022) provided a novel analytical algorithm for generalized fifth-order time-fractional and non-linear evolution. The study designed an efficient algorithm for solving time-fractional Caudrey-Dodd-Gibbon, and Kaup-Kupershmidt equations in the form of a convergent conformable time-fractional series and the results were presented in two-and three-dimensional graphs, while dynamic behaviors of fractional parameters were reported for several values. Kumar (2021) introduced a new numerical method for the solution of the fractional SEIR epidemic of measles. The study used wavelet-based numerical scheme for fractional order SEIR epidemic of measles by Genocchi polynomial. Simulations were done and the results were compared to the existing ones. Djennadi (2021) considered an inverse backward and source problems for time-space fractional diffusion equation by applying the Tikhonov regularization technique. The error estimates between the exact and its regularized solutions were obtained. The convergence of the regularized solutions also obtained validate the results which were adjudged okay. Gu (2021) studied a starting point in fractional equation in revealing the memory of length by an inverse problem. The study, through existence analysis of the inverse problem, obtained the range of the initial value points and the memory length of fractional differential equations and the yielded results favorably agreed with those in the literatures. Oyedepo (2022) used the least square method to solve fractional order integro-differential equations. Alkhalissi (2021) proposed an operational matrix method for fractional differential equations by using the generalized Gegenballer-Hambert polynomials. There are many other researchers that have employed numerical techniques to solve various fractional differential or integro-differential equations. They include among others; Sadabad (2020) that studied the eigenvalue and eigenvector of fractional Sturm-Liouville problems via Laplace transform, Yang (2021) that used numerical approach to investigated and analyzed intermediate value problems of fractional differential equations, Uwaheren (2022) that find the Numerical solution of Volterra integro-differential equations by modified Akbari-Ganij's method, Hao (2017) who proposed nuemrical solution for a class of multi-order fractional differential equation with error correction and convergence analysis and Wei (2021) applied extrapolation technique to solve two-point fractional order boundary value problems (BVP). The work presented a stable numerical method for solving fractional differential equation with end-point singularities by finite difference method. In this study, Least Squares and Akbari-Ganji's Methods solution of singular multi-order fractional Volterra integro-differential equations using Legendre polynomials basis functions is proposed.

2. Materials and Methods

Least Squares Method

Consider a singular multi-order fractional inegro-differential equation of the form:

$$D^{\alpha}y(x) = \sum_{i=0}^{n} p_i y^{(i)}(x) + \lambda \int_a^b k(x,t)\frac{y(t)}{\sqrt{x-t}}dt + f(x) \tag{2.1}$$

together with the initial conditions

$$y(0) = \alpha,\ y'(0) = \beta \tag{2.2}$$

To solve equations (11) and (12), we assume a trial solution of the form:

$$y_N(x) = \sum_{i=0}^{N} a_i L_i(x) \tag{2.3}$$

where $L_i(x)$ is Legendre polynomials
we substitute (13) into (11)

$$D^{\alpha}\left\{\sum_{i=0}^{N} a_i L_i(x)\right\} = \Big\{\sum_{i=0}^{n} p_i \sum_{i=0}^{N} a_i L_i(x)\Big\} + \Big\{\lambda \int_a^x \frac{k(x,t)}{\sqrt{x-t}} \sum_{i=0}^{N} a_i L_i(t)dt\Big\} + \{f(x)\} \tag{2.4}$$

apply J^{α} operator on both sides of equation, we have

$$J^{\alpha}D^{\alpha}\left\{\sum_{i=0}^{N} a_i L_i(x)\right\} = J^{\alpha}\Big\{\sum_{i=0}^{n} Q_i L^{(i)}(x)\Big\} + J^{\alpha}\Big\{\lambda \int_a^x \frac{k(x,t)}{\sqrt{x-t}} \sum_{i=0}^{N} a_i L_i(t)dt\Big\} + J^{\alpha}\{f(x)\} \tag{2.5}$$

where $Q_i = p_i a_i$

$$\sum_{i=0}^{N} a_i L_i(x) = J^{\alpha}\Big\{\sum_{i=0}^{n} Q_i L^{(i)}(x)\Big\} + J^{\alpha}\Big\{\lambda \int_a^x \frac{k(x,t)}{\sqrt{x-t}} y_N(t)dt\Big\} + J^{\alpha}\{f(x)\} \tag{2.6}$$

and the residual equation $R(a_0, a_1, \cdots, a_N)$ is written as

$$\begin{aligned} R(a_0, a_1, \cdots, a_N) &= \sum_{j=0}^{N} a_j L_j(x) - J^{\alpha}\Big\{\sum_{i=0}^{n} Q_i L^{(i)}(x)\Big\} \\ &- \lambda J^{\alpha}\Big\{\int_a^x \frac{k(x,t)}{\sqrt{x-t}}\Big(\sum_{j=0}^{N} a_j L_j(t)\Big)\Big\} - J^{\alpha}\{f(x)\} = 0 \end{aligned} \tag{2.7}$$

Now, we let the sum of the squares of errors $s(a_0, a_1, \cdots, a_N)$ to be

$$s(a_0, a_1, \cdots, a_N) = \int_0^1 \Big[w(x)R(a_0, a_1, \cdots, a_N)\Big]^2 dx \tag{2.8}$$

where $w(x)$ is the positive weight function defined in the interval $[a, b]$.
Thus,

$$\begin{aligned} s(a_0, a_1, \cdots, a_N) \\ &= \int_0^1 w(x)\Big[\sum_{j=0}^{N} a_j L_j(x) - J^\alpha\Big\{\sum_{i=0}^{n} Q_i L^{(i)}(x)\Big\} \\ &- \lambda J^\alpha \int_a^x \frac{k(x,t)}{\sqrt{x-t}}\Big(\sum_{j=0}^{N} a_j L_j(t)\Big)dt - J^\alpha\{f(x)\}\Big]^2 dx = 0 \end{aligned} \tag{2.9}$$

The necessary condition for a minimum is that:

$$\frac{\partial s}{\partial a_j} = 0; \quad j = 0, 1, 2, \cdots, N \tag{2.10}$$

so taking the partial derivative of equation (19) successively with respect to $a_j; \quad j = 0, 1, 2, \cdots, N$, we have

$$\begin{bmatrix} \int_0^1 w(x)\Big[\sum_{j=0}^N a_j L_j(x) - \Big[J^\alpha\Big\{\sum_{j=0}^N Q_j L_j^{(j)}(x)\Big\} + \lambda J^\alpha\Big\{\int_a^x \frac{k(x,t)}{\sqrt{x-t}}\Big(\sum_{j=0}^N a_j L_j(t)\Big)dt\Big\} - \\ J^\alpha f(x)\Big]\Big] d_1 dx = 0 \\ \int_0^1 w(x)\Big[\sum_{j=0}^N a_j L_j(x) - \Big[J^\alpha\Big\{\sum_{i=0}^n Q_i L^{(i)}(x)\Big\} + \lambda J^\alpha\Big\{\int_a^x \frac{k(x,t)}{\sqrt{x-t}}\Big(\sum_{j=0}^N a_j L_j(t)\Big)dt\Big\} - \\ J^\alpha f(x)\Big]\Big] d_2 dx = 0 \\ \vdots \\ \int_0^1 w(x)\Big[\sum_{j=0}^N a_j L_j(x) - \Big[J^\alpha\Big\{\sum_{i=0}^n Q_i L^{(i)}(x)\Big\} + \lambda J^\alpha\Big\{\int_a^x \frac{k(x,t)}{\sqrt{x-t}}\Big(\sum_{j=0}^N a_j L_j(t)\Big)dt\Big\} - \\ J^\alpha f(x)\Big]\Big] d_n dx = 0 \end{bmatrix} \tag{2.11}$$

where $d_1, d_2, \cdots d_n$ are the derivatives of the inner terns of the integral sign with respect to $a_j = 0; \quad j = 0, 1, 2, \cdots, N$. Thus, we have (N+1) algebraic linear system of equations in (N+1) unknown constants $a_i's$ which are then solved by maple 18 to obtain the unknown constants. Substituting the constant values into equation (13) we get the required approximate solution.

Akbari-Ganji Method (AGM)

Consider a multi-order fractional singular inegro-differential equation of the form:

$$D^\alpha y(x) + \sum_{i=0}^{n} p_i y^{(i)}(x) + \lambda \int_a^x \frac{k(x,t)y(t)}{\sqrt{x-t}} dt = f(x) \tag{2.12}$$

subject to initial conditions

$$y(0) = \alpha, \; y'(0) = \beta \tag{2.13}$$

To solve equations (22) and (23), we assume a trial solution of the form:

$$y_N(x) = \sum_{i=0}^{N} a_i L_i(x) \tag{2.14}$$

where $L_i(x)$ is Legendre polynomials
we apply the initial conditions in equation (24) we obtain two equations

$$y_N(0) = \sum_{i=0}^{n} a_i L_i(0) = \alpha \tag{2.15}$$

$$y'_N(0) = \sum_{i=0}^{n} a_i L'_i(x) = \beta \tag{2.16}$$

Next, substituting equation (26) into equation (24) we obtain

$$D^{\alpha}\Big\{\sum_{j=0}^{n} a_i L_i(x)\Big\} + \sum_{i=0}^{n} Q_j L_j^{(i)}(x) + \lambda \int_a^b \frac{k(x,t)}{\sqrt{x-t}} \sum_{j=0}^{N} a_j L_j(t)dt = f(x) \tag{2.17}$$

Equation (27) is differentiated $n-2$ times to obtain.

$$\begin{bmatrix} D^{\alpha}\Big\{\sum_{j=0}^{N} a_j L'_j(x)\Big\} + \sum_{i=0}^{n} Q_j L_j^{(i+1)}(x) + \lambda \int_a^b \frac{k(x,t)}{\sqrt{x-t}} \sum_{j=0}^{N} a_j L'_j(t)dt = f'(x) \\ D^{\alpha}\Big\{\sum_{j=0}^{N} a_j L''_j(x)\Big\} + \sum_{i=0}^{n} Q_j L_j^{(i+2)}(x) + \lambda \int_a^b \frac{k(x,t)}{\sqrt{x-t}} \sum_{j=0}^{N} a_j L''_j(t)dt = f''(x) \\ \vdots \\ D^{\alpha}\Big\{\sum_{j=0}^{N} a_j L_j^{(n-2)}(x)\Big\} + \sum_{i=0}^{n} Q_j L_j^{(i+n-2)}(x) + \lambda \int_a^b \frac{k(x,t)}{\sqrt{x-t}} \sum_{j=0}^{N} a_j L_j^{(n-2)}(t)dt = f^{(n-2)}(x) \end{bmatrix} \tag{2.18}$$

The equations (28) are evaluated at $x = 0$ to obtained $n-2$ algebraic linear equations. The system of algebraic linear equations after the evaluation are solved together with equations (25) and (26) by Gaussian elimination method or any mathematical software readily available to obtain the constant coefficients. Then substitute the constants back into the assumed or trial solution, equation (24) to obtain the required approximate solution.

3. Numerical Examples

Problem 1
Consider a third order singular fractional integro-differential equation of the form

$$y'''(x) + D^{2.75}y(x) + xy(x) + \int_0^x \frac{y(t)}{\sqrt{x-t}}dt = 6 + 12x^2 + \frac{32}{35}x^{\frac{7}{2}} + 6.61957590800x^{0.25} \tag{3.1}$$

$$y(0) = 0, y'(0) = 0, y''(0) = 0$$

The exact solution is $y(x) = x^3$
Let

$$\begin{aligned} y_N(x) &= \sum_{k=0}^{N} a_k L_k(x) \\ &= a_0 + (2x-1)a_1 + (6x^2-6x+1)a_2 + (20x^3-30x^2+12x-1)a_3 \\ &+ (70x^4-140x^3+90x^2-20x+1)a_4 + (252x^5-630x^4+560x^3-210x^2 \\ &\qquad + 30x - 1)a_5 \end{aligned} \quad (3.2)$$

where $N = 5$, and L_k are the Legendre polynomials.

Least Square Method

Following the algorithm of the methodology, we got the values of the unknown constants a_i $i = 0, 1, 2...N$ to be:

$$a_0 = 0.2678334737, a_1 = 0.4890789570, a_2 = 0.2832738628, a_3 = 0.06516880404,$$
$$a_4 = 0.003367235102, a_5 = 0.0002268105466$$

Substituting the constants into the trial solution (34), we got the required approximate solution as

$$y_5(x) = 2.7510^{-11} - 1.10^{-10}x - 1.10^{-10}x^2 + 1.080000000x^3 - .4467412829x^4 + .46893151883x^5$$

similarly

Akbari-Ganji's Method

Following the algorithm of the methodology, we got the values of the unknown constants a_i $i = 0, 1, 2...N$ to be:

$$a_0 = 0.1754736632, a_1 = 0.3030647436, a_2 = 0.1488273474, a_3 = 0.01767747155,$$
$$a_4 = -0.003205892655, a_5 = 0.0003529028525$$

Substituting the constants into the trial solution (34), we got the required approximate solution as

$$y_5(x) = 1.5410^{-11} - 1.10^{-8}x + 1.09589770727x^3 - 0.928158127e^{-1}x^4 + 0.5715625774e^{-1}x^5$$

TABLE 1. Example 1 with N=5

x	Exact Solution	LSM	AGM	LSM Error	AGM Error
0.0	0.0000000000	0.0000000000	0.0000000001	1.5400e-11	5.7500e-11
0.1	0.0010000000	0.00083848841	0.00080121640	3.1170e-10	4.3785e-10
0.2	0.0080000000	0.00792672905	0.00768840666	1.6139e-10	6.8633e-09
0.3	0.0270000000	0.02690151847	0.02658009817	2.1692e-10	3.4025e-09
0.4	0.0640000000	0.06287318108	0.06387009884	3.3590e-08	1.0526e-09
0.5	0.1250000000	0.1241749968	0.12468972318	2.4593e-08	2.5142e-09
0.6	0.2160000000	0.2134463243	0.21537083210	7.6124e-08	5.0982e-09
0.7	0.3430000000	0.36424022056	0.3418607301	1.7820e-08	9.2316e-09
0.8	0.5120000000	0.5031401961	0.5118701464	3.5743e-06	1.5384e-09
0.9	0.7290000000	0.7129523105	0.7287030884	6.4741e-06	2.4059e-09
1.0	1.0000000000	0.9972993995	0.9987996812	1.0895e-06	3.5781e-08

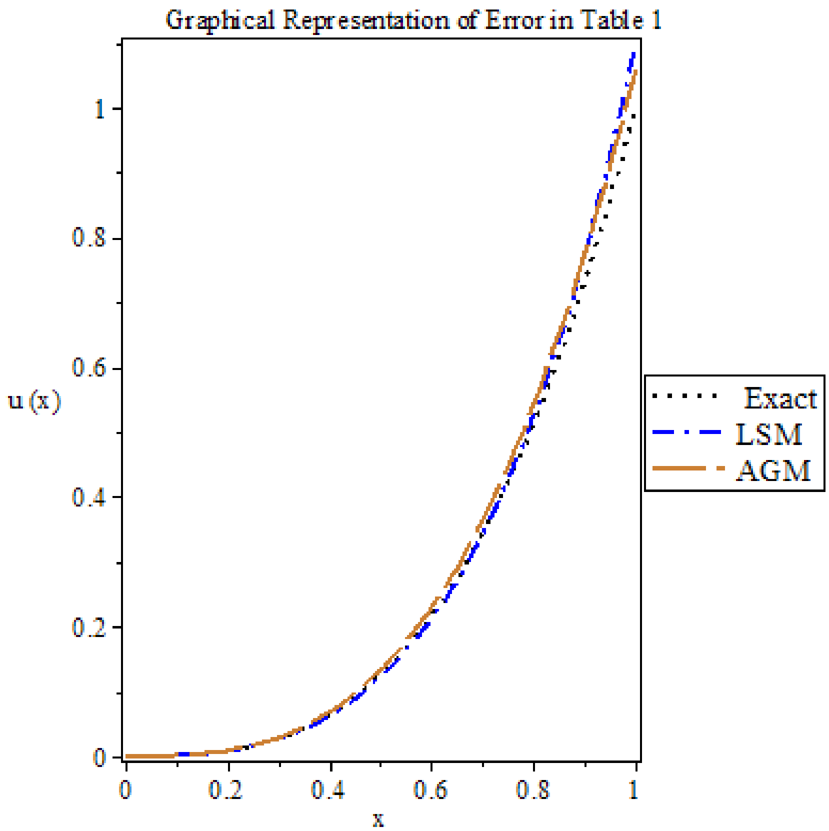

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FIGURE 1. Example 1

Problem 2

Consider a second order singular fractional integro-differential equation of the form

$$y''(x)+D^{1.5}y(x)-\int_0^x \frac{y(t)}{\sqrt{x-t}}dt = 2.256758334x^{0.5}+7.221626669x^{2.5}-\frac{16}{15}x^{\frac{5}{2}}-\frac{256}{315}x^{\frac{9}{2}}+2+12x^2 \quad (3.3)$$

$0 \leq x \leq 1$

$y(0) = 0,\ y'(0) = 0$

The exact solution is given as $y(x) = x^4 + x^2$

Let,

$$\begin{aligned} y_N(x) &= \sum_{k=0}^{N} a_k L_k(x) \\ &= a_0 + (2x-1)a_1 + (6x^2-6x+1)a_2 + (20x^3-30x^2+12x-1)a_3 \\ &+ (70x^4-140x^3+90x^2-20x+1)a_4 + (252x^5-630x^4+560x^3-210x^2 \\ &+ 30x-1)a_5 \end{aligned} \quad (3.4)$$

where $N = 5$, and L_k are the Legendre polynomials.

Least Square Method

Following the algorithm of the methodology, we got the values of the unknown constants a_i $i = 0, 1, 2...N$ to be:

$a_0 = 0.1754736632, a_1 = 0.3030647436, a_2 = 0.1488273474, a_3 = 0.01767747155,$
$a_4 = -0.003205892655, a_5 = 0.0003529028525$

putting the values of a_k into equation (36) to get an approximate solution

$$y_5(x) = 1.10^{-11} - 2.10^{-6}x + 1.1250000001x^2 - 1.20008923575x^3 + 1.00025275159x^4 + 1.4746422227744x^5$$

Akbari-Ganji's Method

$a_0 = 0.2874537050, a_1 = 0.560149388, a_2 = 0.4971861988, a_3 = 0.2439948375,$
$a_4 = 0.05242454420, a_5 = -0.00294532836$

putting the values of a_k into equation (36) to get an approximate solution

$$y_5(x) = 1.1410^{-11} - 1.10^{-9}x + 1.114053524x^2 - 1.004356229x^3 + 1.0016326121x^4 + 1.1102813744x^5$$

TABLE 2. Example 2 with N=5

x	Exact Solution	LSM	AGM	LSM Error	AGM Error
0.0	0.0000000000	0.0000000001	0.0000000000	6.1400e-11	1.0000e-11
0.1	0.0101000000	0.0093591978	0.0087093202	2.2083e-08	3.6638e-08
0.2	0.0416000000	0.0408508113	0.0407322253	2.2333e-08	2.5868e-08
0.3	0.0981000000	0.0971879442	0.0955504840	2.7188e-07	7.6090e-08
0.4	0.1856000000	0.1624335822	0.1850809722	9.3460e-06	1.5472e-08
0.5	0.3125000000	0.3065009240	0.2953869622	2.4202e-06	2.5398e-05
0.6	0.4896000000	0.4760347808	0.4654262760	5.4726e-06	3.5877e-05
0.7	0.7301000000	0.7272985143	0.6999416230	1.1302e-05	4.4759e-05
0.8	1.0496000000	1.0349672010	1.0163529480	2.1717e-05	4.9343e-05
0.9	1.4661000000	1.4396474940	1.4347921290	3.9259e-05	4.6465e-05
1.0	2.0000000000	1.9546334080	1.9780430520	6.7330e-05	3.2587e-05

Problem 3

Consider a first order singular fractional integro-differential equation of the form

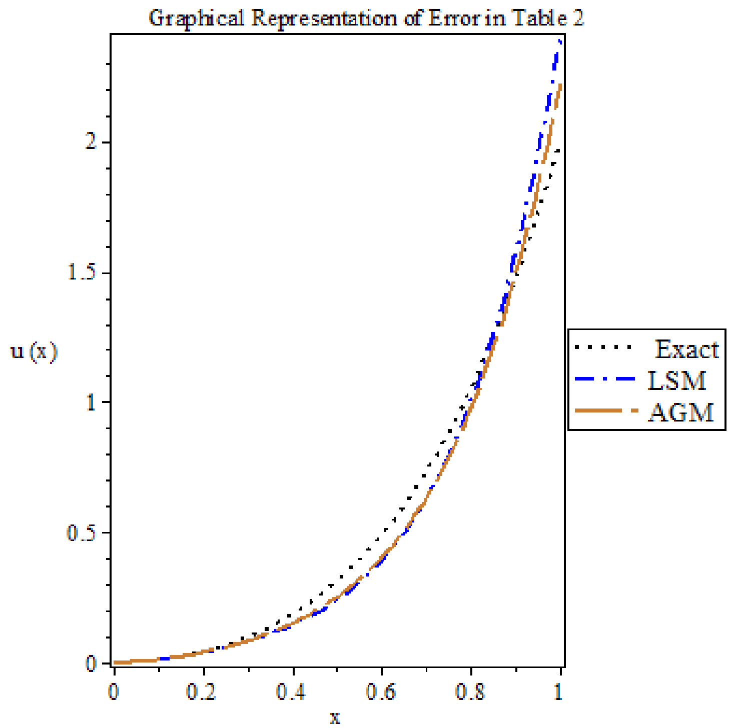
$$y'(x) - D^{0.5}y(x) + \frac{8x^{\frac{3}{2}}}{3\sqrt{\pi}} - \frac{2x^{\frac{1}{2}}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^x \frac{xt}{\sqrt{x-t}}y(t)dt = 0, \ 0 \leq x \leq 1 \quad (3.5)$$

$y(0) = 0$

The exact solution is given as $y(x) = x^2 + x$

Let,

$$\begin{aligned} y_N(x) &= \sum_{k=0}^{N} a_k L_k(x) \\ &= a_0 + (2x-1)a_1 + (6x^2 - 6x + 1)a_2 + (20x^3 - 30x^2 + 12x - 1)a_3 \\ &+ (70x^4 - 140x^3 + 90x^2 - 20x + 1)a_4 + (252x^5 - 630x^4 + 560x^3 - 210x^2 \\ &+ 30x - 1)a_5 \quad (3.6) \end{aligned}$$



2.PNG

FIGURE 2. Example 2

where $N = 5$, and L_k are the Legendre polynomials.

Least Square Method

Following the algorithm of the methodology, we got the values of the unknown constants a_i $i = 0, 1, 2...N$ to be:

$$a_0 = 0.13612632, a_1 = 0.0348927456, a_2 = 0.2531427074, a_3 = 0.01076172315,$$
$$a_4 = -0.006120543515, a_5 = 0.00000000000$$

putting the values of a_k into equation (38) to get an approximate solution

$$y_5(x) = 0.001006300000-x+1.0162500\,x^2+0.0007923575\,x^3+0.000231151\,x^4+0.0000065\,x^5$$

Akbari-Ganji's Method

$$a_0 = 0.2562430691, a_1 = 0.132564931088, a_2 = 0.39567861348, a_3 = 0.22343948115,$$
$$a_4 = 0.0263109403, a_5 = -0.0025345300281$$

putting the values of a_k into equation (38) to get an approximate solution

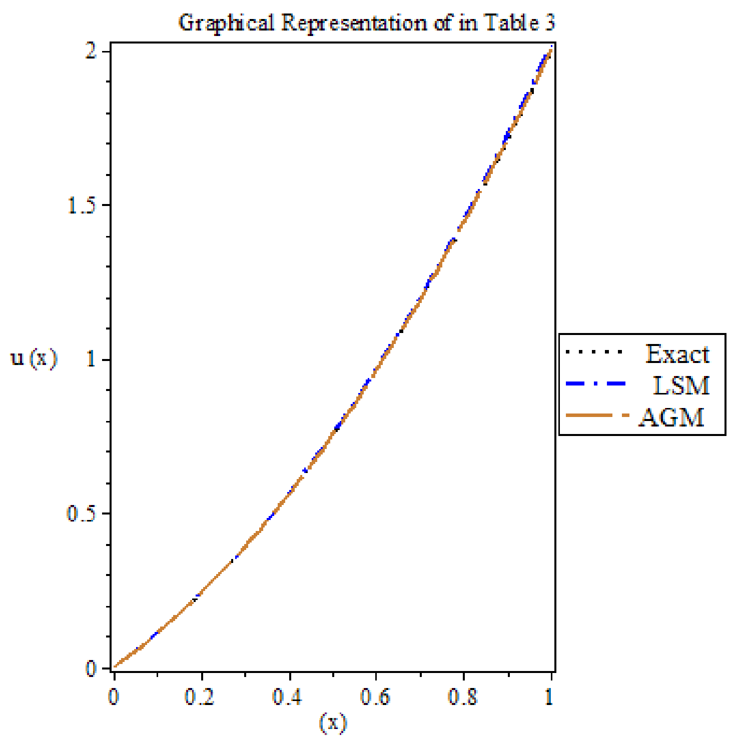
$$y_5(x) = 0.000005000110-x+1.0012580000001\,x^2+0.0008983505\,x^3+0.00025075159\,x^4$$

4. Discussion

In this section, the results obtained using the two proposed numerical methods with Legendre polynomials basis function are discussed. The methods were used to solve some singular multi-order fractional Volterra integro-differential

TABLE 3. Example 3 with N=5

x	Exact Solution	LSM	AGM	LSM Error	AGM Error
0.0	0.0000000000	0.000000000	0.0000000000	3.4800e-11	2.0000e-11
0.1	0.1100000000	0.1100000250	0.1100000412	2.5061e-08	4.1281e-08
0.2	0.2400000000	0.2400000176	0.24000002455	1.7623e-08	2.4557e-08
0.3	0.3900000000	0.3900003273	0.3900000512	3.2731e-07	5.1237e-08
0.4	0.5600000000	0.5600004435	0.5600002641	4.435e-06	2.641e-08
0.5	0.7500000000	0.750002820	0.7500002395	2.8202e-06	2.3695e-05
0.6	0.9600000000	0.960005716	0.9600033421	5.716e-06	3.3421e-05
0.7	1.1900000000	1.190010540	1.190024359	1.0540e-05	2.4359e-05
0.8	1.4400000000	1.440021137	1.440049003	2.1137e-05	4.9003e-05
0.9	1.7100000000	1.710033425	1.710044064	3.3425e-05	4.4064e-05
1.0	2.0000000000	2.000017130	2.000032207	1.7130e-05	3.2207e-05



3.PNG

FIGURE 3. Example 3

equations and the results were presented vis-a-vis the exact solutions. Results obtained are presented in tabular and graphical tables 1, 2 and 3 with the coresponding figures 1, 2 and 3 respectively. It was found that the methods are suitable for the solutions of the class of problem considered. The two methods produced results that converged rapidly to the exact solutions. We can say that the two methods are easy to implement and are effective in solving our class of problem.

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Authors' Conflicts of interest. The authors declare(s) that there are no conflicts of interest regarding the publication of this paper.

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References

[1] O. Abu-arqub, M. Al-Smadi, H. Almusawa, D. Baleanu, T. Haya, M. Alhodaly, M. S. Osman A novel analytical algorithm for generalized fifth-order time-fractional non-linear evolution equations with conformable time derivatives arising in shallow water waves. Alexendria Engineering Journal 61(2021) 5753-5769. https://doi.org/10.1016/j.aej.2021.12.044

[2] J.H.S. Alkhalissi, I. Emirogle, M. Bayyram, A. Secer, A. Tasci A new operational matrix of fractional derivatives based on the generalized Gegenbauer-Humbert polynomials to solve fractional differential equations *Alex. Eng.J.* 60 (2021) 3509-3519.

[3] D. Baleanu, M. H. Abadi, A. Jajarmi, K. Z. Vahid, J.J. Nieto A new comparative study on the general fractional model of covid-19 with isolation and quarantine effects. Alexendria Engineering Journal 61(2002) 4779-4791 . https://www.sciencedirect.com/science/article/pii/S111001682100702

[4] S. Djennadi, N. Shawagfeh, M. S. Osman, J. F. Gmez-Aguilar, O. Abu-Arqub The Tikhonov regularization method for the inverse source problem of time fractional least equation in the view of ABC-fractional technique. *Phys. Scripts.* 96 (2021) 217-232.

[5] C. Y. Gu, G. C. Wu, B. Shiri An inverse problem approach to determine possible memory length of fractional differential equations. *Fract. cal. Appl. Anal.* 24 (2021) 1919-1936.

[6] Z.P. Hao, W. R. Cao An improved algorithm based on finite difference schemes for fractional boundary value problems with non-smooth solution. *J. Sci. Comput* 73(2017) 395-415

[7] S. Kumar, R. Kumar, M. S. Osman, B. Samet A wavelet based numerical scheme of measles by using Genocchi polynomials. *Numer. Math. Part. D.E.* 37(2021) 1250-1268.

[8] T. Oyedepo, C. Y. Ishola, O. A. Uwaheren, M. I. Olaosebikan, M. O. Ajisope, A. A. Victor Least square bernstein method for solving fractional integro-differential equations.

[9] M. K. Sadabad, A. J. Akbarfam, B. Shiri A numerical Study of Eigen values and eigen functions of fractional sturm-Liouville problems via Laplace transform (2020)

[10] O. A. Uwaheren, A. F. Adebisi, C.Y. Ishola., M.T. Raji., A.O. Yekeem, O.J. Peter Numerical solution of volterra integro-differential equations by Akbari Ganji's method. Journal of mathematic and its application 16 (2022) 1123-1130. https://doi.org/10.30598/barekengvol16iss3pp1123-1130

[11] M. Weibeer Efficient Numerical Methods for Fractional Differential Equations and their Analytical Background. Ph.D Thesis. Von der Carl-Friedrich-Gaub-Fakultat fur Mathematik und Informatik der Techniscihen Universitat Braunschweig (2005)

[12] Y. Wei, J. Zhou A two-step algorithm for solving singular linear system with index one,. *Appl. Math and comp.* 174 (2021) 252-270.

[13] G. Yang, B. Shiri, H. Kong, G. C. Wu Intermediate value problems for fractional differential equations. *Comp. Appl. Math.* 195 (2021) 314-341.

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