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ESTIMATES OF SECOND AND THIRD HANKEL DETERMINANTS FOR BAZILEVIC FUNCTION OF ORDER GAMMA

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Abstract. In the recent time, the study of Bazilevic functions became so popular that researchers, especially in Geometric Function Theory, have had to study different subclasses of Bazilevic functions in different directions. However, their study seem to lack full stamina addressing relevant connections of Bazilevic functions to some properties such as coefficient bounds, sharp bounds of the Fekete-Szego functional as well as the Hankel determinants for functions belonging to some specific subclasses of Bazilevic functions. Consequently, in this article, with the aid of Salagean derivative operator, the author derived the Bazilevic class $T_n^{\alpha}(\gamma)$, of order γ, type α, via the convolution of the fractional analytic function $g(z)^{\alpha}$ and the normalized univalent function $f(z)$ in the open unit disk. In the sequel, sharp bounds on the Taylor-Maclaurin coefficients $|a_k(\alpha)|$ for functions belonging to the aforementioned class were obtained while the relationship of these bounds to the classical Fekete-Szego inequality $H_2(1)$ and Hankel determinants $H_2(2)$ and $H_3(1)$ were established using a clear Mathematical approach. Some of the consequences of the results so obtained were discussed as corollaries.

1. Introduction

Let A denote the usual class of analytic functions $f(z)$ with series expansion:

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... \tag{1.1}$$

in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$. Also let denote by S, S^* and C the well-known subclasses of A which consist of normalized univalent, starlike and convex functions respectively. In 1955 a Russian Mathematician Bazilevic ([4]), defined a function *f(z)* (say) in D such that

$$f(z) = \left\{ \frac{\alpha}{1+\sigma^2} \int_0^z \frac{p(v) - i\sigma}{V^{\left(1+\frac{i\alpha\sigma}{(1+\sigma^2)}\right)}} g(v)^{\frac{\alpha}{1+\sigma^2}} dv \right\}^{\frac{1+i\sigma}{\alpha}} \tag{1.2}$$

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where $p \in P$ (class of Caratheodory functions), $\alpha > 0$ and $g \in \Psi^*$ (the class of starlike function). This family of functions became known as Bazilevic functions and is usually, denoted by $B(\alpha, \sigma)$. Very little is known about the said family, except that, he Bazilevic showed that each function $f \in B(\alpha, \sigma)$ is univalent in D. By simplifying (1.2) it is quite possible to understand and investigate the family better. It is noted that with special choices of parameters α, σ and the function $g(z)$, the family $B(\alpha, \sigma)$ degenerates to some well-known subclasses of univalent functions defined and studied by different authors, see [2], [16], [21], [22] and [30] among others. Few of these subclasses are illustrated below.

Illustration 1.

Suppose that we let $\sigma = 0$, then equation (1.2) immediately yields

$$f(z) = \left\{ \alpha \int_0^z \frac{p(v)}{V} g(v)^\alpha dv \right\}^{\frac{1}{\alpha}}. \tag{1.3}$$

Differentiating equation (1.3) we obtain

$$\frac{z\, f'(z)\, f(z)^{\alpha-1}}{g(z)^\alpha} = p(z), \qquad (z \in D) \tag{1.4}$$

or equivalently

$$\Re e \left\{ \frac{z\, f'(z)\, f(z)^{\alpha-1}}{g(z)^\alpha} \right\} > 0, \qquad (z \in D) \tag{1.5}$$

The subclass of Bazilevic functions satisfying equation (1.5) is called Bazilevic function of type α and it is usually denoted by $B(\alpha)$, see [28].

Illustration 2.

If $\alpha = 1$, then the class $B(\alpha)$ reduces to the family of close-to-convex functions; that is,

$$\Re e \left\{ \frac{z f'(z)}{g(z)} \right\} > 0 \qquad (z \in D). \tag{1.6}$$

Illustration 3.

If we decide to choose $g(z) = f(z)$ in inequality (1.6), we have

$$\Re e \left\{ \frac{z f'(z)}{f(z)} \right\} > 0 \qquad z \in D \tag{1.7}$$

which implies that $f(z)$ is starlike.

Illustration 4.

Suppose that $zf'(z)$ replaces $f(z)$ in (1.7),then we obtain

$$\Re \left\{ \frac{z\big(f'(z)\big)'}{zf'(z)} \right\} = \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \qquad (z \in D) \tag{1.8}$$

which shows that $f(z)$ is convex.

Illustration 5.

If $g(z) = z$ in inequality (1.6), then the family $B_1(\alpha)$ of functions satisfying

$$\Re e \left\{ \frac{z\, f'(z)\, f(z)^{\alpha-1}}{z^\alpha} \right\} > 0, \qquad z \in D \tag{1.9}$$

was obtained, see [5], [28] and [29]. In 1992, Abdulhalim ([1]) introduced the Bazilevic class $B(\alpha, n)$, a generalization of (9) which satisfies the geometric condition

$$\Re e \left\{ \frac{D^n f(z)^\alpha}{z^\alpha} \right\} > 0, \qquad z \in D \tag{1.10}$$

where the parameter $\alpha > 0$ and the operator D^n is the famous Sălăgean derivative operator [25], see also [22] and [25]. However, he was able to show that for all $n \in N$, each function of the class $B(\alpha, n)$ is univalent in D. Recently, (1.10) was improved upon such that

$$\Re \left\{ \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \right\} > 0, \qquad (\alpha > 0,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ z \in \mathbb{C}) \tag{1.11}$$

for $\alpha > 0$. For more details on the Bazilevic class of the form (1.11), interested reader can see [2], [9],[10], [11], [12] and [22] among others. Although several researchers have examined certain properties of Bazilevic function of the form (1.11) and their results authenticated diversely in literatures, the study of the Bazilevic class defined in (1.11) as related to Hankel determinants is not famous in literatures and thus the motivation for the present investigation. For recent works on Hankel determinant interested reader can refer to [6], [8], ([9]), [11], [13], [14], [15], [17], [18]-[20], [26] [27], [29] and [31] to mention just few.
Now, in [12], Hamzat and Oladipo considered certain fractional analytic function $g(z)^\alpha$ of the form:

$$g(z)^\alpha = \frac{z^\alpha}{1-z} = z^\alpha + \sum_{k=2}^{\infty} z^{\alpha+k-1} \tag{1.12}$$

for real number $\alpha (\alpha > 0)$ in E (see also [8]). Using the concept of convolution and applying Salagean differential operator respectively, then

$$f(z)^\alpha = f(z) * g(z)^\alpha = z^\alpha + \sum_{k=2}^{\infty} a_k z^{\alpha+k-1} \tag{1.13}$$

and

$$D^n f(z)^\alpha = \alpha^n z^\alpha + \sum_{k=2}^{\infty} (\alpha + k - 1)^n a_k(\alpha) z^{\alpha+k-1}. \tag{1.14}$$

It is observed that

$$\Re \left\{ \frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} \right\} > \gamma, \qquad (0 \le \gamma < 1,\ \alpha > 0,\ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ z \in \mathbb{C}). \tag{1.15}$$

Interestingly, (1.15) coincides with the famous class of Bazilevic functions $T_n^\alpha(\gamma)$ studied by different authors, see [3], [19] and [23] among others.

Moreover, Pommerenke [24] in 1966 defined the q^{th} Hankel determinant of for $q \ge 1$ and $n \ge 1$ as

$$\mathbf{H_q(n)} = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n+q-1} & \cdots & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.16}$$

With various choices of the parameters n and q several functional are obtained. For instance,
[1] if $q = 2$ and $n = 1$, we obtain

$$\mathbf{H_2(1)} = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \left|a_3 - a_2^2\right| \quad (a_1 = 1) \tag{1.17}$$

see [7]-[11] among others.
[2] letting $q = 2$ and $n = 2$, we have

$$\mathbf{H_2(2)} = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = |a_2a_4 - a_3^2| \tag{1.18}$$

[3] suppose that $q = 3$ and $n = 1$, we have

$$\mathbf{H_3(1)} = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2). \tag{1.19}$$

Applying the triangle inequality in (1.19) we obtain

$$|H_3(1)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|. \tag{1.20}$$

2. Preliminary Lemma

Let P denote the class of all functions having the form

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + ... \tag{2.1}$$

which are regular in the open unit disk U and satisfy the geometric condition

$$\Re\{p(z)\} > 0 \quad (z \in U).$$

This class of functions defined in (2.1) is popularly known as the Caratheodory class. They are usually referred to as the class of functions with positive real part.

Lemma 2.1:
Let $p \in P$, then $|p_k| \leq 2$ and the inequality is sharp for the function

$$p(z) = \frac{1+z}{1-z}, \tag{2.2}$$

see [14], and [31] among others.

Lemma 2.2:
The power series for $p(z) = 1 + \sum_{k=1}^{\infty} p_kz^k$ given in (2.1) converges in the unit disk to a function in p if and only if the Toeplitz determinants

$$\mathbf{D_n} = \begin{vmatrix} 2 & p_{-1} & p_{-2} & \cdots & p_{-n} \\ p_{-1} & 2 & p_{-1} & \cdots & p_{-n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{vmatrix}, \quad (n = 1, 2, 3, ...) \tag{2.3}$$

and $p_{-k} = \overline{p_k}$ are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k}z)$, $\rho_k > 0$, t_k is real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$, in this case $D_n > 0$ for $n < (m-1)$ and $D_0 = 0$ for $n \geq m$.

If we assume without restriction that $p_1 > 0$ in (2.3). Then, using lemma 2.2 for $n = 2$ we have that

$$\mathbf{D_2} = \begin{vmatrix} 2 & p_1 & p_2 \\ \overline{p_1} & 2 & p_1 \\ \overline{p_2} & \overline{p_1} & 2 \end{vmatrix} = 8 + 2\Re\{p_1^2 p_2\} - 2|p_2|^2 - 4|p_1|^2 \geq 0 \tag{2.4}$$

which is equivalent to

$$p_2 = \frac{1}{2}\{p_1^2 + (4 - p_1^2)x\}, \quad |x| \leq 1. \tag{2.5}$$

Also for $n = 3$,

$$\mathbf{D_3} = \begin{vmatrix} 2 & p_1 & p_2 & p_3 \\ \overline{p_1} & 2 & p_1 & p_2 \\ \overline{p_2} & \overline{p_1} & 2 & p_1 \\ \overline{p_3} & \overline{p_2} & \overline{p_1} & 2 \end{vmatrix} \geq 0 \tag{2.6}$$

which is equivalent to

$$|(4p_3 - 4p_1p_2 + p_1^3)(4 - p_1^2) + p_1(2p_2 - p_1^2)^2| \leq 2(4 - p_1^2)^2 - 2|2p_2 - p_1^2|^2. \tag{2.7}$$

Using (2.5) on (2.7), we obtain

$$p_3 = \frac{1}{4}\{p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z\} \quad |z| \leq 1 \tag{2.8}$$

see [14] and [31] among others .
Lemma 2.3: Let $p \in P$, then we have sharp inequalities

$$\left|p_2 - \mu\frac{p_1^2}{2}\right| = \left\{ \begin{array}{lll} 2(1-\mu), & if & \mu \leq 0, \\ 2, & if & 0 \leq \mu \leq 2, \\ 2(\mu - 1), & if & \mu \leq 2 \end{array} \right\}$$

see [3] among others. Throughout this work unless otherwise stated, $f(z)^\alpha$ would be written as f^α.

3. Methods and Results

Theorem 3.1: Let $f^\alpha \in T_n^\alpha(\gamma)$. If $\alpha > 0$, $0 \leq \gamma < 1$ and $n \in N_0 = N \cup \{0\}$ then

$$|a_k(\alpha)| \leq \frac{2(1-\gamma)}{\alpha_{n,k}}.$$

where $\alpha_{n,k} = \left(\frac{\alpha+k-1}{\alpha}\right)^n$ $k = 2, 3, ...,$.
Proof: Let $f^\alpha \in T_n^\alpha(\gamma)$. Then, we set

$$\frac{\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \gamma}{1 - \gamma} = p(z). \tag{3.1}$$

It follows that

$$\frac{D^n f(z)^\alpha}{\alpha^n z^\alpha} - \gamma = p(z)(1 - \gamma). \tag{3.2}$$

Equating the coefficients of the like powers of z, z^2, z^3 and z^4 in (3.2), we obtain

$$a_2(\alpha)=\frac{(1-\gamma)p_1}{\alpha_{n,2}},\ a_3(\alpha)=\frac{(1-\gamma)p_2}{\alpha_{n,3}},\ a_4(\alpha)=\frac{(1-\gamma)p_3}{\alpha_{n,4}}\ \textit{and}\ a_5(\alpha)=\frac{(1-\gamma)p_4}{\alpha_{n,5}}. \tag{3.3}$$

Obviously, we conclude that

$$a_k(\alpha)=\frac{(1-\gamma)p_{k-1}}{\alpha_{n,k}}.$$

Also

$$a_3(\alpha)-a_2^2(\alpha)=\frac{(1-\gamma)}{\alpha_{n,3}}\Big(p_2-\frac{2(1-\gamma)\alpha_{n,3}}{\alpha_{n,2}^2}\frac{p_1^2}{2}\Big). \tag{3.4}$$

Applying lemma 2.1 in (3.3) and (3.4) with the fact $\mu=\frac{2(1-\gamma)\alpha_{n,3}}{\alpha_{n,2}^2}$ and $\Big|p_2-\frac{2(1-\gamma)\alpha_{n,3}}{\alpha_{n,2}^2}\frac{p_1^2}{2}\Big|\le 2$ then, we obtain the inequality in Theorem 3.1 and this completes the proof.

Theorem 3.2: Let $f^\alpha\in T_n^\alpha(\gamma)$. If $\alpha>0$, $0\le\gamma<1$ and $n\in N_0=N\cup\{0\}$

$$\big|a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)\big|\le\frac{4\alpha^{2n}(1-\gamma)^2}{(\alpha+2)^{2n}}. \tag{3.5}$$

Proof: Let $f^\alpha\in T_n^\alpha(\gamma)$. Then, from (3.3), we have that

$$a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)=\frac{(1-\gamma)^2}{\alpha_{n,2}\alpha_{n,3}^2\alpha_{n,4}}\Big[\alpha_{n,3}^2p_1p_3-\alpha_{n,2}\alpha_{n,4}p_2^2\Big]. \tag{3.6}$$

Employing lemma 2.2, alongside (2.5) and (2.8), we obtain

$$a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)$$

$$=\frac{(1-\gamma)^2}{4\alpha_{n,2}\alpha_{n,3}^2\alpha_{n,4}}\left\{\alpha_{n,3}^2\big[p_1^4+2(4-p_1^2)p_1^2x-(4-p_1^2)p_1^2x^2+2(4-p_1^2)\big(1-|x|^2\big)p_1z\big]-M\right\},$$

where

$$M=\alpha_{n,2}\alpha_{n,4}\big[p_1^4+2(4-p_1^2)p_1^2x+(4-p_1^2)x^2\big]$$

Supposing that $p=p_1$ and $p\in[0,2]$. Applying triangle inequality with $\delta=|x|$, then

$$\big|a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)\big|\le\frac{(1-\gamma)^2}{4\alpha_{n,2}\alpha_{n,3}^2\alpha_{n,4}}\{(\alpha_{n,3}^2-\alpha_{n,2}\alpha_{n,4})\big(p^4+2(4-p^2)p^2\delta\big)$$

$$+(4-p^2)\big[\alpha_{n,3}^2p^2+\alpha_{n,2}\alpha_{n,4}(4-p^2)\big]\delta^2+2\alpha_{n,3}^2(4-p^2)p-2\alpha_{n,3}^2(4-p^2)p\delta^2\}.$$

If we let

$$F(p,\delta)=(\alpha_{n,3}^2-\alpha_{n,2}\alpha_{n,4})p^4+2p\alpha_{n,3}^2(4-p^2)+2p^2(\alpha_{n,3}^2-\alpha_{n,2}\alpha_{n,4})(4-p^2)\delta$$
$$+\{(4-p^2)\big[\alpha_{n,3}^2p^2+\alpha_{n,2}\alpha_{n,4}(4-p^2)\big]-2p\alpha_{n,3}^2(4-p^2)\}\delta^2. \tag{3.7}$$

Then

$$\big|a_2(\alpha)a_4(\alpha)-a_3^2(\alpha)\big|\le\frac{(1-\gamma)^2}{4\alpha_{n,2}\alpha_{n,3}^2\alpha_{n,4}}F(p,\delta)\ \ \textit{with}\ \ \delta=|x|\le 1. \tag{3.8}$$

Assuming the upper bound for the equation (3.8) occurs at an interior point of the set $\{(p,\delta): p\in[0,2] \;\; and \;\; \delta\in[0,1]\}$. By differentiating equation (3.7) with respect to δ, we obtain

$$F'(p,\delta) = 2p^2(\alpha_{n,3}^2 - \alpha_{n,2}\alpha_{n,4})(4-p^2) + 2\{(4-p^2)\left[\alpha_{n,3}^2 p^2 + \alpha_{n,2}\alpha_{n,4}(4-p^2)\right] - 2p\alpha_{n,3}^2(4-p^2)\}\delta.$$

For $\delta\in(0,1)$ and fixed $p\in(0,2)$, it is observed that $F'(p,\delta)>0$. Therefore $F'(p,\delta)$ is an increasing function of δ which has contradicted our assumption that the maximum value of it occurs at an interior point of the set $\{(p,\delta): p\in[0,2] \;\; and \;\; \delta\in[0,1]\}$. In like manner, for fixed $p\in[0,2]$, we have that

$$\max_{0\le\delta\le1} F(p,\delta) = F(p,1) = G(p).$$

Consequently,

$$\left|a_2(\alpha)a_4(\alpha) - a_3^2(\alpha)\right| \le \frac{(1-\gamma)^2}{4\alpha_{n,2}\alpha_{n,3}^2\alpha_{n,4}} G(p). \tag{3.9}$$

Now, since

$$G(p) = 2(\alpha_{n,2}\alpha_{n,4} - \alpha_{n,3}^2)p^4 - 4(4\alpha_{n,2}\alpha_{n,4}(\alpha) - 3\alpha_{n,3}^2)p^2 + 16\alpha_{n,2}\alpha_{n,4}.$$

Then,

$$G'(p) = 8(\alpha_{n,2}\alpha_{n,4} - \alpha_{n,3}^2)p^3 - 8(4\alpha_{n,2}\alpha_{n,4}(\alpha) - 3\alpha_{n,3}^2)p$$

and

$$G''(p) = 24(a_2(\alpha)a_4(\alpha) - a_{n,3}^2(\alpha))p^2 - 8(4\alpha_{n,2}\alpha_{n,4}(\alpha) - 3\alpha_{n,3}^2).$$

If we set $G'(p)=0$, then

$$8p\left[(\alpha_{n,2}\alpha_{n,4} - \alpha_{n,3}^2)p^2 - (4\alpha_{n,2}\alpha_{n,4}(\alpha) - 3\alpha_{n,3}^2)\right] = 0.$$

Obviously, $G(p)$ attains its maximum value at the point $p=0$ such that $\max G(p) = G(0)$. Therefore,

$$\left|a_2(\alpha)a_4(\alpha) - a_3^2(\alpha)\right| \le \frac{4\alpha^{2n}(1-\gamma)^2}{(\alpha+2)^{2n}}$$

and this clearly ends the proof.

Corollary 3.3: Let $f^\alpha \in T_n^1(\gamma)$. Then

$$\left|a_2(\alpha)a_4(\alpha) - a_3^2(\alpha)\right| \le \frac{4(1-\gamma)^2}{3^{2n}}.$$

Corollary 3.4: Let $f^\alpha \in T_0^1(\gamma)$. Then

$$\left|a_2(\alpha)a_4(\alpha) - a_3^2(\alpha)\right| \le 4(1-\gamma)^2.$$

Corollary 3.5: Let $f^\alpha \in T_n^1(0)$. Then

$$\left|a_2(\alpha)a_4(\alpha) - a_3^2(\alpha)\right| \le \frac{4}{3^{2n}}.$$

Corollary 3.6: Let $f^\alpha \in T_1^1(0)$. Then

$$\left|a_2(\alpha)a_4(\alpha) - a_3^2(\alpha)\right| \le \frac{4}{9}.$$

Theorem 3.7: Let $f^\alpha \in T_n^\alpha(\gamma)$. If $\alpha > 0$, $0 \leq \gamma < 1$ and $n \in N_0 = N \cup \{0\}$

$$\left|a_2(\alpha)a_3(\alpha) - a_4(\alpha)\right| \leq \frac{2\sqrt{6}}{9}\frac{(1-\gamma)[3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}]}{\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}}\left(\sqrt{\frac{3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}}{\alpha_{n,2}\alpha_{n,3}}}\right). \tag{3.10}$$

Proof: Using (3.3), one can write that

$$a_2(\alpha)a_3(\alpha) - a_4(\alpha) = \frac{(1-\gamma)}{\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}}\big[(1-\gamma)\alpha_{n,4}p_1p_2 - \alpha_{n,2}\alpha_{n,3}p_3\big]. \tag{3.11}$$

Applying lemma 2.2, then

$$\left|a_2(\alpha)a_3(\alpha) - a_4(\alpha)\right| \leq \frac{(1-\gamma)}{4\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}}\left|2\alpha_{n,4}(1-\gamma)\big[p_1^2 + (4-p_1^2)x\big]p_1 - E\right|.$$

where

$$E = \alpha_{n,2}\alpha_{n,3}\big[p_1^3 + 2(4-p_1^2)p_1x - (4-p_1^2)p_1x^2 + 2(4-p_1^2)(1-|x|^2)z\big].$$

If $p_1 = p \in [0,2]$ and applying triangle inequality with $\delta = |x| \leq 1$, then

$$\left|a_2(\alpha)a_3(\alpha) - a_4(\alpha)\right| \leq \frac{(1-\gamma)}{4\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}}F(p,\delta) \tag{3.12}$$

where

$$F(p,\delta) = (\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4})p^3 + 2\alpha_{n,2}\alpha_{n,3}(4-p^2) + 2p(4-p^2)(\alpha_{n,2}\alpha_{n,3} + (1-\gamma)\alpha_{n,4})\delta$$
$$+ \alpha_{n,2}\alpha_{n,3}(4-p^2)(p-2)\delta^2. \tag{3.13}$$

Now assuming that the upper bound for (3.12) occurs at an interior point of the set $\{(p,\delta) : p \in [0,2] \ \ and \ \ \delta \in [0,1]\}$. Differentiating (3.13) partially with respect to δ, then we obtain

$$F'(p,\delta) = 2\alpha_{n,2}\alpha_{n,3}(4-p^2)(p-2)\delta + 2p(\alpha_{n,2}\alpha_{n,3} + (1-\gamma)\alpha_{n,4})(4-p^2).$$

For $\delta \in (0,1)$ and for fixed $p \in (0,2)$, we observe that $F'(p,\delta) > 0$. As a result of this, $F'(p,\delta)$ is an increasing function of δ which contradicts our assumption that the maximum value of it occurs at an interior point of the set $\{(p,\delta) : p \in [0,2] \ \ and \ \ \delta \in [0,1]\}$. Also for fixed $p \in [0,2]$, we have that

$$\underset{0\leq\delta\leq1}{F(p,\delta)} = F(p,1).$$

Suppose that we let $F(p,1) = G(p)$. Then

$$G(p) = -2\alpha_{n,2}\alpha_{n,3}p^3 + 4[3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}]p,$$

$$G'(p) = -6\alpha_{n,2}\alpha_{n,3}p^2 + 4[3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}],$$

and

$$G''(p) = -12\alpha_{n,2}\alpha_{n,3}p.$$

Setting $G'(p) = 0$. Then

$$p = \pm\frac{\sqrt{6}}{3}\sqrt{\frac{3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}}{\alpha_{n,2}\alpha_{n,3}}}.$$

But $p \in [0,2]$. Therefore $G(p)$ attains its maximum value at the point

$$p = \frac{\sqrt{6}}{3}\sqrt{\frac{3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}}{\alpha_{n,2}\alpha_{n,3}}}.$$

Hence,

$$G(p) = \frac{8\sqrt{6}}{9}\big[3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}\big]\left(\sqrt{\frac{3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}}{\alpha_{n,2}\alpha_{n,3}}}\right).$$

This implies that the upper bound of (40) which correspond to $\delta = 1$ and $p = \frac{\sqrt{6}}{3}\sqrt{\frac{3\alpha_{n,2}\alpha_{n,3}+2(1-\gamma)\alpha_{n,4}}{\alpha_{n,2}\alpha_{n,3}}}$ is given by

$$\big|a_2(\alpha)a_3(\alpha) - a_4(\alpha)\big| \le \frac{2\sqrt{6}}{9}\frac{(1-\gamma)\big[3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}\big]}{\alpha_{n,2}\alpha_{n,3}\alpha_{n,4}}\left(\sqrt{\frac{3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}}{\alpha_{n,2}\alpha_{n,3}}}\right)$$

where $\alpha_{n,k} = \left(\frac{\alpha+k-1}{\alpha}\right)^n$ and $0 \le \gamma < 1$. This obviously end the proof of theorem 3.7.

Corollary 3.8: Let $f^\alpha \in T_0^\alpha(\gamma)$. Then

$$\big|a_2(\alpha)a_3(\alpha) - a_4(\alpha)\big| \le \frac{2\sqrt{6}}{9}(1-\gamma)\big[3 + 2(1-\gamma)\big]\sqrt{\big[3 + 2(1-\gamma)\big]}.$$

Corollary 3.9: Let $f^\alpha \in T_0^\alpha(0)$. Then

$$\big|a_2(\alpha)a_3(\alpha) - a_4(\alpha)\big| \le \frac{10\sqrt{30}}{9}.$$

Theorem 3.10: Let $f^\alpha \in T_n^\alpha(\gamma)$. Then

$$\big|H_3(1)\big| \le \frac{4(1-\gamma)^2}{\alpha_{n,3}}Q$$

where

$$Q = \left\{\frac{\alpha_{n,3}^2 + 2\alpha_{n,5}(1-\gamma)}{\alpha_{n,3}^2\alpha_{n,5}} + \frac{\sqrt{6}[3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}]}{9\alpha_{n,2}\alpha_{n,4}^2}\left(\sqrt{\frac{3\alpha_{n,2}\alpha_{n,3} + 2(1-\gamma)\alpha_{n,4}}{\alpha_{n,2}\alpha_{n,3}}}\right)\right\}.$$

for $0 \le \gamma < 1$, $\alpha > 0$, $n \in N_0$ and $z \in U$.

Proof: It follows from (20) that

$$|H_3(1)| \le |a_3(\alpha)||a_2(\alpha)a_4(\alpha) - a_3^2(\alpha)| + |a_4(\alpha)||a_4(\alpha) - a_2(\alpha)a_3(\alpha)| + |a_5(\alpha)||a_3(\alpha) - a_2^2(\alpha)|.$$

Now using Theorems 3.1, 3.2 and 3.7, we obtain the desired result.

Corollary 3.11: Let $f^\alpha \in T_n^\alpha(0)$. Then

$$\big|H_3(1)\big| \le \frac{4}{\alpha_{n,3}}\left\{\frac{\alpha_{n,3}^3 + 2\alpha_{n,5}}{\alpha_{n,3}^2\alpha_{n,5}} + \frac{\sqrt{6}[3\alpha_{n,2}\alpha_{n,3} + 2\alpha_{n,4}]}{9\alpha_{n,2}\alpha_{n,4}^2}\left(\sqrt{\frac{3\alpha_{n,2}\alpha_{n,3} + 2\alpha_{n,4}}{\alpha_{n,2}\alpha_{n,3}}}\right)\right\}$$

for $\alpha > 0$, $n \in N_0$ and $z \in U$.
Corollary 3.12: Let $f^\alpha \in T_0^1(0)$. Then

$$\left|H_3(1)\right| \leq \frac{4}{9}\left(27 + 5\sqrt{30}\right).$$

4. Conclusion

In the present study, the author has successfully investigated and studied a class of Bazilevic functions of order gamma. Bounds on the first few Taylor-Maclaurin coefficients for functions belonging to the said class of Bazilevic functions were obtained. The author also considered the estimate of both the second and third Hankel determinants for the class of Bazilevic functions of order gamma while some consequences of the results obtained were considered as corollaries. It is pertinent to say that Hankel determinants play a crucial role in the study of singularities and in the theory of power series with integral coefficients.

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