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THREE-STEP SECOND DERIVATIVE HYBRID BLOCK BACKWARD DIFFERENTIATION FORMULAE FOR SOLVING SYSTEM OF DIFFERENTIAL ALGEBRAIC EQUATIONS (DAES)

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Abstract. This paper presents a new Second Derivative Hybrid Block Backward Differentiation Formulae (SDHBBDF) for solving series of engineering problems that are represented by some sets of Differential-Algebraic Equations (DAEs). The main and complimentary methods, were developed by collocation and interpolation techniques that are combined as a set of block equations. The analysis of the method showed that it is consistent, convergent, and satisfied the L-stability condition. The SDHBBDF was implemented on some physical problems of DAEs with broad intervals and the numerical results demonstrated that the method is accurate, efficient and suitable for solving DAEs. More so, the method compared favorably well with some excellent methods in the literature.

1. Introduction

A differential algebraic equation is a set of differential equations with algebraic constraints that can be written as

$$\begin{aligned} y'(t) &= f_1(y(t), z(t)), \quad y(t_0) = y_0 \\ f_z(y(t), z(t)) &= 0, \quad z(t_0) = z_0 \end{aligned} \tag{1.1}$$

1.1. **Literature Review.** Differential algebraic equations are useful for modeling a wide range of engineering problems (1.1). Numerous scientific and engineering applications such as Circuit Analysis (CA), Computer-Aided Design and Real-Time Simulation of Power Systems (CADRTSPS), Chemical Process Simulation (CPS), and Optimal Control (OC) (Pandya, 1983). Constrained Multibody Systems (MS) [9], Vehicle System Dynamics [22], Space Shuttle Simulation, and

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Incompressible Fluids Dynamics [10], Linear Descriptor Dynamic Control System (LDDCS), Linear-Time Invariant Descriptor Systems (LTIDS), Control Model Linear Mechanical System (CMLMS) and Electric Circuit (EC) which contain certain typical electrical system components use DAE systems.
According to [3, 4] the behavior of the capacitor and inductor is described by the differential equations via a non-classical variational approach to solve the DAE index-2.
Numerous numerical techniques have been proposed for DAEs, including the Adomian Decomposition Method (ADM) for DAEs [17], Implicit Lie-Group Method [8], Sequential Regularization Methods [18] and [19], Pade Series Approximation Method [6, 7, 2], and Implicit Lie-Group Method [13]; [11] [14], [17], [11], [1], and Variational Formulation Technique [4, 3] have been developed for the solution of DAEs. These techniques are only effective for low-index situations and frequently call for a unique problem structure. Although these methods can solve a number of significant applications, the disadvantage is their high computing cost, which might result in non-physical solutions. Backward Differentiation Formula (BDF) [21] [20] presented more all-encompassing strategies.
In this research, we take into account physical model problems like the restricted motion of a particle to a circular track, the linear circuit of the modified modal analysis that directly leads to the system of DAEs, and the mechanical control problem to choose an appropriate controller orbit of the mechanical system. Also a descriptor index-2 control model of linear mechanical system. It provides hybrid methods based on BDF approach assembled in block form to solve not only lower index DAEs but efficiently provide the solution for higher index ones.

2. Derivation of the Method

The new Second Derivatives Hybrid Block Backward Differentiation Formula (SDHBBDF) on the interval $[x_n,\ x_n+3h]$, where h is the step-length, is developed in this section.

Particularly, we assume that local y, denoted by $Y(x)$, approximates the exact solution given by $y(x)$ on the interval $[x_n,\ x_n+3h]$.

$$Y(x) = \sum_{j=0}^{r+s-1} a_j \phi_j(x) \tag{2.1}$$

where the number of interpolation points r and the number of unique collocation points s are respectively chosen to satisfy $r = 2k+5, \quad Ss > 3$, a_j are unknown coefficients to be calculated and $\phi_j(x)$ are polynomial basis function of degree $r+s-1$. To derive SDHBBDF $\phi(x_{n+j}) = x_{n+i}^j$, $s = 3$ and $k = 3$. Now, we impose the following restrictions

$$\sum_{j=0}^{13} a_j x_{n+i}^{j} = y_{n+i}, \qquad i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2} \tag{2.2}$$

$$\sum_{j=1}^{13} j\ a_j x_{n+i}^{j-1} = f_{n+i}, \qquad i = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3 \tag{2.3}$$

$$\sum_{j=2}^{13} j(j-1) a_j x_{n+i}^{j-2} = g_{n+i}, \qquad i = 3 \tag{2.4}$$

Assuming that $y_{n+i} = Y\left(x_n + ih\right)$ denotes the numerical approximation to the exact solution, and that $y\left(x_{n+i}\right), f_{n+i} = y'\left(x_n + ih\right)$, $g_{n+i} = y''\left(x_n + ih\right)$ denotes the approximation to $y''(x_{n+i})$, and n is the grid index. Equations (2.2), (2.3), and (2.4) create a system of 2k+8 euation that must be solved in order to determine the coefficient $a_j's$. The values of the $a_j's$ are subsequently substituted into equation (2.1). After some algebraic processing, The approach produces the continuous equation in the form.

$$y(x) = \sum_{j=0}^{5} \alpha_{\frac{j}{2}}(x) y_{n+\frac{j}{2}} + h \sum_{j=0}^{6} \beta_{\frac{j}{2}}(x) f_{n+\frac{j}{2}} + h^2 \gamma_k(x) g_{n+k} \tag{2.5}$$

Where $\alpha_{\frac{j}{2}}(x)$, $\beta_{\frac{j}{2}}(x)$ and $\gamma_k(x)$, are continuous coefficients.

The main discrete Second Derivative Hybrid Backward Differentiation Formula (SDHBDF) is then produced by evaluating equation (2.5) at the point $x = x_n + 3$ to obtain.

$$\begin{aligned} y_{n+3} = &-\frac{450}{48587} h^2 g_{n+3} + \frac{h}{48587} \Big(300 f_n + 101250 f_{n+1} + 202500 f_{n+2} + 8820 f_{n+3} + 12960 f_{n+\frac{1}{2}} \\ &+ 240000 f_{n+\frac{3}{2}} + 64800 f_{n+\frac{5}{2}} \Big) + \frac{71712}{48587} y_{n+\frac{1}{2}} + \frac{160000}{48587} y_{n+\frac{3}{2}} - \frac{203040}{48587} y_{n+\frac{5}{2}} \\ &+ \frac{3040}{48587} y_n + \frac{286875}{48587} y_{n+1} - \frac{270000}{48587} y_{n+2} \end{aligned} \tag{2.6}$$

To obtain the complementary (SDHBDF), equation (2.5) is differentiated twice with respect to x to get

$$y''(x) = \frac{1}{h^2} \left\{ \sum_{i=0}^{5} \alpha''_{\frac{j}{2}}(x) y_{n+\frac{j}{2}} - h \sum_{i=0}^{6} \beta''_{\frac{j}{2}}(x) y_{n+\frac{j}{2}} - h^2 \gamma_k g_{n+k} \right\} \tag{2.7}$$

By evaluating (2.7), at point $x = \left\{ x_{n+\frac{1}{2}}, x_{n+1}, x_{n+\frac{3}{2}}, x_{n+2}, x_{n+\frac{5}{2}} \right\}$ the discrete complementary (SDHBDF)are obtained as

$$
\begin{aligned}
h^2 g_{n+\frac{5}{2}} = &-\frac{590}{48587} h^2 g_{n+3} + \frac{h}{48587}\left(12960 f_n + 101250 f_{n+1} + 202500 f_{n+2} + 8820 f_{n+3}\right. \\
&\left. + 12960 f_{n+\frac{1}{2}} + 240000 f_{n+\frac{3}{2}} + 64800 f_{n+\frac{5}{2}}\right) + \frac{71712}{48587} y_{n+\frac{1}{2}} + \frac{160000}{48587} y_{n+\frac{3}{2}} - \frac{203040}{48587} y_{n+\frac{5}{2}} \\
&+ \frac{3040}{48587} y_n + \frac{286875}{48587} y_{n+1} - \frac{270000}{48587} y_{n+2}
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
h^2 g_{n+2} = &\frac{216}{242935} h^2 g_{n+3} + \frac{1}{109320750}\left(26121120 f_{n+\frac{1}{2}} + 82923200 f_{n+\frac{3}{2}} - 5596848 f_{n+\frac{5}{2}}\right. \\
&\left. + 8184 f_n + 240511500 f_{n+1} + 553903500 f_{n+2} + 933380 f_{n+3}\right) h + \frac{201008}{145761} y_{n+\frac{1}{2}} \\
&+ \frac{16357376}{437283} y_{n+\frac{3}{2}} + \frac{75871504}{18220125} y_{n+\frac{5}{2}} + \frac{5661101}{109320750} y_n + \frac{1054360}{145761} y_{n+1} - \frac{7978027}{291522} y_{n+2}
\end{aligned}
\tag{2.9}
$$

$$
\begin{aligned}
h^2 g_{n+\frac{3}{2}} = &-\frac{157}{485870} h^2 g_{n+3} + \frac{1}{72880500}\left(13796730 f_{n+\frac{1}{2}} + 12560000 f_{n+\frac{3}{2}} - 9727290 f_{n+\frac{5}{2}}\right. \\
&\left. + 258635 f_n + 169279875 f_{n+1} - 153383625 f_{n+2} + 218645 f_{n+3}\right) h + \frac{56142}{48587} y_{n+\frac{1}{2}} \\
&- \frac{3157604}{145761} y_{n+\frac{3}{2}} + \frac{5819526}{6073375} y_{n+\frac{5}{2}} + \frac{2701813}{72880500} y_n + \frac{483732}{48587} y_{n+1} + \frac{1857213}{194348} y_{n+2}
\end{aligned}
\tag{2.10}
$$

$$
\begin{aligned}
h^2 g_{n+1} = &-\frac{103}{242935} h^2 g_{n+2} + \frac{1}{54660375}\left(35650080 f_{n+\frac{1}{2}} - 376336000 f_{n+\frac{3}{2}} - 8323680 f_{n+\frac{5}{2}}\right. \\
&\left. + 501320 f_n - 249867375 f_{n+1} - 98892000 f_{n+2} + 211295 f_{n+3}\right) h \\
&+ \frac{675296}{54660375} y_n - \frac{3871856}{145761} y_{n+1} + \frac{934660}{145761} y_{n+2}
\end{aligned}
\tag{2.11}
$$

$$
\begin{aligned}
h^2 g_{n+\frac{1}{2}} = &-\frac{102}{48587} h^2 g_{n+3} + \frac{1}{4372830}\left(44630004 f_{n+\frac{1}{2}} - 92278000 f_{n+\frac{3}{2}} + 3050910 f_{n+\frac{5}{2}}\right. \\
&\left. + 491990 f_n - 10725525 f_{n+1} - 32309250 f_{n+2} + 8275 f_{n+3}\right) h - \frac{4443325}{97174} y_{n+\frac{1}{2}} \\
&+ \frac{10043800}{437283} y_{n+\frac{3}{2}} + \frac{6830531}{1457610} y_{n+\frac{5}{2}} + \frac{2897641}{2186415} y_n + \frac{1019600}{145761} y_{n+1} + \frac{1153475}{48587} y_{n+2}
\end{aligned}
\tag{2.12}
$$

The block hybrid second derivative backward differentiation formula is created by combining the methods (2.6), (2.8), (2.9), (2.10), (2.11), and (2.12).

3. Order of Accuracy and Stability of SDHBBDF

Using a matrix finite difference equation in block form, the three-step second derivative hybrid block backward differentiation formulas are expressed as

$$A^{(1)}Y_w = A^{(0)}Y_{w-1} + hB^{(1)}f_w + hB^{(0)}f_{w-1} + h^2\left(C^{(1)}g_w\right) \tag{3.1}$$

where

$$y_w = \left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}\right)^T, y_{w-1} = \left(y_{n-\frac{5}{2}}, y_{n-2}, y_{n-\frac{3}{2}}, y_{n-1}, y_{n-\frac{1}{2}}, y_n\right)^T$$

$$f_w = \left(f_{n+\frac{1}{2}}, f_{n+1}, f_{n+\frac{3}{2}}, f_{n+2}, f_{n+\frac{5}{2}}, f_{n+3}\right)^T, f_{w-1} = \left(f_{n-\frac{5}{2}}, f_{n-2}, f_{n-\frac{3}{2}}, f_{n-1}, f_{n-\frac{1}{2}}, f_n\right)^T$$

$$g_w = \left(g_{n+\frac{1}{2}}, g_{n+1}, g_{n+\frac{3}{2}}, g_{n+2}, g_{n+\frac{5}{2}}, g_{n+3}\right)^T, \quad g_{w-1} = \left(g_{n-\frac{5}{2}}, g_{n-2}, g_{n-\frac{3}{2}}, g_{n-1}, g_{n-\frac{1}{2}}, g_n\right)^T$$

$w = 0, 1, 2, 3, \cdots$ and $n = 0, 4, ..., N-4$

Additionally, the matrices $A^{(1)}, A^{(0)}, B^{(1)}, B^{(0)}, C^{(1)}$ are six by six matrices whose matrices are given by the coefficient of (3.1) given as

$$A^{(1)} = \begin{pmatrix} \frac{4443325}{97174} & \frac{1019600}{145761} & \frac{-10043800}{437283} & \frac{-1153475}{48587} & \frac{-6830531}{457610} & 0 \\ \frac{-675296}{145761} & \frac{3871856}{145761} & \frac{-6285056}{437283} & \frac{934660}{145761} & \frac{1905884}{18220125} & 0 \\ \frac{-56142}{48587} & \frac{-483732}{48587} & \frac{3157604}{145761} & \frac{-1857213}{194348} & \frac{-5819526}{6073375} & 0 \\ \frac{-201008}{145761} & \frac{-1054360}{145761} & \frac{-6357376}{437282} & \frac{7978027}{219522} & \frac{-78571504}{18220125} & 0 \\ \frac{-2206375}{291522} & \frac{-1590800}{48587} & \frac{-11605400}{437283} & \frac{-2276675}{145761} & \frac{-125206261}{2429350} & 0 \\ \frac{-71712}{48587} & \frac{-286875}{48587} & \frac{-160000}{48587} & \frac{270000}{48587} & \frac{203040}{48587} & 1 \end{pmatrix}$$

$$B^{(1)} = \begin{pmatrix} \frac{7438334}{728805} & \frac{1191725}{48587} & \frac{9227800}{437283} & \frac{1076975}{145761} & \frac{33899}{48587} & \frac{5911}{312345} \\ \frac{-792224}{1214675} & \frac{666313}{145761} & \frac{3010688}{437283} & \frac{87904}{48587} & \frac{554912}{3644025} & \frac{6037}{1561725} \\ \frac{-459891}{2429350} & \frac{-451413}{194348} & \frac{-251201}{145761} & \frac{409023}{194348} & \frac{324243}{2429350} & \frac{-6247}{2082300} \\ \frac{-870704}{145761} & \frac{-106894}{48587} & \frac{-3316928}{437283} & \frac{-738538}{145761} & \frac{-621872}{1214675} & \frac{-1334}{1561725} \\ \frac{-65579}{48587} & \frac{-1612925}{145761} & \frac{-12549400}{437283} & \frac{-1480175}{48587} & \frac{-8756798}{728805} & \frac{7927}{62469} \\ \frac{-12960}{48587} & \frac{-101250}{48587} & \frac{-240000}{48587} & \frac{-202500}{48587} & \frac{-64800}{48587} & \frac{-1260}{6941} \end{pmatrix}$$

$$B^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-49199}{437283} \\ 0 & 0 & 0 & 0 & 0 & \frac{-100264}{10932075} \\ 0 & 0 & 0 & 0 & 0 & \frac{-51727}{14576100} \\ 0 & 0 & 0 & 0 & 0 & \frac{-55067}{10932075} \\ 0 & 0 & 0 & 0 & 0 & \frac{-66287}{2186415} \\ 0 & 0 & 0 & 0 & 0 & \frac{-300}{48587} \end{pmatrix} \qquad C^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \frac{102}{48587} \\ 0 & 1 & 0 & 0 & 0 & \frac{-103}{242935} \\ 0 & 0 & 1 & 0 & 0 & \frac{157}{485870} \\ 0 & 0 & 0 & 1 & 0 & \frac{216}{242935} \\ 0 & 0 & 0 & 0 & 1 & \frac{590}{48587} \\ 0 & 0 & 0 & 0 & 0 & \frac{450}{48587} \end{pmatrix}$$

$$A^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \frac{-2897641}{2186415} \\ 0 & 0 & 0 & 0 & 0 & \frac{-5403956}{54660375} \\ 0 & 0 & 0 & 0 & 0 & \frac{-2701813}{72880500} \\ 0 & 0 & 0 & 0 & 0 & \frac{-5661101}{109320750} \\ 0 & 0 & 0 & 0 & 0 & \frac{-3374737}{109320750} \\ 0 & 0 & 0 & 0 & 0 & \frac{-3040}{48587} \end{pmatrix}$$

3.1. **Local Truncation Error.**

Theorem 3.1. The SDHBBDF has a local truncation error (LTE) $= C_{14}h^{14}y^{14}(x_n) + O(h^{15})$.

Proof. Supposed $y(x_n)$ is an adequately differentiable function, and consider the Taylor series expansion of $y\left(x_n + \frac{jh}{2}\right), j = 0, 1, , 5, \quad y'\left(x_n + \frac{j}{2}h\right), \quad j = 0, 1, 6$ and $y''(x_n{+}3h)$. we assume that $y_n + \frac{j}{2} = y\left(x_n + \frac{jh}{2}\right), f\left(x_n + \frac{j}{2}\right) = y'\left(x_n + \frac{j}{2}\right) = y'\left(x_n + \frac{jh}{2}\right)$ and $g\left(x_{n+2}\right) = y''\left(x_{n+k}\right) = y''\left(x_n + kh\right)$ and substitute the coefficients $\alpha_{\frac{j}{2}}, \quad j = 0, 1, \cdots, 5, \quad \beta_{\frac{j}{2}}, \quad j = 0, 1, \cdots, 6$ and k, $k = 3$ into the equivalent expression in (2.6); after simplification, we discover the linear difference operator to represent the local truncation error associated with the SDHBBDF as

$$L\left\{y(x_n); h\right\} = y\left(x_n + \frac{j}{2}h\right) - \sum_{j=0}^{5} a_j 2 y_n + j2 - h\sum_{j=0}^{6} \beta_{\frac{j}{2}} f_{n+\frac{j}{2}} - h^2\gamma_k g_{n+k} \quad (3.2)$$

$$LTE = C_{14}h^{14}y^{14}(x_n) + O(h^{15})$$

Where $\frac{j}{2}h, \quad j = 6; \quad \alpha_{\frac{j}{2}} \quad j = 0, 1, , 5; j2, \quad j = 0, 1, , 6$ and γ_k, are constant coefficients, $k = 3$. If $y(x)$ is sufficiently differentiable, then we can write the term in (3.2) as a Taylor series expression of

$$y\left(x_n + \frac{j}{2}h\right), f\left(x_n + \frac{j}{2}\right) = y'\left(x_n + \frac{jh}{2}\right), g\left(x_{n+2h}\right) = y''\left(x_{n+3h}\right)$$

$$\begin{aligned} y\left(x_{n+\frac{j}{2}}\right) &= \sum_{j=0}^{\infty} h^2 \frac{\left\{\frac{jh}{2}\right\}^p}{P!} y^p(x_n) \\ y'\left(x_{n+\frac{j}{2}}\right) &= \sum_{j=0}^{\infty} h^2 \frac{\left\{\frac{jh}{2}\right\}^p}{P!} y^{p+1}(x_n) \\ y''\left(x_{n+3j}\right) &= \sum_{j=0}^{\infty} h^2 \frac{\left\{3jh\right\}^p}{P!} y^{p+2}(x_n) \end{aligned} \quad (3.3)$$

Substituting (3.3) and (3.2), we obtain the equation

$$L\left[y\left(x_n\right) : h\right] = c_0 y(x) + c_1 h y' x + c_2 h^2 y^u(x) + \cdots + c_p h^p y^p(x) +$$

Where the constants c_p, $p = 0, 1, 2, \cdots$ are given as follows

$$c_0 = \sum_{j=0}^{6}(\alpha_{\frac{j}{2}})$$

$$c_1 = \sum_{j=1}^{6}(\alpha_{\frac{j}{2}}) - \sum_{j=0}^{6}(\beta_{\frac{j}{2}})$$

$$c_2 = \sum_{j=1}^{6}(\alpha_{\frac{j}{2}}) - \sum_{j=1}^{6}(\alpha_{\frac{j}{2}}) + \gamma_k$$

$$c_p = \frac{1}{p!}\left[\sum_{j=1}^{6}\left(\frac{j}{2}\right)^p\left(\alpha_{\frac{j}{2}}.\left(\frac{j}{2}\right)^p\right)\right] - \frac{1}{(p-1)!}\left[\sum_{j=1}^{6}\beta_{\frac{j}{2}}.\left(\frac{j}{2}\right)^{p-1}\right] + \frac{1}{(p-2)!}\gamma_k k^{(p)}$$

□

The SDHBBDF in (3.1) has a maximum order accuracy of p if,

$$L\{y(x_n);h\} = c_{p+1}h^{p+1}y^{p+1}(x_n) + O(h^{p+2})$$

And

$$c_0 = c_1 = c_2 = c_p = 0, \text{ and } c_{p+1} \neq 0 \tag{3.4}$$

Therefore, $c_{p+1}$is the error constant and $c_{p+1}h^{p+1}y^{p+1}(x_n)$ the major local truncal at the point x_n. Consequently, the SDHBBDF calculated error constant's value is given as

$$c_p = \left\{ \frac{-542502203}{7355496529920}, \frac{-22116398777623}{99140324428873728}, \frac{983640934949}{77453378460057600}, \frac{133520585749}{22949149173350400}, \frac{221626415977}{38726689230028800}, \frac{2435209734119}{99140324428873728T} \right\}$$

with order $\{13,13,13,13,13,13\}^T$and T is the transpose.

3.2. **Zero Stability.** It is worth noting that zero-stability is concerned with the stability of the difference system (3.1) in the limit $h \to 0$. Thus, as $h \to 0$, (3.1), becomes.

$$A^{(1)}Y_w = A^{(0)}Y_{w-1}$$

Whose first characteristics polynomial $p(R)$ given by $|R_j| \leq 1$, $j = 1,,6$

$$\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] = \frac{287539200}{631}R^5(1-R) \tag{3.5}$$

The SDHBBDF (3.1) is zero stable for $p(R) = 0$ and satisfies $|R_j| < 1$ and for those roots with $|R_j| = 1$, $j = 1, \cdots, 6$ the multiplicity doesn't exceed 1.

3.3. **Consistency and Convergence.** It was observed that the block method (3.1) is consistent because order $P > 1$ and it is zero stable. According to [15], convergence is defined as zero stability plus consistency, hence the method (2.9) converges.

3.4. **Linear Stability.** The stability characteristics of the block method (3.1), are discussed and determined through the use of the test equation of the form:

$$y' = \lambda y \qquad y'' = \lambda^2 y, \qquad \lambda < 0 \tag{3.6}$$

Applying (3.1) on (3.6) yields

$$Y_{\varpi} = N(z)Y_{w-1} \tag{3.7}$$

Where $N(z)$ is the amplification matrix with $z = h\lambda$ given by

$$N(z) = \left(A^{(1)} + zB^{(1)} + z^2c^{(1)}\right)^{-1} . \left(A^{(0)} + B^{(0)} + c^{(0)}\right)$$

The matrix N(z) has eigenvalue $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) = (0,0,0,0,0,\xi_6)$, where the dominant eigenvalue ξ_6 is a rational function of z given by:

$$\xi_6(z) = \frac{4\left(210660000z^{11} + 3358719250z^{10} + 10093800300z^9 - 27805037455z^8 - 128955838245z^7 + .. + 373621248000\right)}{3275601240z^{12} + 19653607440z^{11} + 30055003596z^{10} - 330433086180z^9 - 2620982571494z^8 + \; + 14944899} \tag{3.8}$$

which is the stability function and $z = h\lambda \in C$. The stability domain of the method (or region of absolute stability) S is defined as

$$S = [z \epsilon \mathbb{C} : R(z) \leq 1] \tag{3.9}$$

Specifically, when the left-half complex plane is contained in S, the method is said to be A-stable. (3.8) corresponds to the stability zone S. In Figure 1, the plot depicted in blue is the stability region which encompasses the entire left half complex plane thus the method is said to be $A - stable$ and as in Cash [?] the requirement that $Max_{z\leq 0}|R(z)| \leq 1$, z real and $\lim_{z\to -\infty} R(z) = 0$ is satisfied.Thus it is L-stable.

4. Implementation of the Method (SDBHBDF)

The technique is effectively implemented as a three-step block numerical integrator for: (1.1) obtaining approximations simultaneously for $\left(y_{n+\frac{1}{2}}, y_{n+1}, y_{n+\frac{3}{2}}, y_{n+2}, y_{n+\frac{5}{2}}, y_{n+3}\right)^T$. Without the need for back values or predictors.

Step 1. Choose N for $k = 3$, $h = \frac{b-a}{N}$ the number of blocks $\pi = \frac{N}{3}$ using (10) $n = 0, \omega = 1$ the values $(y_1, y_2, y_3)^T$ are generated simultaneously over the subinterval $[t_0, t_3]$ as y_0 are known from the IVP (1).
Step 2. for $n = 3, \omega = 2$, $(y_3 \ldots, y_6)^T$ are obtained over the subinterval $[t_3, t_6]$ since y_3 is known from the first block
Step 3. The process is continued for $n = 2k, \ldots, N-k$ and $\omega = 3, \ldots, \pi$ to obtain approximate solutions to (1) on sub-intervals $[t_0, t_k], \ldots, [t_{N-k}, t_N]$ N is a positive integer and n the grid index.

5. Numerical Examples

To demonstrate the accuracy of the SDHBBDF, we provide physical examples in this section. MAPLE, 2016, was used to carry out all the calculations with a wide range of closed intervals.

Example 5.1. We considered the nonlinear index-three Hessenberg DAEs system below, which described the restricted motion of a particle to a circular track (see [1]).

$$\begin{aligned} y_1''(t) &= 2y_2(t) - 2y_2^3(t) - 2y_1(t)y_3(t) \\ y_1''(t) &= 2y_1(t) - 2y_1^3(t) - y_2(t).y_3(t) \\ 0 &= y_1^2(t) + y_2^2(t) - 1, \qquad t \geq 0 \end{aligned}$$

DAEs system above is supplied with the following consistent initial conditions

$$y_1(0) = 1, \quad y_1'(0) = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1$$

Exact solution is

$$y_1(t) = \cos(t), \;\; y_2(t) = \sin(t), \;\; v(t) = 1 + \sin(2t)$$

Example 5.2. We consider the linear circuit which the modified nodal analysis leads directly to the system (see [6])

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} c_1 e_1(t) \\ -c_2 e_1(t) + c_2 e_2(t) \end{pmatrix} + \begin{pmatrix} G & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ j(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ v(t) \end{pmatrix}$$

by choosing

$$c_1(t) = 1 + \frac{1}{4}\sin t + \cos t, c_2 = 1, G = 2$$

And the input voltage

$$v(t) = 4\sin t + \frac{1}{4}\sin 2t$$

By Substitution, we obtain the following DAE

$$\left[2 + \frac{1}{4}\sin\sin t + \cos\cos t\right] e_1(t) + \left[2 + \frac{1}{4}(\sin\sin t + \cos\cos t)\right] e_1'(t) - e_2'(t) = 0$$

$$-e_1'(t) + e_2'(t) - j(t) = 0$$

$$e_2(t) = 4\sin\sin t + \frac{1}{4}\sin\sin(2t)$$

The above system of DAE yields exact solution

$$e_1(t) = \sin(t) + \cos(t), \quad e_2(t) = (t) + \frac{1}{4}\sin\sin(2t), \quad j(t) = 3\cos t + \frac{1}{2}\cos(2t) + \sin t$$

the reliable initial values

$$e_1(0) = 1, \qquad e_2(0) = 0, \qquad j(0) = \frac{7}{2}$$

Remark 5.3. The graph showed that for (Celik and Bayram, 2005), the exact and numerical solutions did not show a perfect relationship, while the current method showed a perfect relationship based on the diagram presented above

Example 5.4. (Index-2 Linear Time-varying DAE with given $U(t)$ over admissible class). We considered the linear time invariant index-2 semi explicit DAE problem [**?**] and [3] as

$$\begin{pmatrix} y_{11} \\ y_{12}' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{11} \\ y_{12}' \end{pmatrix} + \begin{pmatrix} 0 \\ 1 + 2t \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} U(t)$$

$$0 = (1\ 1) \begin{pmatrix} y_{11} \\ y_{12}' \end{pmatrix} - e^{-t} + U(t), \qquad t \in [0, 1]$$

The exact solution that taken from [**?**] is

$$y_{11}(t) = e^{-t}, \ \ y_{12}(t) = \sin t, \ \ y_{21}(t) = \frac{\cos(t)}{1 + 2t}$$

for a given $U(t) = -sin(t)$

Example 5.5 (The algebraic equation appears as a system of equations). We considered differential and algebraic equations linear time invariant descriptor system as a set of equations (see in [3]).

$$x_1 = A_{11}x_1 + A_{12}x_2 + B_1u(t) + f_1$$
$$0 = A_{21}x_1 + B_2u(t) + f_2$$

where,

$$x_1 = (x_{11}, x_{12})^T, x_2 = (x_{21}, x_{22})^T, A_{11} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f_1 = (f_{11}, f_{12})^T$$

$$f_2 = (f_{21}, f_{22})^T, B_1 = (1,0)^T, B_2 = (-1,0)^T$$

$u(t) \in \Delta v$, where Δv is the class of admissible control? Exact solutions are

$$x_{11}(t) = x^5 + x^2 + 1, \qquad x_{12}(t) = x^4 + x^3 + 2$$
$$x_{21}(t) = x^5 + x^4 + x, \qquad x_{22}(t) = x^2 + x^3 - 2x^5$$

Example 5.6. We considered an index-three system of differential algebraic equations of second order, which can be viewed as a mechanical control problem. See [19] and [8].

$$u_1''(t) = 2u_2(t) + \lambda(t)u_1(t), \qquad u_1(0) = 0$$
$$u_2''(t) = -2u_1(t) + \lambda(t)u_2(t), \qquad u_2(0) = 1$$
$$u_1^2(t) + u_2^2(t) - 1 = 0, \qquad \lambda(0) = 0$$

With exact solution

$$u_1(t) = t^2, \qquad u_2(t) = \cos t^2, \qquad \lambda(t) = -4t^2$$

Example 5.7. We considered evaluated the descriptor index-2 of the linear mechanical system's two-control model as (see [28] and [3])

$$EX' = AX + Bu(t) + f(t)$$

Where,

$$E = \begin{pmatrix} I & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & I & I \\ -K & -D & -J \\ H & G & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ L \\ 0 \end{pmatrix}, \quad X = \begin{pmatrix} Z \\ Z' \\ \mu \end{pmatrix}$$

The representative models are represented by the matrices in the table below. For additional information about this mechanical model, [28] says that all these matrices are known with the proper dimensions. Consequently, the semi-explicit descriptor system can be rewritten as follows using these matrices:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ -2 & 0 & -1 & -1 & 1 \\ 0 & -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} t+1 \\ t \\ 0 \\ 0 \\ t^2+t \end{pmatrix}$$

First, this system needs to be converted into DAEs. The mechanical system will undergo the transition and become a differential algebraic control system as follows:

$$A_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & -2 \\ -1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}$$

$$f_1(t) = \begin{pmatrix} t \\ 0 \\ 0 \\ t+1 \end{pmatrix} \qquad f_2(t) = t2 + t, \quad x_1(t) = \begin{pmatrix} x_{11}(t) \\ x_{12}(t) \\ x_{13}(t) \\ x_{14}(t) \end{pmatrix}, \qquad x_2(t) = x_{21}(t)$$

Which is an index-2 system of exact solution obtained by MAPLE 2016 as

$$x_{11}(t) = -\frac{1}{3}t2 - \frac{5}{9} + \frac{2}{5}e^-t - \frac{\sqrt{6}}{5}\sin(\sqrt{6}t) - \frac{38}{45}\cos(\sqrt{6}t)$$

$$x_{12}(t) = \frac{1}{9}t^2 - \frac{11}{27}t - \frac{19}{81} + \frac{1}{2}e^{-t} + \frac{257}{1134}e^{-3t} - \frac{89\sqrt{6}}{189}\sin(\sqrt{6}t) - \frac{32}{63}\cos(\sqrt{6}t)$$

$$x_{13}(t) = \frac{1}{9}t^2 - \frac{20}{27}t + \frac{62}{81} + \frac{3}{10}e^{-t} + \frac{257}{1134}e^{-3t} + \frac{353\sqrt{6}}{945}\sin(\sqrt{6}t) - \frac{218}{315}\cos(\sqrt{6}t)$$

$$x_{14}(t) = -\frac{1}{6}t^2 - \frac{4}{9} - \frac{2}{5}e^{-t} + 65\sin(\sqrt{6}t) + \frac{38}{45}\cos(\sqrt{6}t)$$

$$x_{15}(t) = -\frac{1}{9}t^2 - \frac{25}{27}t - \frac{62}{81} - \frac{1}{10}e^{-t} - \frac{257}{1134}e^{-3t} + \frac{46\sqrt{6}}{9}45\sin(\sqrt{6}t) + \frac{29}{315}\cos(\sqrt{6})$$

With initial condition

$$x_{11}(0) = -1, \;\; x_{12}(0) = 1, \;\; x_{13}(0) = 0, \;\; x_{14}(0) = 0, \;\; x_{15}0 = -1$$

6. Conclusion

The main objective of this study was to develop a hybrid second derivative block approach for the numerical solution of DAEs with a wide range of near intervals. The numerical solution satisfactorily approximates the exact solution, as shown in figures 17, and clearly distinguishes itself from certain other methods used in the literature. In this study, a few numerical examples were tested with a level of convergence and consistency that other methods could not match.

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Authors Contributions. Contribution of each author

Authors' Conflicts of interest. Authors must declare all potential interests in a Conflicts of interest section, which should explain why the interest may be a conflict. If there are none, the authors could state The author(s) declare(s)

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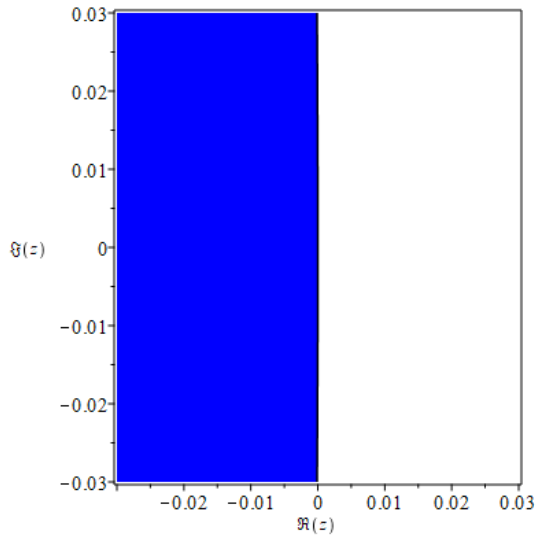

FIGURE 1. Region of Absolute Stability for $k = 3$

TABLE 1. **Comparison between the proposed method and exact solution**

T **method** **solution** **error** $y_1(t)$ **method** **solution** **error** $y_2(t)$ **method** **solution** **error** $y_3(t)$	[c]**Proposed** [c]**Exact** [c]**Abs** [c]**Propose** [c]**Exact** [c]**Abs** [c]**Proposed** [c]**Exact** [c]**Abs**								
0	0	0	0	0	0	0	0	0	0
0.1	0.9998	0.9998	1.650×10^{-28}	0.0410	0.0410	1.0500×10^{-29}	1.0998	1.0998	1.50×10^{-28}
0.2	0.9950	0.9950	9.660×10^{-28}	0.0998	0.0998	1.5800×10^{-27}	1.1987	1.1987	3.85×10^{-27}
0.3	0.9888	0.9888	1.873×10^{-27}	0.1494	0.1494	1.0720×10^{-27}	1.2955	1.2955	5.40×10^{-27}
0.4	0.9801	0.9801	6.581×10^{-27}	0.1987	0.1987	1.3660×10^{-27}	1.3894	1.3894	5.40×10^{-27}
0.5	0.9689	0.9689	1.064×10^{-26}	0.2474	0.2474	6.7290×10^{-27}	1.4794	1.4794	1.54×10^{-26}
0.6	0.9553	0.9553	1.255×10^{-26}	0.2955	0.2955	9.9490×10^{-27}	1.5646	1.5646	3.525×10^{-27}
0.7	0.9394	0.9394	1.639×10^{-26}	0.3429	0.3429	2.0190×10^{-27}	1.6442	1.6442	5.650×10^{-26}
0.8	0.9211	0.9211	2.542×10^{-26}	0.3894	0.3894	1.2200×10^{-26}	1.7174	1.7174	4.899×10^{-26}
0.9	0.9004	0.9004	3.421×10^{-26}	0.4350	0.4350	1.5620×10^{-26}	1.7833	1.7833	1.705×10^{-26}
1.0	0.8776	0.8776	4.283×10^{-26}	0.4794	0.4794	2.7060×10^{-26}	1.8415	1.8415	1.187×10^{-26}

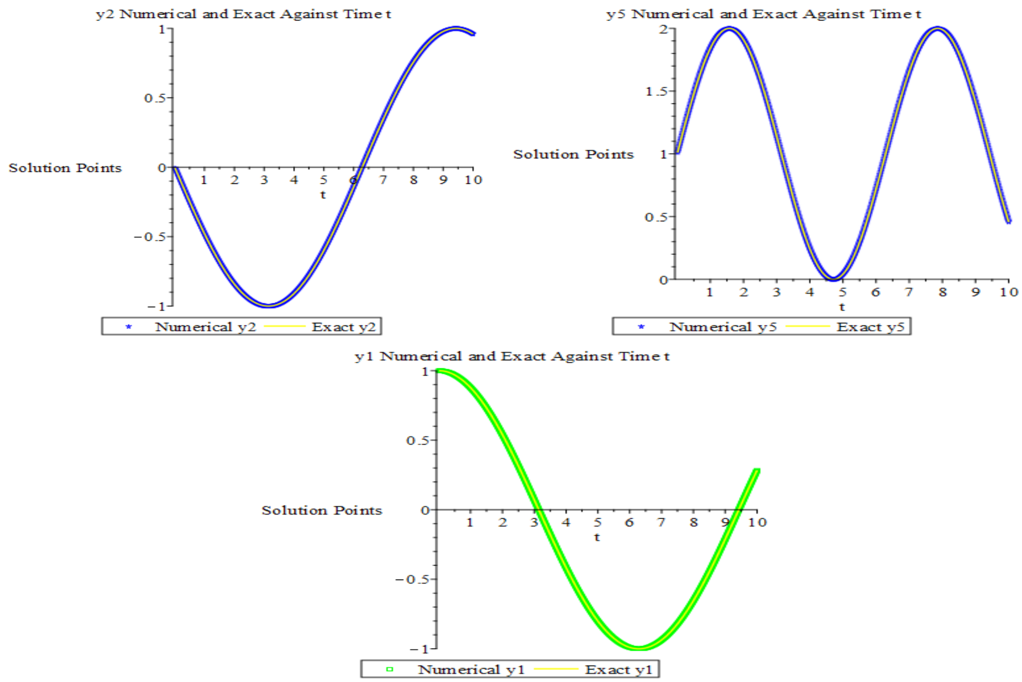

FIGURE 2. Shows that there is a perfect relationship between numerical and exact solutions and the range of interval can be increase to [0, 10] for example 5.1.

TABLE 2. Comparison between Numerical Solution, Exact solution and Absolute Errors In SDHBBDF

t	**Numerical Solution**			**Exact Solution**			**Absolute Error in SDHBBDF**		
T	$Num\ 1$	$Num\ 2$	$Num\ 3$	Ex_1	Ex_2	Ex_3	Err_1	Err_2	Err_3
1.0	1.3570	2.11281	3.3823	1.3570	2.1281	3.8323	2.930×10^{-27}	1.38×10^{-26}	7.160×10^{-27}
2.0	1.3818	3.5932	2.2543	1.3818	3.5932	2.2543	1.590×10^{-27}	4.825×10^{-26}	2.910×10^{-27}
3.0	1.0682	4.0253	0.7147	1.0682	4.0253	0.7147	2.220×10^{-27}	9.145×10^{-26}	4.342×10^{-27}
4.0	0.4132	3.4980	−0.6660	0.4932	3.4480	−0.6660	4.254×10^{-27}	1.3624×10^{-25}	4.562×10^{-22}
5.0	−0.2027	2.1542	−1.6631	−0.2027	2.1542	−1.6631	5.866×10^{-27}	1.694×10^{-25}	7.800×10^{-28}
6.0	−08489	0.4946	−2.3488	−0.4946	0.4946	−2.3488	4.653×10^{-27}	1.838×10^{-25}	2.270×10^{-27}
7.0	−1.2872	−1.2389	−2.7832	−1.2872	−1.2389	−2.7832	1.220×10^{-27}	1.792×10^{-25}	6.610×10^{-17}
8.0	−1.4105	−2.7799	−2.7905	−1.4105	−2.7799	−2.7405	4.700×10^{-27}	1.579×10^{-25}	1.275×10^{-26}
9.0	−1.1883	−3.8071	−2.0655	−1.1883	−3.8071	−2.0655	9.340×10^{-27}	1.304×10^{-25}	8.030×10^{-27}
10.0	−0.6753	−3.9717	−0.5275	−0.6753	−3.9717	−0.5275	1.016×10^{-25}	1.028×10^{-25}	5.7490×10^{-27}

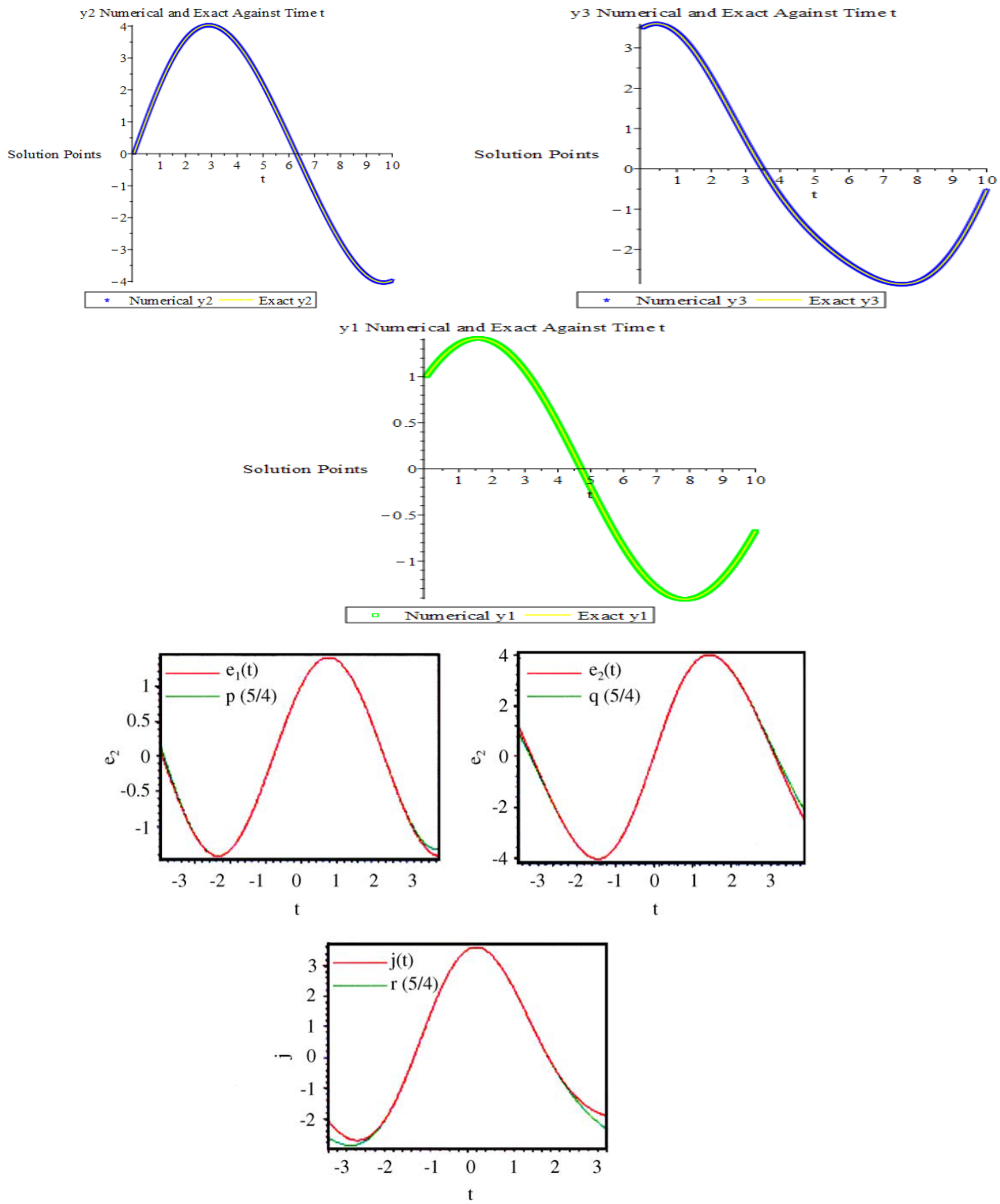

FIGURE 3. Comparison of graph between present method and [6] which shows the perfect relationship between present numerical solution and exact solution from starting point to the end point in which [6] did not.

TABLE 3. **Comparison of Absolute errors $|y_{ij}(t) - y_{ij}(t)|$ between GRBA in [3] and SDHBBDF**

GRBA [3]				**SDHBBDF**		
t	Abs Errx_{11}	Abs Errx_{12}	Abs Errx_{21}	Abs Erry_{11}	Abs Erry_{12}	Abs Erry_{21}
0	0	0	0	0	0	0
0.1	1.496×10^{-07}	1.299×10^{-7}	2.329×10^{-7}	1.622×10^{-26}	6.9343×10^{-25}	1.5702×10^{-23}
0.2	5.443×10^{-07}	3.842×10^{-7}	6.633×10^{-7}	7.205×10^{-26}	1.6612×10^{-24}	1.7159×10^{-23}
0.3	3.639×10^{-07}	3.052×10^{-7}	4.182×10^{-7}	1.744×10^{-25}	2.6886×10^{-24}	$1,5610 \times 10^{-23}$
0.4	3.007×10^{-07}	1.859×10^{-7}	2.703×10^{-7}	3.224×10^{-25}	3.7075×10^{-24}	1.4335×10^{-23}
0.5	6.213×10^{-07}	4.818×10^{-7}	5.516×10^{-7}	5.116×10^{-25}	4.6958×10^{-24}	1.2961×10^{-23}
0.6	2.583×10^{-07}	2.461×10^{-7}	2.293×10^{-7}	7.394×10^{-25}	5.6506×10^{-24}	1.1721×10^{-23}
0.7	3.840×10^{-07}	2.770×10^{-7}	2.754×10^{-7}	1.002×10^{-24}	6.5728×10^{-24}	1.0662×10^{-23}
0.8	5.108×10^{-07}	4.324×10^{-7}	3.628×10^{-7}	1.295×10^{-24}	9.7449×10^{-24}	9.7449×10^{-24}
0.9	1.605×10^{-07}	1.139×10^{-7}	9.804×10^{-7}	1.618×10^{-24}	8.3278×10^{-24}	8.9484×10^{-24}
1.0	3.043×10^{-07}	2.77×10^{-10}	1.93×10^{-10}	1.002×10^{-24}	9.2482×10^{-24}	8.1932×10^{-24}

T	**Numerical Solution**			**Exact Solution**		
0	0	0	0	0	0	0
0.1	0.9950	0.0050	0.9901	0.9950	0.0050	0.9901
0.2	0.9048	0.0998	0.8292	0.9048	0.0998	0.8292
0.3	0.8607	0.1494	0.7606	0.8607	0.1494	0.7606
0.4	0.8187	0.1987	0.7001	0.8187	0.1987	0.7001
0.5	0.7788	0.2474	0.6459	0.7788	0.2474	0.6459
0.6	0.7408	0.2955	0.5971	0.7408	0.2955	0.5971
0.7	0.7047	0.3429	0.5526	0.7047	0.3429	0.5526
0.8	0.6703	0.3894	0.5117	0.6703	0.3894	0.5117
0.9	0.6376	0.4350	0.4739	0.6376	0.4350	0.4739
1.0	0.6065	0.4794	0.4388	0.6065	0.4794	0.4388

TABLE 4. **Comparisons between absolute errors among differential and constraints in the proposed method and existing method** [3]

ZABOON AND ADB ERROR, 2021					**SDHBBDF**			
T	E_{rr1}	E_{rr2}	E_{rr3}	E_{rr4}	E_{rr1}	E_{rr2}	E_{rr3}	E_{rr4}
0.1	7×10^{-13}	0	0	0	0	8×10^{-29}	8×10^{-31}	6.28×10^{-30}
0.2	0	0	0	0	0	8×10^{-29}	2×10^{-30}	4.10×10^{-30}
0.3	1×10^{-13}	0	0	0	0	8×10^{-29}	3×10^{-30}	2.30×10^{-30}
0.4	0	0	0	0	1×10^{-29}	8×10^{-29}	3×10^{-30}	3.80×10^{-30}
0.5	0	0	0	0	1×10^{-29}	2×10^{-28}	2×10^{-30}	1.00×10^{-30}
0.6	0	0	0	0	1×10^{-29}	2×10^{-28}	4×10^{-30}	4.00×10^{-30}
0.7	6×10^{-13}	0	0	0	0	2×10^{-28}	6×10^{-30}	6.00×10^{-30}
0.8	0	0	0	0	1×10^{-29}	2×10^{-28}	6×10^{-30}	1.90×10^{-29}
0.9	1×10^{-13}	0	0	0	1×10^{-29}	2×10^{-28}	3×10^{-30}	4.00×10^{-29}
0.1	0	0	0	0	0	2×10^{-28}	5×10^{-30}	4.30×10^{-29}

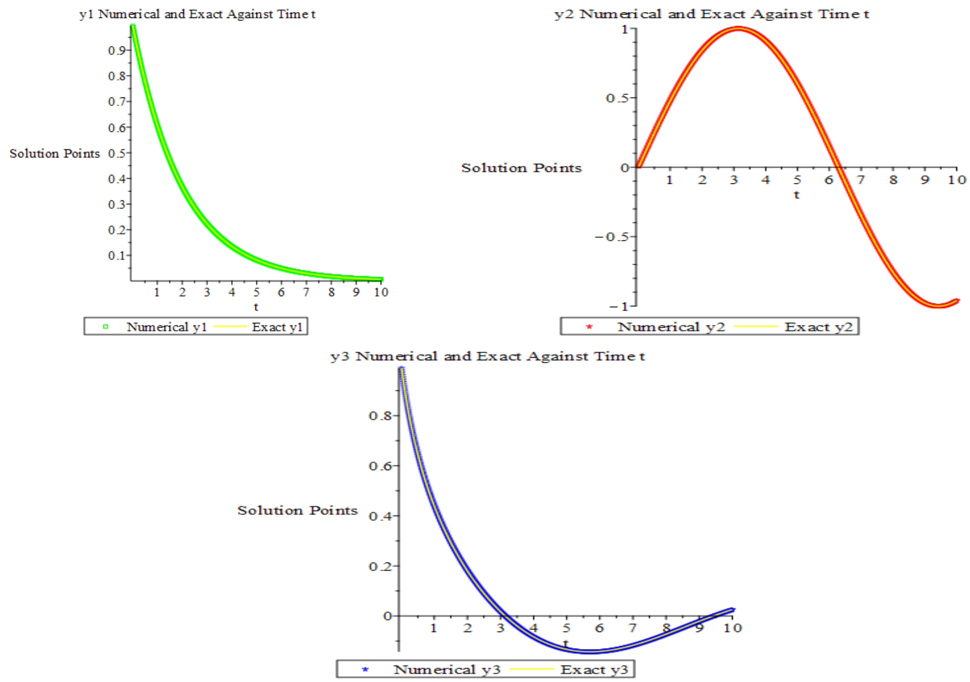

FIGURE 4. Shows the efficient and the level of accuracy between numerical and exact solutions in SDHBBDF and the range of interval can be increased to [0 ,10].

Numerical Solution				Exact Solution			
1.009	2.0000	0.0300	0.0009	1.0009	2.0000	0.0300	0.0009
1.0056	2.0005	0.0750	0.0060	1.0056	2.0005	0.0750	0.0060
1.0169	2.0025	0.1303	0.0190	1.0169	2.0025	0.1303	0.0190
1.0308	2.0063	0.1761	0.0357	1.0308	2.0063	0.1761	0.0357
1.0535	2.0150	0.2334	0.0638	1.0535	2.0150	0.2334	0.0638
1.0772	2.0250	0.2823	0.0933	1.0772	2.0250	0.2823	0.0933
1.1128	2.0478	0.3458	0.1370	1.1128	2.0478	0.3458	0.1370
1.1428	2.0725	0.4022	0.1785	1.1480	2.0725	0.4789	0.2350
1.2498	2.1581	0.5501	0.2844	1.2498	2.1581	0.5501	0.2844

TABLE 5. A Comparison of methods for Example 5.5 with $0 \leq t \leq 1$ and $h = 0.1$

	ILGM (2013)					SDHBBDF				
T	E_{rr1}	E_{rr2}	E_{rr3}	E_{rr4}	E_{rr5}	E_{rr1}	E_{rr2}	E_{rr3}	E_{rr4}	E
0.0	1×10^{-19}	1×10^{-13}	1.0×10^{-13}	1.0×10^{-17}	1×10^{-6}	3×10^{-36}	4.6×10^{-32}	8.1×10^{-26}	9.4×10^{-40}	1.4 ×
0.2	1×10^{-12}	1×10^{-10}	1.0×10^{-10}	1.0×10^{-11}	$1. \times 10^{-5}$	3×10^{-28}	5.6×10^{-27}	2.8×10^{-26}	1.0×10^{-28}	1.9 ×
0.4	1×10^{-11}	1.0×10^{-10}	1.0×10^{-10}	1.0×10^{-10}	$1. \times 10^{-5}$	2×10^{-27}	2.2×10^{-26}	5.6×10^{-26}	1.6×10^{-27}	1.5 ×
0.6	1×10^{-11}	1.0×10^{-9}	1.0×10^{-10}	1.0×10^{-10}	1.0×10^{-5}	8.0×10^{-27}	4.1×10^{-26}	8.3×10^{-26}	8.3×10^{-27}	5.0 ×
0.8	1×10^{-11}	1.0×10^{-9}	1.0×10^{-9}	1.0×10^{-10}	1.0×10^{-5}	2×10^{-26}	8.7×10^{-26}	1.1×10^{-25}	2.6×10^{-26}	1.2 ×
1.0	1×10^{-11}	1.0×10^{-9}	1.0×10^{-9}	1.0×10^{-9}	$1. \times 10^{-5}$	3×10^{-26}	1.3×10^{-25}	1.3×10^{-25}	6.3×10^{-26}	2.4 ×

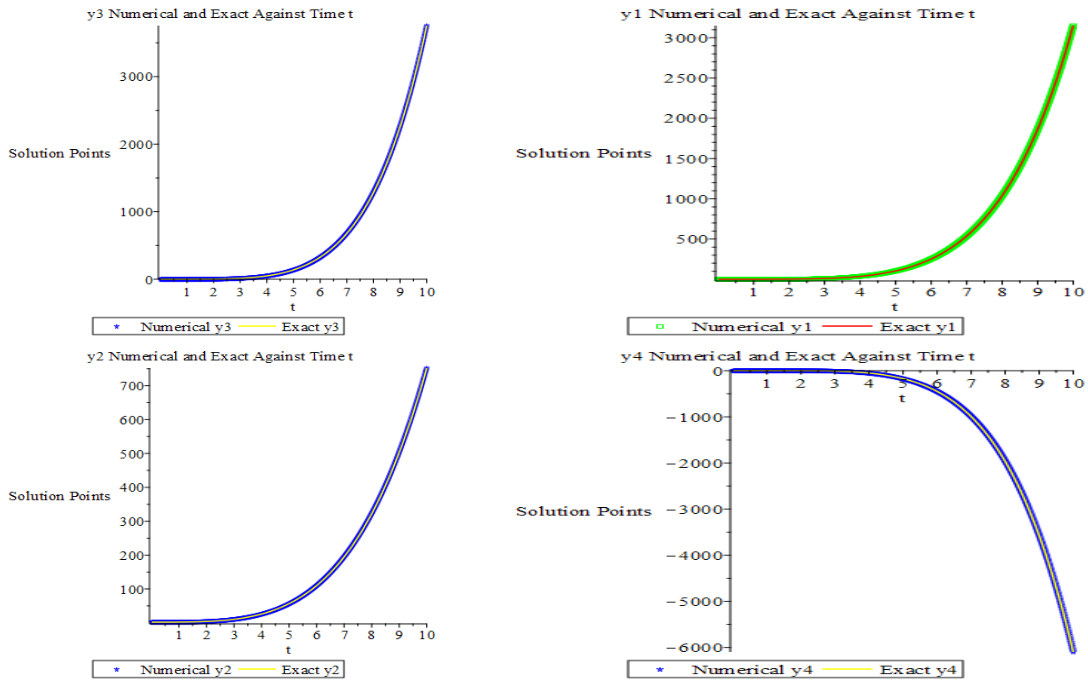

FIGURE 5. Shows that the range of interval can be increase to [0, 10] and perfect relationship between exact and numerical solutions for example 5.5

Numerical Solution					**Exact Solution**				
0	0.0001	1.0000	0	0	0	0.0001	0	0	0
0.0100	0.1995	0.1000	-0.0020	-0.0398	0.0100	0.1995	0.1000	-0.0020	-0.0398
0.0399	0.3992	0.9992	-0.0159	-0.1596	0.0399	0.3992	0.9992	0.0159	-0.1596
0.0897	0.5971	0.9960	-0.0538	- 0.3594	0.0897	0.5971	0.9960	-0.0538	- 0.3594
0.1591	0.7893	0.9873	-0.1272	-0.6392	0.1591	0.7893	0.9873	-0.1272	-0.6392
0.2472	0.9685	0.9690	-0.2470	-0.9990	0.2472	0.9685	0.9690	-0.2470	-0.9990

Matrix	Embody in a mechanical model
$Z \in R^n$	displacement vector
$N \in R^q$	vector of Lagrange multiplier
U	input force
M	inertial matrix
D	damping matrix
K	the stiffness
L	Matrix of force distribution
G, H	coefficient matrices

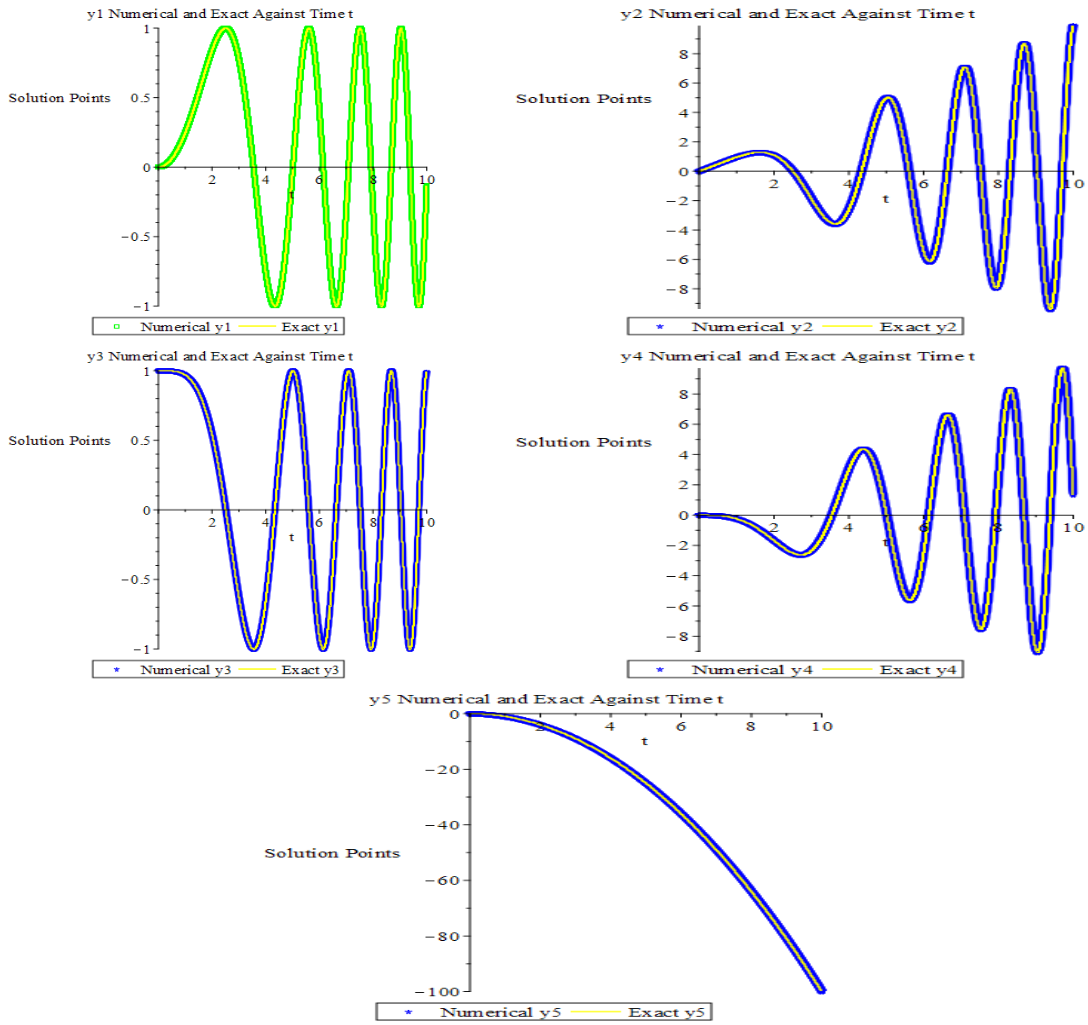

FIGURE 6. Shows that the range of interval can be increase to [0, 10] in SDHBBDF for example 5.6

TABLE 5A. **Comparisons among differential states and equality constraint absolute errors $|x_{ij}(t) - x_{ij}|$ in the proposed method and exact solutions with $h = 10^{-04}$**

	Exact solution in SDHBBDF					Absolute errors in SDHBBDF				
T	Ext_{11}	Ext_{12}	Ext_{13}	Ext_{14}	Ext_{15}	E_{rr11}	E_{rr12}	E_{rr13}	E_{rr14}	E_{rr15}
0.0	-1.0000	0.9999	0.0000	0.0000	-1.0000	4×10^{-15}	4×10^{-15}	4×10^{-15}	3.39×10^{-15}	7.489×10^{-10}
0.1	-1.0485	1.8531	0.0036	0.0473	-1.0030	3×10^{-08}	3×10^{-08}	3×10^{-08}	2.87×10^{-08}	1.120×10^{-06}
0.2	-1.0949	1.7095	0.0138	0.0899	-1.0949	1×10^{-07}	1×10^{-08}	1×10^{-08}	1.14×10^{-07}	3.086×10^{-06}
0.3	-1.1383	0.5710	0.0298	0.1271	-1.0250	3×10^{-07}	2×10^{-07}	2×10^{-07}	2.25×10^{-07}	3.086×10^{-06}
0.4	-1.1794	0.4355	0.0509	0.1594	-1.0430	4×10^{-07}	3×10^{-07}	3×10^{-07}	4.41×10^{-07}	3.953×10^{-06}
0.5	-1.2179	0.3030	0.0767	0.1866	-1.0652	7×10^{-07}	4×10^{-07}	5×10^{-07}	6.77×10^{-07}	4.766×10^{-06}
0.6	-1.2532	0.1758	0.1058	0.2082	-1.0906	1×10^{-07}	6×10^{-07}	7×10^{-07}	9.59×10^{-07}	5.510×10^{-06}
0.7	-1.2863	0.0502	0-.1387	0.2250	-1.1197	1×10^{-06}	7×10^{-07}	8×10^{-07}	1.28×10^{-06}	6.230×10^{-06}
0.8	-1.3164	-0.0709	0.1740	0.2364	-1.1515	2×10^{-06}	8×10^{-07}	1×10^{-07}	1.65×10^{-06}	6.900×10^{-06}
0.9	-1.3439	-0.1888	0.2115	0.2427	-1.1861	2×10^{-06}	9×10^{-07}	1×10^{-06}	2.05×10^{-06}	7.550×10^{-06}
1.0	-1.3689	-0.3041	0.2511	0.2439	-1.2233	3×10^{-06}	9×10^{-07}	1×10^{-06}	2.49×10^{-06}	8.170×10^{-06}

TABLE 5B. **Comparisons among differential states and equality constraint absolute errors $|x_{ij}(t) - x_{ij}|$ in SD-HBBDF and exact solutions with $h = 10^{-04}$**

NUMERICAL SOLUTION IN SDHBBDF				
Num_{11}	Num_{12}	Num_{13}	Num_{14}	Num_{15}
-1.0000	0.9999	0.0000	0.0000	-1.0000
-1.0485	1.8531	0.0036	0.0473	-1.0030
-1.0949	1.7095	0.0138	0.0899	-1.0949
-1.1383	0.5710	0.0298	0.1271	-1.0250
-1.1794	0.4355	0.0509	0.1594	-1.0430
-1.2179	0.3030	0.0767	0.1866	-1.0652
-1.2532	0.1758	0.1058	0.2082	-1.0906
-1.2863	0.0502	0.1387	0.2250	-1.1197
-1.3164	-0.0709	0.1740	0.2364	-1.1515
-1.3439	-0.1888	0.2115	0.2427	-1.1861