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HYERS-ULAM-RASSIAS STABILITY OF CERTAIN PERTURBED NONLINEAR LIENARD TYPE OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper the Hyers-Ulam-Rassias stability of certain perturbed nonlinear second order differential equations of Lienard type was studied, using some new modifications of Grownwall-Bellman-Bihari Integral inequality. Some examples are given to illustrate our results.

1. Introduction

There have been few investigations on the Hyers-Ulam-Rassias stability of nonlinear first order differential equations in the literature, see for instance Qarawani[16], Rus[17]. The stability problem of functional equation started with the question concerning stability of group homomorphism proposed by Ulam [21] in 1940 during a talk before a Mathematical Colloquium at the University of Wincosin, Madison. In 1941, Hyers [10] gave a solution of Ulam's problem for the case of approximate additive mappings in the context of Banach spaces. The result obtained by Hyers opened up research in Hyers-Ulam stability. Rassias [18] in 1978 generalised the theorem of Hyers stability to Hyers-Ulam-Rassias stability.

However, the study on the Hyers-Ulam-Rassias stability of a perturbed nonlinear second order Lienard equation is yet to be considered and this is our major concern in this paper. The following authors employed different approach to study the properties of solutions to a generalised Lineard equation: Kroopnick [12, 13] studied properties of solutions to a generalised Lienard equations with forcing term and also the bounded L^p-solutions, Ogundare and Afuwape[15] studied conditions which guarantee boundedness and stability properties of solutions. Tunc[19, 20] considered new stability and boundedness results for such type equations with multiple deviating arguments.In addition, Olutimo and Adams [1]

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studied the stability and boundedness of solutions of delay differential equations. While Bicer and Tunc[2] considered new theorems for Hyers-Ulam stability of Lienard equation with variable time lags using Banach's contraction principle. Further more, the recent publications on the Hyers-Ulam and Hyers-Ulam-Rassias stability of the second order nonlinear differential equations include:Fakunle[4, 5, 6, 7, 8, 9].

In this paper, we consider Hyers-Ulam-Rassis stability of the following certain perturbed nonlinear Lienard type differential equations using some newly modified forms of Gronwall-Bellman-Bihari inequality:

$$u'' + Y(t,u(t),u'(t))u'(t) + q(t,u(t)) = P(t,u(t),u'(t)) \tag{1.1}$$

and its special case

$$u'' + c(t)f(u(t))u'(t) + a(t)g(u(t)) = P(t,u(t),u'(t)), \tag{1.2}$$

with initial conditions

$$u(t_0) = u'(t_0) = 0, \quad \forall t \geq t_0 \geq 1.$$

Equation (1.1) is considered under the following cases:

i $P(t,u(t),u'(t)) \neq Y(t,u(t),u'(t))$,
where$|Y(t,u(t),u'(t))| \leq \phi(t)g(|u(t)|)h(|u'(t)|)$
and $P(t,u(t),u'(t)) \leq \alpha(t)\omega(|u(t)||u'(t)|^n$,

ii $P(t,u(t),u'(t)) = Y(t,u(t),u'(t))$,
where $|P(t,u(t),u'(t))| = |Y(t,u(t),u'(t))|$
$\leq \phi(t)g(|u(t)|)h(|u'(t)|)$

and equation (1.2) is also considered under the following cases:

iii $P(t,u(t),u'(t)) \neq 0$,
iv $P(t,u(t),u'(t)) = P(t,u(t))$,
v $P(t,u(t),u'(t)) = 0$,

where $c,a,\alpha,\phi \in C(\mathbb{I},\mathbb{R}_+), \omega,\kappa,f,h,g \in C(\mathbb{R}_+,\mathbb{R}_+),$ for $\mathbb{R}_+ = [t_0,\infty)$, $\mathbb{I} = (t_0,\infty), \mathbb{R} = (-\infty,\infty)$ and $Y,P \in C(\mathbb{I}\times\mathbb{R}^2,\mathbb{R}), P(t_0,0,0) = 0, g(0) = 0, Y(t_0,0,0) = 0, P(t_0,0) = 0$.

2. Preliminaries

Definitions, lemmas and theorems are presented here.

Definition 2.1. The equation (1.1) has the Hyers-Ulam-Rassias stablility for any positive function $\varphi(t)$, defined as $\varphi : \mathbb{I} \to \mathbb{R}_+$, if there exist real constant $C_\varphi > 0$ for each solution $u(t) \in C^2(\mathbb{I},\mathbb{R}_+)$ of

$$|u'' + Y(t,u(t),u'(t))u'(t) + q(t,u(t)) - P(t,u(t),u'(t))| \leq \varphi(t), \tag{2.1}$$

there exists a solution $u_0(t) \in C^2(\mathbb{I},\mathbb{R}_+)$ of (1.1) with

$$|u(t) - u_0(t)| \leq C_\varphi\varphi(t), \quad \forall t \in \mathbb{I}.$$

Definition 2.2. Equation (1.2) is said to be Hyers-Ulam-Rassias stable, if there exists a constant $C_\varphi > 0$ such that for $u(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$, satisfying

$$|u'' + c(t)f(u(t))u'(t) + a(t)g(u(t)) - P(t, u(t), u'(t))| \leq \varphi(t) \tag{2.2}$$

$\forall t \in \mathbb{I}$ for a positive function $\varphi(t)$ where $\varphi : \mathbb{I} \to [0, \infty)$, there exists a solution $u_0(t) \in C^2(\mathbb{I}, \mathbb{R}_+)$ of the equation (1.2), such that

$$|u(t) - u_0(t)| \leq C_\varphi \varphi(t), \quad \forall t \in \mathbb{I},$$

where C_φ is called Hyers-Ulam-Rassias constant.

Definition 2.3. [4] A function $\omega : [0, \infty) \to [0, \infty)$ is said to belong to a class Ψ if

i $\omega(u)$ is nondecreasing and continuous for $u \geq 0$,
ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$,
iii there exists a function ϕ, continuous on $[0, \infty)$ with $\omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha \geq 0$.

Theorem 2.4. [3]Let

i $u(t), r(t), h(t) : \mathbb{I} \to \mathbb{R}_+$ and be continuous
ii $f, \omega \in \Psi$

If

$$u(t) \leq K + \int_{t_0}^{t} r(s)f(u(s))ds + \int_{t_0}^{t} h(s)\omega(u(s))ds \tag{2.3}$$

then

$$u(t) \leq \Omega^{-1}\left(\Omega(K) + \int_{t_0}^{t} h(s)\omega\left(F^{-1}\left(F(1) + \int_{t_0}^{s} r(\delta)d\delta\right)\right)ds\right) F^{-1}\left(F(1) + \int_{t_0}^{t} r(s)ds\right), \tag{2.4}$$

where $(0, b) \subset (0, \infty)$, where

$$F(u) = \int_{u_0}^{u} \frac{ds}{\omega(s)}, \quad 0 < u_0 \leq u \tag{2.5}$$

and

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{f(t)}, \quad 0 < u_0 < u, \tag{2.6}$$

F^{-1}, Ω^{-1} are the inverses of F, Ω respectively and t is in the subinterval $(0, b) \in \mathbb{R}_+$ so that

$$F(1) + \int_{t_0}^{t} r(s)ds \in Dom(F^{-1})$$

and

$$\left(\Omega(K) + \int_{t_0}^{t} h(s)\omega\left(F^{-1}\left(F(1) + \int_{t_0}^{s} r(\delta)d\delta\right)\right)ds\right) \in Dom(\Omega^{-1}).$$

Theorem 2.5. [3] Let

i $u(t), r(t), \in C(\mathbf{R}_+, \mathbf{R}_+)$

ii $\omega \in \Re$

iii $n > 0$ be monotonic, nondecreasing and continuous on $\mathbf{R}_+$

if

$$u(t) \leq n(t) + \int_{t_0}^{t} f(s)\omega(u(s))ds, \quad t \in \mathbb{I}, \tag{2.7}$$

then

$$u(t) \leq n(t)\Omega^{-1}\left(\Omega(1) + \int_{t_0}^{t} f(s)ds\right), \tag{2.8}$$

where $(0, b) \subset (0, \infty)$ and $\Omega(u)$ is defined by equation (2.6) and Ω^{-1} is the inverse of Ω and t is in the subinterval $(0, b)$is so chosen that

$$\Omega(1) + \int_{t_0}^{t} f(s)ds \in Dom(\Omega^{-1}).$$

Theorem 2.6. [14](Generalised First Mean Value Theorem). If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbb{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that

$$\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds.$$

Lemma 2.7. [11] Let $r(t)$ be an integrable function then the n-successive integration of r over the interval $[t_0, t]$ is given by

$$\int_{t_0}^{t} \cdots \int_{t_0}^{t} ds^n = \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1} r(s)ds. \tag{2.9}$$

3. Result

The extensions of the nonlinear Gronwall-Bellman-Bhari inequality are developed as follow to establish our results.

Theorem 3.1. Let $u(t), r(t), h(t)$ be defined as in Theorem 2.4 and $, \omega(u), f(u) \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nonnegative, monotonic and nondecreasing functions. Furthermore, the function $\beta(t) > 0$ be a nondecreasing in t, continuous on $\mathbb{R}_+$ and $\omega(u)$ be submultiplicative for $u > 0$. If

$$u(t) \leq \beta(t) + A\int_{t_0}^{t} r(s)f(u(s))ds + T\int_{t_0}^{t} h(s)\omega(u(s))ds, \tag{3.1}$$

where $K,\ A$ and $T > 0$ holds.
Then,

$$u(t) \leq \beta(t)\Omega^{-1}\left(V(t)\right)F^{-1}(B(t)), \tag{3.2}$$

where

$$V(t) = \Omega(1) + T\int_{t_0}^{t} h(s)\omega(F^{-1}\left(B(s)\right))\,ds \tag{3.3}$$

and

$$B(t) = F(1) + A\int_{t_0}^{t} r(s)ds, \tag{3.4}$$

where F and Ω are defined in equations (2.5) and (2.6) with F^{-1}, Ω^{-1} as their inverses respectively, and t is in the subinterval $(0, b) \subset \mathbb{I}$ so that

$$B(t) \in Dom(F^{-1})$$

and

$$V(t) \in Dom(\Omega^{-1}).$$

Proof. Since $\beta(t)$ is monotonic and nondecreasing on $\mathbb{R}_+$, equation (3.1) yields

$$\frac{u(t)}{\beta(t)} \leq 1 + A\int_{t_0}^{t} r(s)f\left(\frac{u(s)}{\beta(s)}\right)ds + T\int_{t_0}^{t} h(s)\omega\left(\frac{u(s)}{\beta(s)}\right)ds. \tag{3.5}$$

It is clear that

$$\alpha(t) \leq 1 + A\int_{t_0}^{t} r(s)f(\alpha(s))ds + T\int_{t_0}^{t} h(s)\omega(\alpha(s))ds, \tag{3.6}$$

where

$$\frac{u(t)}{\beta(t)} = \alpha(t). \tag{3.7}$$

Use Theorem 2.4 on equation (3.6) to obtain

$$\alpha(t) \leq \Omega^{-1}\left(V(t)\right)F^{-1}\left(B(t)\right). \tag{3.8}$$

By applying equation (3.7) we have (3.2). □

Theorem 3.2. Let $u(t), r(t), h(t) : \mathbb{I} \to \mathbb{R}_+$ be real valued nonnegative continuous functions and $\omega(u), f(u), \gamma(u)$ be positive, monotonic, nondecreasing continuous functions on $\mathbb{R}_+$ and belong to class Ψ. If $\gamma(u)$ is submultiplicative for $u > 0$, the following inequality

$$u(t) \leq K + A\int_{t_0}^{t} r(s)f(u(s))ds + T\int_{t_0}^{t} h(s)\omega(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds, \tag{3.9}$$

for K, A, T and $L > 0$ and $t \in \mathbb{I}$, then

$$u(t) \leq G^{-1}\left[G(K) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(V(s)F^{-1}(B(s))\right)\right]ds\right] \\ \Omega^{-1}\left(V(t)\right)F_{-1}\left(B(t)\right), \tag{3.10}$$

where $F,\ \Omega,\ B(t), and\ V(t)$ are defined in (2.5), (2.6), (3.4) and (3.3) respectively. The function G is defined as

$$G(r) = \int_{r_0}^{r} \frac{ds}{\gamma(s)} \quad 0 < r_0 \leq r, \tag{3.11}$$

and F^{-1}, Ω^{-1} and G^{-1} are the inverses of the functions F, Ω, and G respectively, then

$$G(K) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(V(s)\right)F^{-1}\left(B(s)\right)\right]ds \in Dom(G^{-1}),$$

for t is in the subinterval $(t_0, t_1) \subset \mathbb{R}_+$

$$V(t) \in Dom(\Omega^{-1}),$$

for t is in the subinterval $(t_0, t_2) \subset \mathbb{R}_+$ and

$$B(t) \in Dom(F^{-1})$$

for t is in the subinterval$(t_0, t_3) \subset (R_+)$.

Proof. Define

$$n(t) = K + L\int_{t_0}^{t} g(s)\gamma(u(s))ds, \quad t \in \mathbb{I}, \tag{3.12}$$

then, we re-write (3.9) to get

$$u(t) \leq n(t) + A\int_{t_0}^{t} r(s)f(u(s))ds + B\int_{t_0}^{t} h(s)\omega(u(s))ds. \tag{3.13}$$

Let $n(t)$ be monotonic, nondecreasing on $C(\mathbb{R}_+)$, using Theorem 3.1 to obtain

$$u(t) \leq n(t)\Omega^{-1}\left(V(t)\right)F^{-1}\left(B(t)\right). \tag{3.14}$$

Since $\gamma(u)$ is submultiplicative, it is clear that

$$\frac{d}{dt}G(n(t)) \leq Lg(t)\gamma\left[\Omega^{-1}\left(V(t)\right)F^{-1}\left(B(t)\right)\right],$$

and

$$G(n(t)) \leq G(K) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(V(s)\right)F^{-1}\left(B(s)\right)\right]ds.$$

Hence,

$$n(t) \leq G^{-1}\left[G(n(t_0)) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(V(s)\right)F^{-1}\left(B(s)\right)\right]ds\right]. \tag{3.15}$$

Use (3.15) in (3.14) we arrive at the result. □

Theorem 3.3. Suppose that

i $u(t), r(t), h(t), g(t)$ and $\beta(t) \in C(\mathbb{R}_+$ be defined as in Theorem 3.2
ii $\omega(u), f(u), \gamma(u)$ be nonnegative, monotonic, nondecreasing continuous functions on $\mathbb{R}_+$.
iii $\gamma(u)$ is submultiplicative for $u > 0$.

If

$$u(t) \leq \beta(t) + A\int_{t_0}^{t} r(s)f(u(s))ds + T\int_{t_0}^{t} h(s)\omega(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds, \tag{3.16}$$

for K, A, T and L are positive constants, then

$$u(t) \leq \beta(t)G^{-1}\left[G(K) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(V(s)\right)F^{-1}\left(B(s)\right)\right]ds\right] \Omega^{-1}\left(V(t)\right)F^{-1}\left(B(t)\right), \tag{3.17}$$

where $F, \Omega, G, V(t)$ and $B(t)$ are defined in (2.5),(2.6),(3.11),(3.3) and (3.4), and F^{-1}, Ω^{-1} and G^{-1} are the inverses of F, Ω, G, choosing t as in Theorem 3.1 so that

$$G(K) + L \int_{t_0}^{t} g(s)\gamma \left[\Omega^{-1}\left(V(s)\right) F^{-1}\left(B(s)\right)\right] ds \in Dom(G^{-1}).$$

Proof. Since $\beta(t)$ is monotonic, nondecreasing and nonnegative continuous function on $C(\mathbb{R}_+)$, equation(3.16) becomes

$$\frac{u(t)}{\beta(t)} \leq 1 + A \int_{t_0}^{t} r(s) f(\frac{u(s)}{\beta(s)}) ds + T \int_{t_0}^{t} h(s)\omega(\frac{u(s)}{\beta(s)}) ds$$
$$+ L \int_{t_0}^{t} g(s)\gamma(\frac{u(s)}{\beta(s)}) ds.$$

Applying Theorem 3.1 to have

$$z(t) \leq G^{-1}\left[G(1) + L \int_{t_0}^{t} g(s)\gamma \left[\Omega^{-1}\left(V(s)\right) F^{-1}\left(B(s)\right)\right] ds\right]$$
$$\Omega^{-1}\left(V(t)\right) F^{-1}\left(B(t)\right),$$

where $\frac{u(t)}{\beta(t)} = z(t)$ to arrive at the result (3.17). □

Now, the Hyers-Ulam-Rassias stability of equation (1.1) is considered by using case (i).

Theorem 3.4. Let

- i $|Y(t, u(t), u'(t))| \leq \phi(t) g(|u(t)|) h(|u'(t)|)$,
- ii $|P(t, u(t), u'(t))| \leq \alpha(t)\omega(|u(t)||u'(t)|^n$,
- iii $|q(t, u(t))| \leq r(t)\kappa(|u(t)|)$,
- iv there exist $\varrho > 0$ such that $\int_{t_0}^{t} \varphi(t) dt \leq \varrho\varphi(t)$,
- v there exists $\eta > 0$ such that $|u'(t)| \geq \eta$
- vi $\lim_{t\to\infty} \int_{t_0}^{t} \alpha(s) ds = s_1 < \infty, \quad s_1 > 0$,
- vii $\lim_{t\to\infty} \int_{t_0}^{t} r(s) ds = s_2 < \infty, \quad s_2 > 0$,
- viii $\lim_{t\to\infty} \int_{t_0}^{t} \phi(s) ds = s_3 < \infty, \quad s_1 > 0, \forall t_0 \geq 1$,

where $r(t), \phi(t), \alpha(t)$ are all nonnegative functions on $C(\mathbb{R}_+)$ and the functions g, h, ω, κ are nonnegative, monotonic, nondecreasing on $C(\mathbb{R}_+)$. Furthermore, let g, ω, κ belong to class of Ψ and $\varphi : \mathbb{I} \to \mathbb{R}_+$ be an increasing positive function, then equation (1.1) is Hyers-Ulam-Rassis stable with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_1} = \varrho G^{-1}\left[G(1) + s_1\eta^n\omega\left[\Omega^{-1}\left(V_1^*\right) F^{-1}\left(B_1^*\right)\right]\right]$$
$$\Omega^{-1}\left(V_1^*\right) F^{-1}\left(B_1^*\right). \quad (3.18)$$

where V_1^* and B_1^* are constants defined as

$$V_1^* = \Omega(1) + s_2\kappa\left(F^{-1}\left(B_1^*\right)\right)$$

and

$$B_1^* = F(1) + s_3 h(\eta)\eta,$$

Proof. From inequality (2.1) with Lemma 2.7 we have

$$u(t) \leq t\int_{t_0}^{t}\varphi(s)ds - t\int_{t_0}^{t}Y(s,u(s),u'(s))(u'(s))ds$$
$$-t\int_{t_0}^{t}q(s,u(s))ds + t\int_{t_0}^{t}P(s,u(s),u'(s))ds.$$

It follows that

$$|u(t)| \leq \int_{t_0}^{t}\varphi(s)ds + \int_{t_0}^{t}|Y(s,u(s),u'(s))||(u'(s))|ds$$
$$+\int_{t_0}^{t}|q(s,u(s))|ds + \int_{t_0}^{t}|P(s,u(s),u'(s))|d, \quad \forall t \geq t_0.$$

We use conditions (i),(ii), (iii) of Theorem 3.4 to get

$$|u(t)| \leq \int_{t_0}^{t}\varphi(s)ds + \int_{t_0}^{t}\phi(s)g(|u(s)|)h(|u'(s)|)|(u'(s))|ds + \int_{t_0}^{t}r(s)\kappa(|u(s)|)ds$$
$$+\int_{t_0}^{t}\alpha(s)\omega(|u(s)||u'(s)|^n ds,$$

and condition(iv) to obtain

$$|u(t)| \leq \int_{t_0}^{t}\varphi(s)ds + h(\eta)\eta\int_{t_0}^{t}\phi(s)g(|u(s)|)ds + \int_{t_0}^{t}r(s)\kappa(|u(s)|)ds$$
$$+\eta^n\int_{t_0}^{t}\alpha(s)\omega(|u(s)|ds.$$

Apply Theorem 3.3 to get

$$|u(t)| \leq \int_{t_0}^{t}\varphi(s)dsG^{-1}\left[G(1) + \eta^n\int_{t_0}^{t}\alpha(s)\omega\left[\Omega^{-1}\left(V_1(s)\right)F^{-1}\left(B_1(s)\right)\right]ds\right]$$
$$\Omega\left(V_1(t)\right)F^{-1}\left(B_1(t)\right),$$

where

$$V_1(t) = \Omega(1) + \int_{t_0}^{t}r(s)\kappa\left(F^{-1}\left(B_1(s)\right)\right)ds$$

and

$$B_1(t) = F(1) + h(\eta)\eta\int_{t_0}^{t}\phi(s)ds.$$

Using conditions (iv),(vi),(vii) and (viii) to arrive at

$$|u(t)| \leq \varrho\varphi(t)G^{-1}\left[G(1) + s_1\eta^n\omega\left[\Omega^{-1}\left(V_1^*\right)F^{-1}\left(B_1^*\right)\right]\right]\Omega^{-1}\left(V_1^*\right)F^{-1}\left(B^*\right),$$

where

$$V_1^* = \Omega(1) + s_2\kappa\left(F^{-1}(B_1^*)\right),$$

and

$$B_1^* = F^{-1}\left(F(1) + s_3 h(\eta)\eta\right).$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_1}\varphi(t),$$

where

$$C_{\varphi_1} = \varrho G^{-1}\left[G(1) + s_1\eta^n\omega\left[\Omega^{-1}\left(V_1^*\right)F^{-1}\left(B_1^*\right)\right]\right]\Omega^{-1}\left(V_1^*\right)F^{-1}\left(B_1^*\right).$$

□

Example 3.5. Consider the problem

$$u'' + \frac{1}{t^2}u^2(t)u'^2(t) + \frac{1}{t^4}u^4(t) = \frac{1}{t^3}u^2(t)u'^4(t), \quad t \geq t_0,$$

$$u(t_0) = u'(t_0) = 0$$

and $\alpha(t) = \frac{1}{t^2}$, $r(t) = \frac{1}{t^4}$, $\alpha(t) = \frac{1}{t^3}$. By applying the conditions of Theorem 3.4, the above problem is Hyers-Ulam-Rassias stable.

We consider the case(ii).

Theorem 3.6. Let all the conditions of Theorem 3.4 remain valid and if

$$|Y(t, u(t), u'(t))| = |P(t, u(t), u'(t))| \leq \phi(t)g(|u(t)|)h(|u'(t)|)$$

then, equation (1.1) is Hyers-Ulam-Rassias stable with its initial conditions and Hyers-Ulam-Rassias constant is given as:

$$C_{\varphi_2} = \varrho\Omega^{-1}\left(V_2^*\right)F^{-1}\left(B_2^*\right), \tag{3.19}$$

where the constants V_2^* and B_2^* are defined as

$$V_2^* = \Omega(1) + s_2(F^{-1}(B_2^*))$$

and

$$B_2^* = F(1) + (\eta + 1)h(\eta)s_3.$$

Proof. It easy to see from inequality (2.1) together with application of Lemma 2.7 that

$$|u(t)| \leq \int_{t_0}^t \varphi(s)ds + \int_{t_0}^t |Y(s, u(s), u'(s))|(|u'(s)| + 1)ds + \int_{t_0}^t |q(s, u(s))|ds, \quad \forall t \geq t_0$$

and by conditions (i), (iii),(v) of Theorem 3.4, it is clear that

$$|(u(t)| \leq \int_{t_0}^t \varphi(s)ds + (\eta + 1)h(\eta)\int_{t_0}^t \phi(s)g(|u(s)|)ds + \int_{t_0}^t r(s)\kappa(|u(s)|)ds.$$

By applying Theorem 3.1 we have

$$|u(t)| \leq \int_{t_0}^t \varphi(s)ds\Omega^{-1}\left(V_2(t)\right)F^{-1}\left(B_2(t)\right),$$

where

$$V_2(t) = \Omega(1) + \int_{t_0}^{t} r(s)\kappa(F^{-1}(B_2(s)))ds$$

and

$$B_2(t) = F(1) + (\eta+1)h(\eta)\int_{t_0}^{t} \phi(s)ds.$$

We use conditions (iv),(vi), (vii) to arrive at

$$|u(t)| \le \varrho\varphi(t)\Omega^{-1}\left(V_2^*\right)F^{-1}\left(B_2^*\right).$$

Therefore,

$$|u(t) - u_0(t)| \le |u(t)| \le C_{\varphi_2}\varphi(t),$$

where C_{φ_2} is well defined in (3.19) □

The next theorem is given as

Theorem 3.7. Let all the conditions of Theorem 3.4 remain valid. If

$$|P(t,u(t),u'(t))| = |Y(t,u(t),u'(t))| \le \alpha(t)\omega(|u(t)||u'(t)|^n,$$

then equation (1.1) has Hyers-Ulam-Rassias stability with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_3} = \varrho\Omega^{-1}\left(V_3^*\right)F^{-1}\left(B_3^*\right), \tag{3.20}$$

where

$$V_3^* = \Omega(1) + s_2(F^{-1}(B_3^*))$$

and

$$B_3^* = F(1) + s_1(\eta+1)\eta^n.$$

Proof. Evaluating inequality (2.1)and applying Lemma 2.7 together with conditions (ii), (iii) and (v) we have

$$|(u(t)| \le \int_{t_0}^{t} \varphi(s)ds + (\eta+1)(\eta)^n \int_{t_0}^{t} \alpha(s)\omega(|u(s)|)ds + \int_{t_0}^{t} r(s)\kappa(|u(s)|)ds$$

Applying Theorem 3.1 we obtain

$$|u(t)| \le \int_{t_0}^{t} \varphi(s)ds\Omega^{-1}\left(V_3(t)\right)F^{-1}\left(B_3(t)\right),$$

where

$$V_3(t) = \Omega(1) + \int_{t_0}^{t} r(s)\kappa(F^{-1}(B_3(s)))ds$$

and

$$B_3(t) = F(1) + (\eta+1)\eta^n \int_{t_0}^{t} \alpha(s)ds.$$

We use conditions (iv), (vii), (iv) to get

$$|u(t)| \le \varrho\varphi(t)\Omega^{-1}\left(V_3^*\right)F^{-1}\left(B_3^*\right),$$

where

$$V_3^* = \Omega(1) + s_2(F^{-1}(B_3^*))$$

and

$$B_3^* = F(1) + s_1(\eta + 1)\eta^n.$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_3}\varphi(t),$$

where C_{φ_3} is given in (3.20). □

Theorem 3.8. Suppose all the conditions of Theorem 3.4 remain valid. Then equation

$$u''(t) + Y(t, u(t), u'(t))(u'(t)) + q(t, u(t)) = 0, \tag{3.21}$$

is Hyers-Ulam-Rassis stable with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_4} = \Omega^{-1}\left(V_4^*\right) F^{-1}\left(B_4^*\right), \tag{3.22}$$

where

$$V_4^* = \Omega(1) + s_2\kappa\left(F^{-1}\left(B_4^*\right)\right)$$

and

$$B_4^* = F^{-1}\left(F(1) + s_3 h(\eta)\eta\right),$$

Proof. Simplify inequality (2.1) with the application of Lemma 2.7 and condition (v) of Theorem 3.4, we obtain

$$|u(t)| \leq \int_{t_0}^{t} \varphi(s)ds + h(\eta)\eta \int_{t_0}^{t} \phi(s)g(|u(s)|)ds + \int_{t_0}^{t} r(s)\kappa(|u(s)|)ds. \tag{3.23}$$

By applying Theorem 3.1, we obtain

$$|u(t)| \leq \int_{t_0}^{t} \varphi(s)ds\Omega^{-1}\left(V_4(t)\right) F^{-1}\left(B_4(t)\right),$$

where

$$V_4(t) = \Omega(1) + \int_{t_0}^{t} r(s)\kappa(F^{-1}(B_4(s)))ds$$

and

$$B_4(t) = F(1) + h(\eta)\eta \int_{t_0}^{t} \phi(s)ds.$$

By conditions (vi) and (viii)of Theorem 3.4 we have

$$|u(t)| \leq \varrho\varphi(t)\Omega^{-1}\left(V_4^*\right) F^{-1}\left(B_4^*\right),$$

where

$$V_4^* = \Omega(1) + s_2\kappa(F^{-1}(B_4^*))$$

and

$$B_4^* = F(1) + s_3 h(\eta)\eta.$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_4}\varphi(t),$$

where C_{φ_4} is given in (3.22) □

Finally, we consider Hyers-Ulam-Rassias stability of equation (1.2) which is a special case of equation (1.1).

Theorem 3.9. Let $a(t)$ be nondecreasing function on $C(\mathbb{R}_+)$ then, there exists $a'(t) \geq 0$, $\delta > 0$ such that $a(t) > \delta$. Suppose that

ix $\lim_{t\to\infty} \int_{t_0}^{t} c(s)ds = b < \infty, \quad b > 0,$

x $G(u(t)) = \int_{u(t_0)}^{u(t)} g(s)ds < \infty,$

xi let $|G(u(t)| \geq |u(t)|$.

Then, equation(1.2) is Hyer-Ulam-Rassias stable with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_5} = \varrho(\eta + \frac{\eta^2}{2})\Omega^{-1}(V_5^*)F^{-1}(B_5^*), \tag{3.24}$$

where

$$V_5^* = \Omega(1) + \eta^{(n+1)} s_1 \omega\left(F^{-1}(B_5^*)\right)$$

and

$$B_5^* = F(1) + \eta^2 b.$$

Proof. From inequality (2.2) with condition (x)of Theorem 3.9 we obtain

$$\begin{aligned} -u'(t)\varphi(t) \leq u''(t)u'(t) + c(t)f(u(t))(u'(t))^2 + a(t)\frac{d}{dt}G(u(t)) \\ -P(t, u(t), u'(t))u'(t) \leq \varphi(t)u'(t) \end{aligned}$$

By applying conditions (ii), (v) of Theorem 3.4 and (xi) of Theorem 3.9 we obtain

$$\begin{aligned} |u(t)| \leq \frac{(2\eta + \eta^2)}{2\delta} \int_{t_0}^{t} \varphi(s)ds + \frac{\eta^2}{\delta} \int_{t_0}^{t} c(s)f(|u(s)|)ds \\ + \frac{\eta^{(n+1)}}{\delta} \int_{t_0}^{t} \alpha(s)\omega(|u(s)|)ds \end{aligned}$$

and with the application of Theorem 3.1, we arrive

$$|u(t)| \leq \frac{(2\eta + \eta^2)}{2\delta} \int_{t_0}^{t} \varphi(s)ds\Omega^{-1}(V_5(t))F^{-1}(B_5(s)),$$

where

$$V_5(t) = \Omega(1) + \frac{\eta^{(n+1)}}{\delta} \int_{t_0}^{t} \alpha(s)\omega\left(B_5(s)\right)ds,$$

and

$$B_5(t) = F(1) + \frac{\eta^2}{\delta} \int_{t_0}^{t} c(s)ds.$$

Using conditions (ix) of Theorem 3.9 and (iv),(vi) of Theorem 3.4 to obtain

$$|u(t)| \leq \varrho\varphi(t)\frac{(\eta + \eta^2)}{2\delta}\Omega^{-1}(V_5^*)F^{-1}(B_5^*),$$

where

$$V_5^* = \Omega(1) + \frac{\eta^{(n+1)} s_1}{\delta}\omega\left(F^{-1}(B_5^*)\right)$$

and

$$B_5^* = F(1) + \frac{\eta^2 b}{\delta}.$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq C_{\varphi_5}\varphi(t),$$

where

$$C_{\varphi_5} = \varrho\frac{(\eta + \eta^2)}{2\delta}\Omega^{-1}(V_5^*)F^{-1}(B_5^*).$$

□

Example 3.10. Consider the equation

$$u''(t) + (t+1)^{-2}u^2u' + t^4u^4 = (t+1)^{-5}u^2(t)u'^4(t), \quad t \geq t_0,$$

where $|P(t, u(t), u'(t))| \leq (t+1)^{-4}u^2(t)u'^4(t)$ and $n = 3$, then the nonlinear differential equation is Hyers-Ulam-Rassias stable by the conditions of the Theorem 3.9.

Next, we consider equation (1.2) under case (iv).

Theorem 3.11. Let all the conditions of Theorem 3.9 remain valid. Then, equation

$$u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t))u'(t) = P(t, u(t)) \tag{3.25}$$

where $|P(t, u(t))| \leq A|u(t)|$, $\int_{t_0}^{\infty} |u'(s)|ds \leq \nu$ for $\nu, \eta > 0$ and $P(t, u(t)) \in \mathbb{R}$, is Hyers-Ulam-Rassias stable with Hyers-Ulam-Rassias constant given by

$$C_{\varphi_6} = \frac{\varrho(\eta^2 + LA|u(\xi)| + \eta)}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{b\eta^2)}{\delta}\right) \tag{3.26}$$

Proof. Simplify inequality (2.2), using conditions (x),(xi) of theorem 3.9 and (v) of Theorem 3.4 together with hypothesis of Theorem 3.11 and by Theorem 2.6, there exists $\xi \in [t_0, t]$ such that

$$|u(t)| \leq \frac{(\frac{\eta^2}{2} + \nu A|u(\xi)| + \eta)}{\delta}\int_{t_0}^{t}\varphi(s)ds + \frac{\eta^2}{\delta}\int_{t_0}^{t}c(s)f(|u(s)|)ds \tag{3.27}$$

By applying Theorem 2.5 we obtain

$$|u(t)| \leq \frac{(\frac{\eta^2}{2} + \nu A|u(\xi)| + \eta)}{\delta}\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{\eta^2}{\delta}\int_{t_0}^{t}c(s)ds\right).$$

By applying conditions (iv) of Theorem 3.4 and (xi) of Theorem 3.9 to arrive at

$$|u(t)| \leq \frac{\varrho\varphi(t)(\frac{\eta^2}{2} + \nu A|u(\xi)| + \eta)}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{b\eta^2}{\delta}\right).$$

Hence,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_6}\varphi(t).$$

Therefore,

$$C_{\varphi_6} = \frac{\varrho(\eta^2 + \nu A|u(\xi)| + \eta)}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{b\eta^2}{\delta}\right).$$

□

Example 3.12. Consider the nonlinear differential equation

$$u''(t) + (t+1)^{-2}u^2u' + t^4u^4 = 2u^2(t), \quad \forall t \geq t_0,$$

where $c(s) = \dfrac{1}{(t+1)^2}$, $f(u(t)) = u^2(t)$, $P(t, u(t)) \leq 2u^2(t)$. Then, the nonlinear differential equation is Hyers-Ulam-Rassias stable by the conditions of the theorem 3.11

Theorem 3.13. Let all the conditions of Theorem 3.9 remain valid, besides, let the equation (1.2) becomes

$$u''(t) + c(t)f(u(t))u'(t) + a(t)g(u(t))u'(t) = 0, \tag{3.28}$$

where $P(t, u(t), u'(t)) = 0$ in equation (1.2), then, equation (3.28) is Hyers-Ulam-Rassias stable with Hyers-Ulam-Rassias constant given as

$$C_{\varphi_7} = \varrho(\eta + \eta^2)\Omega^{-1}\left(\Omega(1) + b\eta^2\right). \tag{3.29}$$

Proof. From inequality (2.2), by conditions (v) of Theorem 3.4 and (ix) of Theorem 3.9, we have

$$|u(t)| \leq \frac{(2\eta + \eta^2)}{2\delta}\int_{t_0}^{t}\varphi(s)ds\Omega^{-1}\left(\Omega(1) + \frac{\eta^2}{\delta}\int_{t_0}^{t}c(s)ds\right). \tag{3.30}$$

By using Theorem 2.2 and applying conditions (x) of theorem 3.9,(v)of Theorem 3.4 to have

$$|u(t)| \leq \frac{\varrho\varphi(t)(2\eta + \eta^2)}{2\delta}\Omega^{-1}\left(\Omega(1) + \frac{b\eta^2}{\delta}\right). \tag{3.31}$$

Therefore,

$$|u(t) - u_0(t)| \leq |u(t)| \leq C_{\varphi_7}\varphi(t)$$

where

$$C_{\varphi_7} = \frac{\varrho(2\eta + \eta^2)}{2\delta}\Omega^{-1}\left(\Omega(1) + b\eta^2\right)$$

□

Example 3.14. Consider the nonlinear differential equation

$$u'' + t^{-2}u^2u' + t^{-4}u^2 = 0, \quad \text{for} \quad \forall t \geq t_0$$

where $c(t) = \dfrac{1}{t^2}$ and $f(u) = u^2(t)$. This equation is Hyers-Ulam-Rassias stable by all the properties of the Theorem 3.13.

Remark 3.15. The results in Theorems 3.4, 3.6, 3.7, 3.8, 3.9, 3.11 are established by making use of Theorems 3.1,3.2, 3.3. The results here genaralized the results of many authors who concentrated on Hyers-Ulam and Hyers-Ulam-Rassias stability of linear differential equations.

4. Conclusion

In this work, the results are exemplified by giving examples at the end of the proofs of the theorems.

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