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POSITIVE SOLUTIONS FOR A NONLINEAR FRACTIONAL BOUNDARY VALUE PROBLEM

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ABSTRACT. In this paper, we establish the existence of positive solutions to a nonlinear fractional differential equation with integral boundary conditions. Our approach is based on the linear operator theory and the application of Krasnosel'skii fixed-point theorem in a cone. We present two examples to illustrate the practicability of our main results.

1. INTRODUCTION

In recent times, fractional boundary value problems have gained much attention and importance in the fields of engineering and applied sciences due to the fact that fractional order models are more realistic and practical. Such fields include chemical engineering, aerodynamics, electrochemistry, thermodynamics, fluid mechanics, plasma physics, polymer science, population dynamics and so forth.

For more details on fractional boundary value problems, we refer the readers to [1], [2], [4], [5], [7], [8], [11], [12], [13], [14], [16], [17], [18], [22], [24], [27], [28], [29], [30], [31], [34], [35], [36], [38] and the references cited therein.

Cabada and Wang [11], by means of Guo-Krasnosel'skii fixed point theorem, investigated the existence of positive solutions to the following boundary value problem with integral boundary conditions

$$\left.\begin{aligned} {}^cD^\alpha u(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \\ u(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds, \end{aligned}\right\} \tag{1.1}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$ and ${}^cD^\alpha$ is the Caputo fractional derivative. Zhao *et al.*[32] applied the Krasnosel'skii fixed-point theorem to obtain the existence

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and nonexistence of positive solutions to the following fractional boundary value problem

$$\left.\begin{aligned} D^{\alpha}u(t)+\lambda h(t)f(u(t))=0,\ \ t\in(0,1),\\ u(0)=u'(0)=0,\ \ u(1)=\int_0^1 g(s)u(s)ds, \end{aligned}\right\} \tag{1.2}$$

where $2<\alpha\leq 3$, D^{α} is the standard Riemann-Liouville fractional derivative, h and f are continuous functions, $g\in L^1(0,1)$ and λ is a positive parameter.

Furthermore, Feng *et al.*[15] studied the existence of positive solutions to the following integral boundary value problem (BVP for short) of nonlinear fractional differential equation

$$\left.\begin{aligned} D^{\alpha}x(t)+g(t)f(t,x)=0,\ \ t\in(0,1),\\ x(0)=0,\ x'(1)=\int_0^1 h(t)x(t)dt, \end{aligned}\right\} \tag{1.3}$$

where $1<\alpha\leq 2$, D^{α} is the standard Riemann-Liouville fractional derivative, $g\in C([0,1],\ I\!R^+)$, $h\in L^1[0,1]$ is non-negative and $f\in C([0,1]\times I\!R^+, I\!R^+)$. The fixed point theorem in cones was used to establish the existence of positive solutions to the BVP (1.3).

Inspired greatly by the works in [11], [15], [32], this paper is designed to study the existence of positive solutions to the following boundary value problem of nonlinear fractional differential equation

$$\left.\begin{aligned} D^{\alpha}u(t)+a(t)f(t,u(t))=0,\ \ t\in(0,1),\\ u(0)=0,\ \ \gamma u(1)+\beta u'(1)=\int_0^1 p(t)u(t)dt, \end{aligned}\right\} \tag{1.4}$$

where $1<\alpha\leq 2$, D^{α} is the standard Riemann-Liouville fractional derivative, $a\in C([0,1],\ I\!R^+)$, $p\in L^1[0,1]$ is non-negative, $f\in C([0,1]\times I\!R^+, I\!R^+)$ and $\gamma,\ \beta\in(0,1)\subset I\!R^+$. This paper employs the fixed-point theorem due to Krasnosel'skii to establish the existence of positive solutions to the BVP (1.4). For the case of $\gamma=0$ and $\beta=1$, the BVP (1.4) reduces to the BVP (1.3) which was studied by Feng *et al.*[15]. Here, we consider the case $\gamma,\ \beta\in(0,1)\subset I\!R^+$. Many authors have focused attention on the existence of positive solutions for singular and non-singular cases of fractional boundary value problem. However, to the best of our knowledge, no work has been done on the existence of positive solutions for the BVP (1.4) in the literature. Our approach and methodology are different from those in [7], [9], [15], [19], [23], [33], [36] and [37].

Throughout this work, the following conditions will be assumed:

$\mathbf{C_1}$. $f:[0,1]\times I\!R^+\longrightarrow I\!R^+$ is continuous.

$\mathbf{C_2}$. There exists a constant $L>0$ such that

$$|f(t,u)-f(t,v)|\leq L|u-v|,\ \forall\ (t,u),(t,v)\in[0,1]\times I\!R^+.$$

$\mathbf{C_3}$. $a : (0,1) \longrightarrow I\!\!R^+$ is continuous, $a(s) \not\equiv 0$ on $(0,1)$ and $0 < \displaystyle\int_0^1 a(s)ds < \infty$.

$\mathbf{C_4}$. $p : [0,1] \longrightarrow I\!\!R^+$ is continuous with $\displaystyle\int_0^1 p(s)ds > 0$ and

$$0 < c_o = \frac{1}{\mu}\int_0^1 p(s)s^{\alpha-1}ds < 1, \quad \mu \in I\!\!R^+.$$

The rest of the paper is outlined as follows: In Section 2, some basic definitions and lemmas are presented. The main existence results are stated and proved in Section 3. Finally, we give two examples in Section 4 to illustrate the application of our main results.

2. Preliminaries

In this section, we give some basic definitions and lemmas which will be needed in the sequel.

Definition 2.1. [6], [20] - The Riemann-Liouville fractional integral of order $\alpha > 0$ for a given continuous function $f : (0,\infty) \longrightarrow I\!\!R$ is defined to be

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds,$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. [6], [20] - The Riemann-Liouville fractional derivative of order $\alpha > 0$ for a given continuous function $f : (0,\infty) \longrightarrow I\!\!R$ is defined to be

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)ds,$$

$n-1 < \alpha \leq n$, provided the right side is pointwise defined on $(0,\infty)$, where $n = [\alpha]+1$ and $[\alpha]$ is the integer part of the number α.

Lemma 2.3. [25] - *If $u \in C(0,1) \cap L(0,1)$, then*

$$D^{\alpha} I^{\alpha} u(t) = u(t).$$

Lemma 2.4. [6], [20] - *Let $\alpha > 0$ and $u \in C(0,1) \cap L(0,1)$.*
Then the unique solution of $D^{\alpha}u(t) = 0$ is given by
$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, for $c_i \in I\!\!R$ and $i = 1,2,\ldots,n$.

Lemma 2.5. [6], [20] - *Let $\alpha > 0$ and $u, D^{\alpha}u \in C(0,1) \cap L(0,1)$. Then*

$$\left.\begin{aligned} I^{\alpha}D^{\alpha}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \\ for\ c_i \in I\!\!R\ \ and\ \ i = 1,2,\ldots,n, \quad n \geq \alpha \end{aligned}\right\} \tag{2.1}$$

Lemma 2.6. *Let $1 < \alpha \leq 2$, $0 < c_o < 1$, $w = \beta(\alpha-1)$ and*
$\mu = [w+\gamma] > 0$. If $h \in L^1[0,1]$ is a given function, then the unique

solution of the BVP

$$\left.\begin{aligned} D^{\alpha}u(t)+h(t)=0, \quad 0<t<1, \\ u(0)=0, \quad \gamma u(1)+\beta u'(1)=\int_0^1 p(t)u(t)dt \end{aligned}\right\} \tag{2.2}$$

is given by

$$u(t)=\int_0^1 G(t,s)h(s)ds,$$

where

$$G(t,s)=G_1(t,s)+G_2(t,s), \tag{2.3}$$

$$G_1(t,s)=\begin{cases} \dfrac{wt^{\alpha-1}(1-s)^{\alpha-2}+\gamma t^{\alpha-1}(1-s)^{\alpha-1}-\mu(t-s)^{\alpha-1}}{\mu\Gamma\alpha}, & s\leq t, \\ \\ \dfrac{wt^{\alpha-1}(1-s)^{\alpha-2}+\gamma t^{\alpha-1}(1-s)^{\alpha-1}}{\mu\Gamma\alpha}, & t\leq s, \end{cases} \tag{2.4}$$

and

$$G_2(t,s)=\frac{c_o}{1-c_o}G_1(t,s), \quad 0<c_o<1. \tag{2.5}$$

Proof. The proof is similar to that of Lemma 2.6 in [3] and so we omit details. □

Lemma 2.7. [3] - *The function $G_1(t,s)$ defined by (2.4) is continuous and satisfies the following conditions:*

(i) $G_1(t,s)\geq 0$ *for all* $t,s\in[0,1]$ *and* $G_1(t,s)>0$ *for all* $t,s\in(0,1)$.

(ii) $G_1(t,s)\leq G_1(s,s)=\dfrac{s^{\alpha-1}[w(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}]}{\mu\Gamma\alpha}$, *for all* $t,s\in[0,1]$.

(iii) $\min\limits_{\frac14\leq t\leq\frac34} G_1(t,s)\geq m(s)\max\limits_{0\leq t\leq 1} G_1(t,s)=m(s)G_1(s,s)$, *for* $\frac14\leq t\leq\frac34$, $s\in(0,1)$ *and* $0<m(s)<1$, *where* $m(s)\in C((0,1),\ \mathbb{R}^+)$ *and*

$$m(s)=\begin{cases} \dfrac{(\frac34)^{\alpha-1}[w(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}]-\mu(\frac34-s)^{\alpha-1}}{s^{\alpha-1}[w(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}]}, & s\in(0,\frac34], \\ \\ \dfrac{1}{(4s)^{\alpha-1}}, & s\in[\frac34,1). \end{cases} \tag{2.6}$$

Lemma 2.8. [3] - *Suppose* $0<c_o<1$. *Then* $G_2(t,s)$ *defined by (2.5) is continuous and satisfies the following conditions:*

(i) $G_2(t,s)\geq 0$ *for all* $t,s\in[0,1]$ *and* $G_2(t,s)>0$ *for all* $t,s\in(0,1)$.

(ii) $G_2(t,s)\leq\lambda_0 G_1(s,s)=\dfrac{\lambda_0 s^{\alpha-1}[w(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}]}{\mu\Gamma\alpha}$,

$\forall\ t,s\in[0,1]$, *where* $\lambda_0=\left(\dfrac{c_o}{1-c_o}\right)>0$.

Lemma 2.9. [3] - *The Green function $G(t,s)$ defined by (2.3) is continuous and satisfies the following conditions:*

(i) $G(t,s) \geq 0$ *for all* $t,s \in [0,1]$ *and* $G(t,s) > 0$ *for all* $t,s \in (0,1)$.

(ii) $$G(t,s) \leq \sigma G_1(s,s) = \frac{\sigma s^{\alpha-1}[w(1-s)^{\alpha-2} + \gamma(1-s)^{\alpha-1}]}{\mu\Gamma\alpha}, \tag{2.7}$$

$\forall\ t,s \in [0,1]$, where $\sigma = (1+\lambda_0) > 0$.

(iii) $\min\limits_{\frac{1}{4}\leq t\leq\frac{3}{4}} G(t,s) \geq \sigma m(s) G_1(s,s)$, for $\frac{1}{4} \leq t \leq \frac{3}{4}$, $s \in (0,1)$ and $0 < m(s) < 1$.

In view of Lemma 2.6, $u \in C[0,1] \cap L^1[0,1]$ is said to be the solution of BVP (1.4) if and only if u satisfies the integral equation

$$u(t) = \int_0^1 G(t,s)a(s)f(s,u(s))ds, \tag{2.8}$$

where $G(t,s)$ is the Green's function defined by (2.3).

Let $E = C[0,1]$ be a Banach space with the maximum norm $\|u\| = \max\limits_{0\leq t\leq 1} |u(t)|$. Define a cone $K \subset E$ by

$$K = \left\{ u \in E : u(t) \geq 0 \ \text{ and } \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq m(s)\|u\| \right\}.$$

Define an integral operator $T : K \longrightarrow E$ by

$$Tu(t) = \int_0^1 G(t,s)a(s)f(s,u(s))ds, \quad u \in K, \tag{2.9}$$

$$= \int_0^1 G_1(t,s)a(s)f(s,u(s))ds + \int_0^1 G_2(t,s)a(s)f(s,u(s))ds.$$

The fixed points of the operator T in the cone K are the positive solutions of the BVP (1.4). Let the operators A and B be defined as follows:

$$Au(t) = \int_0^1 G_1(t,s)a(s)f(s,u(s))ds \tag{2.10}$$

and

$$Bu(t) = \int_0^1 G_2(t,s)a(s)f(s,u(s))ds. \tag{2.11}$$

Lemma 2.10. *Suppose conditions C_1, C_3 are satisfied and let the operator $T : K \longrightarrow E$ be defined as in (2.9). Then $T : K \longrightarrow K$ is completely continuous.*

Proof. Obviously, the operator $T : K \longrightarrow K$ is continuous since the functions G, a and f are continuous and nonnegative.
Next, we prove that T maps bounded sets into bounded sets in K :
Let $\Omega \subset K$ be a bounded set. Then there exists a constant $\eta > 0$ such that $\|u\| \leq \eta$, for all $u \in \Omega$.

Let $L_1 = \max_{0\le t\le 1,\, 0\le u\le L_1} |f(t,u(t))| + 1$ and $q = \int_0^1 G(s,s)a(s)ds$.
Then by (2.7) and for each $t \in [0,1]$, we have

$$\begin{aligned}
|Tu(t)| &= \left|\int_0^1 G(t,s)a(s)f(s,u(s))ds\right|, \\
&\le \int_0^1 \sigma G_1(s,s)a(s)|f(s,u(s))|ds, \\
&\le \sigma L_1 \int_0^1 G(s,s)a(s)ds, \\
\|Tu\| &\le \sigma L_1 q.
\end{aligned}$$

$\Longrightarrow T(\Omega)$ is bounded.
Finally, we show that T maps bounded sets into equicontinuous sets of K :
Let $\Omega \subset K$ be a bounded set and $t_1, t_2 \in [0,1]$ with $t_2 > t_1$.
Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous.
Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|t_2 - t_1| < \delta$, we have $|G(t_2,s) - G(t_1,s)| < \dfrac{\varepsilon}{L_1 \int_0^1 a(s)ds}$. Therefore, for any $u \in \Omega$, we have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &= \left|\int_0^1 \left[G(t_2,s) - G(t_1,s)\right] a(s)f(s,u(s))ds\right|, \\
&\le \int_0^1 |G(t_2,s) - G(t_1,s)|a(s)|f(s,u(s))|ds, \\
&\le L_1 \int_0^1 a(s)ds|G(t_2,s) - G(t_1,s)| < \varepsilon, \\
\Longrightarrow \|Tu(t_2) - Tu(t_1)\| &< \varepsilon,
\end{aligned}$$

which shows that the family of functions $\{Tu : u \in \Omega\}$ is equicontinuous. Therefore, in view of the Arzela-Ascoli theorem, we conclude that $T : K \longrightarrow K$ is equicontinuous and hence completely continuous. □

We state the following Krasnosel'skii fixed point theorems which are fundamental to prove the existence of positive solutions for the BVP (1.4).

Theorem 2.11. [1], [26] - *Let M be a bounded, closed and convex nonempty subset of a Banach space $(E, \|\cdot\|)$. Suppose that A and B map M into E such that*

(i) $x,\ y \in M \Longrightarrow Ax + By \in M$,
(ii) *A is a contraction and*
(iii) *B is completely continuous.*

Then there exists $z \in M$ with $z = Az + Bz$.

Theorem 2.12. [10], [21] - *Let E be a Banach Space and $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$.*
If $T : K \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \longrightarrow K$ is a completely continuous operator such that either
(i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$, *or*
(ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$,
is satisfied, then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

To apply the Krasnosel'skii fixed-point Theorem 2.11, we need to construct two mappings: one a contraction and the other completely continuous. Thus we have the following Lemmas:

Lemma 2.13. *Assume conditions C_1, C_2 and C_3 are satisfied. Suppose there exist constants $m > 0$, $L > 0$ such that $|a(s)| \leq m$ and $|G_1(s,s)| \leq \dfrac{\varepsilon}{mL}$, for $0 < \varepsilon < 1$, $t, s \in [0,1]$. Then the operator $A : K \longrightarrow K$ is a contraction.*

Proof. In view of (2.10), we have $Au(t) = \displaystyle\int_0^1 G_1(t,s)a(s)f(s,u(s))ds$.
Let $u, v \in K$. Then $\forall\ t \in [0,1]$ and $0 < \varepsilon < 1$, we have

$$\begin{aligned}
\|Au(t) - Av(t)\| &= \left\| \int_0^1 G_1(t,s)a(s)f(s,u(s))ds \right. \\
&\quad \left. - \int_0^1 G_1(t,s)a(s)f(s,v(s))ds \right\| \\
&\leq m \int_0^1 |G_1(s,s)|ds \|f(s,u(s)) - f(s,v(s))\| \\
&\leq m \int_0^1 \frac{\varepsilon}{mL} ds \|f(s,u(s)) - f(s,v(s))\| \\
&\leq m \cdot \frac{\varepsilon}{mL} \cdot L\|u - v\| \\
&\leq \varepsilon\|u - v\|.
\end{aligned}$$

Hence A is a contraction. □

Lemma 2.14. *Assume conditions C_1, C_2 and C_3 are satisfied.*
Suppose there exist constants $m > 0$, $\varepsilon > 0$ such that $|a(s)| \leq m$ and
$|G_2(t_1,s) - G_2(t_2,s)| \leq \dfrac{\varepsilon}{mN}$, for $0 < \varepsilon < 1$, $t_1, t_2 \in [0,1]$ and
$N = \displaystyle\sup_{s\in[0,1]} |f(s,u_n(s))|$, $n = 1, 2, \ldots$.
Then the operator $B : K \longrightarrow K$ is compact.

Proof. In view of equation (2.11), we have

$$Bu(t) = \int_0^1 G_2(t,s)a(s)f(s,u(s))ds.$$

Let $\{u_n\}$ be a bounded sequence in K. Then there exists $\sigma^* > 0$ such that $\|u_n\| \leq \sigma^*$. Let $y_n = Bu_n,\ n = 1, 2, \ldots$
Then $\|y_n\| = \|Bu_n\| \leq \|B\|\|u_n\| \leq \sigma^*\|B\|$. Hence $\{y_n\}$ is also bounded.
Next, we show that $\{y_n\}$ is equicontinuous:
Since $G_2(t, s)$ is continuous on $[0, 1] \times [0, 1]$, it is uniformly continuous.
Hence given any $\varepsilon > 0$, there exists $\delta > 0$ such that $\forall\ t_1, t_2 \in [0, 1]$ and $s \in [0, 1]$, $|t_1 - t_2| < \delta$.
Then for every n, we have

$$\begin{aligned}
\|Bu_n(t_1) - Bu_n(t_2)\| &= \left\| \int_0^1 G_2(t_1, s)a(s)f(s, u_n(s))ds \right. \\
&\quad \left. - \int_0^1 G_2(t_2, s)a(s)f(s, u_n(s))ds \right\| \\
&\leq \int_0^1 |G_2(t_1, s) - G_2(t_2, s)|ds \cdot |a(s)|\|f(s, u_n(s)\| \\
&\leq \frac{\varepsilon}{mN} \cdot mN = \varepsilon. \\
\implies \|Bu_n(t_1) - Bu_n(t_2)\| &\leq \varepsilon.
\end{aligned}$$

Thus $\{y_n\}$ is equicontinuous and by Ascoli-Arzela's theorem, $\{y_n\}$ has a uniformly convergent subsequence. Hence $\{y_n\}$ is relatively compact and so B is compact. □

3. Main Results

In this section, we established the existence of positive solutions to the BVP (1.4).

Theorem 3.1. *Assume that conditions $C_1 - C_3$ are satisfied and let* $L_1 = \max\limits_{0 \leq t \leq 1} |f(t, u(t))| + 1$, *for all* $u \in [0, \infty)$ *and* $\sigma = (1 + \lambda_0) > 0$.
Suppose there exist positive constants k, m and r such that $|a(t)| \leq m$ and $kL_1m\sigma = r$ where

$$k = \int_0^1 G_1(s, s)ds.$$

Then the BVP (1.4) has at least one positive solution in K.

Proof. Let $K_r = \{u \in K : \|u\| \leq r\}$, for $r > 0$.
Obviously K_r is a bounded, closed and convex subset of the Banach space E.
We shall show that if $u,\ v \in K_r$, then $(Au + Bv) \in K_r$.
Let $u, v \in K_r$. Then $\|u\| \leq r$ and $\|v\| \leq r$.

$$
\begin{aligned}
\|Au+Bv\| &= \left\| \int_0^1 G_1(t,s)a(s)f(s,u(s))ds + \int_0^1 G_2(t,s)a(s)f(s,v(s))ds \right\| \\
&\leq \left| \int_0^1 G_1(s,s)a(s)f(s,u(s))ds \right| \\
&\qquad\qquad + \left| \int_0^1 \lambda_0 G_1(s,s)a(s)f(s,v(s))ds \right| \\
&\leq \int_0^1 G_1(s,s)|a(s)||f(s,u(s))|ds \\
&\qquad\qquad + \int_0^1 \lambda_0 G_1(s,s)|a(s)||f(s,v(s))|ds \\
&\leq mL_1 \int_0^1 G_1(s,s)ds + mL_1 \int_0^1 \lambda_0 G_1(s,s)ds \\
&\leq mL_1(1+\lambda_0) \int_0^1 G_1(s,s)ds \\
&\leq mL_1(1+\lambda_0)k \\
&\leq kL_1 m\sigma = r.
\end{aligned}
$$

Hence $(Au+Bv) \in K_r$.
By Lemma 2.13, the operator $A : K \longrightarrow K$ is a contraction and Lemma 2.14 implies that operator $B : K \longrightarrow K$ is compact. Thus, all the hypotheses of Krasnosel'skii fixed-point Theorem 2.11 are satisfied and so there exists $u \in K_r$ such that $u = Au + Bu$. This fixed point is the positive solution to the BVP (1.4) and the proof is completed. □

Theorem 3.2. *Assume that conditions C_1, C_3 and C_4 are satisfied and let* $L = \left(\int_0^1 G_1(s,s)a(s)ds \right)^{-1} > 0$ *and* $M = \left(\sigma \int_{\frac{1}{4}}^{\frac{3}{4}} m(s)G_1(s,s)a(s)ds \right)^{-1} > 0.$ *Suppose there exist constants $r^* > 0$ and $R > 0$ with $R > r^*$ such that the following hypotheses hold:*

(i) $f(t,u) \leq LR$, *for* $(t,u) \in [0,1] \times [0,R]$,
(ii) $f(t,u) \geq Mr^*$, *for* $(t,u) \in [\frac{1}{4}, \frac{3}{4}] \times (0, r^*]$.

Then, the BVP (1.4) has at least one positive solution.

Proof. Let $u \in K$ and $t \in [0,1]$ with $\|u\| = R$. Then by hypothesis (i) and for $0 \leq u \leq R$, we have

$$\begin{aligned}
|Tu(t)| &= \left| \int_0^1 G(t,s)a(s)f(s,u(s))ds \right|, \\
&\leq \int_0^1 G_1(s,s)a(s)LRds, \\
&\leq L\int_0^1 G_1(s,s)a(s)\cdot Rds, \\
&\leq R = \|u\|, \\
\Longrightarrow \|Tu\| &\leq \|u\|.
\end{aligned}$$

Setting $\Omega_1 = \{u \in K : \|u\| < R\}$, then we have

$$\|Tu(t)\| \leq \|u\|, \text{ for } u \in K \cap \partial\Omega_1.$$

On the other hand, let $u \in K$ with $\|u\| = r^*$. Then by hypothesis (ii) and for any $t \in [\frac{1}{4}, \frac{3}{4}]$, we have

$$\begin{aligned}
|Tu(t)| &= \left| \int_0^1 G(t,s)a(s)f(s,u(s))ds \right|, \\
&\geq \sigma \int_0^1 m(s)G_1(s,s)a(s)f(s,u(s))ds, \\
&\geq \sigma \int_{\frac{1}{4}}^{\frac{3}{4}} m(s)G_1(s,s)a(s)Mr^*ds, \\
&\geq \sigma M \int_{\frac{1}{4}}^{\frac{3}{4}} m(s)G_1(s,s)a(s)\cdot r^*ds, \\
&\geq r^* = \|u\|, \\
\Longrightarrow \|Tu\| &\geq \|u\|.
\end{aligned}$$

Setting $\Omega_2 = \{u \in K : \|u\| < r^*\}$, then we have

$$\|Tu\| \geq \|u\|, \text{ for } u \in K \cap \partial\Omega_2.$$

In view of part (i) of Theorem 2.12, we conclude that the BVP (1.4) has one positive solution in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$. $\square$

4. Example

1. Consider the nonlinear boundary value problem:

$$\left.\begin{aligned} D^{\frac{3}{2}}u(t) + \frac{(t+1)}{30}\left[\frac{340e^{-t}u}{(7+e^t)(1+u)}\right] = 0, \quad t \in (0,\ 1), \\ u(0) = 0, \quad \frac{3}{16}u(1) + \frac{5}{8}u'(1) = \int_0^1 \frac{3t}{20} \cdot u(t)dt. \end{aligned}\right\} \tag{4.1}$$

Here, $\alpha = \dfrac{3}{2}$, $\gamma = \dfrac{3}{16}$, $\beta = \dfrac{5}{8}$, $a(t) = \dfrac{(t+1)}{30}$ and $p(t) = \dfrac{3t}{20}$.

Now, condition C_3 holds since $a(t)$ is continuous with $a(t) \not\equiv 0$ on $[0,1]$ and $0 < \displaystyle\int_0^1 a(s)ds < \infty$. Let $u \in [0, \infty)$ and $t \in [0,1]$.

Then $f(t,u) = \left[\dfrac{340e^{-t}u}{(7+e^t)(1+u)}\right]$ is continuous and condition C_1 holds.

$$\begin{aligned} \text{Also,} \quad |f(t,u) - f(t,v)| &= \frac{340e^{-t}}{(7+e^t)}\left|\frac{u}{(1+u)} - \frac{v}{(1+v)}\right| \\ &= \frac{340e^{-t}|u-v|}{(7+e^t)(1+u)(1+v)} \\ &\leq \frac{340e^{-t}}{(7+e^t)}|u-v| \\ &\leq \frac{85}{2}|u-v|. \end{aligned}$$

Hence, condition C_2 holds with $L = \dfrac{85}{2}$.

By simple computation, we have

$$w = \beta(\alpha - 1) = \frac{5}{8}\left(\frac{1}{2}\right) = \frac{5}{16}.$$

$$\mu = [w + \gamma] = \frac{5}{8}\left(\frac{1}{2}\right) + \frac{3}{16} = \frac{1}{2}.$$

$$\mu\Gamma\alpha = \frac{1}{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4}.$$

$$c_o = \frac{1}{\mu}\int_0^1 p(t)t^{\alpha-1}dt = 2 \cdot \frac{3}{20}\int_0^1 t^\alpha dt = \frac{3}{10}\left(\frac{2}{5}\right) = \frac{3}{25}.$$

$$\lambda_0 = \frac{c_o}{1 - c_o} = \frac{\left(\frac{3}{25}\right)}{1 - \left(\frac{3}{25}\right)} = \frac{3}{25}\left(\frac{25}{22}\right) = \frac{3}{22}.$$

$$\sigma = 1 + \lambda_0 = 1 + \frac{3}{22} = \frac{25}{22} = 1.136363636.$$

$$
\begin{aligned}
\text{Also, } k &= \int_0^1 G_1(s,s)ds \\
&= \int_0^1 \frac{s^{\alpha-1}[w(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}]}{\mu\Gamma\alpha}ds \\
&= \frac{4}{\sqrt{\pi}}\left[\frac{5}{16}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-2}ds + \frac{3}{16}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}ds\right] \\
&= \frac{5\sqrt{\pi}}{8}+\frac{3\sqrt{\pi}}{32} \\
&= 1.273951205.
\end{aligned}
$$

Moreover, for $t \in [0,1]$ and $u \in [0,\ 5]$, we obtain

$$
\begin{aligned}
L_1 &= \max_{0\le t\le 1}\left|\frac{340e^{-t}u}{(7+e^t)(1+u)}\right| \\
&= \left|\frac{340e^{-1}(5)}{(7+e^1)(1+5)}\right| \\
&= 7.570874104.
\end{aligned}
$$

For all $t \in [0,1]$, we have $a(t) \le \frac{1}{15} = m$ and

$$r = kL_1 m\sigma = 0.730676067.$$

Thus, all the hypotheses of Theorem 3.1 are satisfied and the BVP (4.1) has at least one positive solution.

2. Consider the nonlinear boundary value problem:

$$
\left.
\begin{aligned}
D^{\frac{3}{2}}u(t) + \frac{(1-t)}{2}\left[\frac{u^2}{3}+\frac{7t}{2}+1\right] = 0, \quad t \in (0,\ 1), \\
u(0) = 0, \quad \frac{3}{16}u(1) + \frac{5}{8}u'(1) = \int_0^1 \frac{3t}{20}\cdot u(t)dt.
\end{aligned}
\right\} \tag{4.2}
$$

Here, $\alpha = \frac{3}{2}$, $\gamma = \frac{3}{16}$, $\beta = \frac{5}{8}$, $a(t) = \frac{(1-s)}{2}$ and $p(t) = \frac{3t}{20}$.

$$
\begin{aligned}
&\int_0^1 G_1(s,s)a(s)ds \\
&\quad = \int_0^1 \frac{s^{\alpha-1}[w(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}]}{\mu\Gamma\alpha}\cdot\frac{1-s}{2}ds \\
&\quad = \frac{1}{\Gamma\alpha}\left[\frac{5}{16}\int_0^1 s^{\alpha-1}(1-s)^{\alpha-1}ds + \frac{3}{16}\int_0^1 s^{\alpha-1}(1-s)^{\alpha}ds\right]
\end{aligned}
$$

$$= \frac{1}{\Gamma\alpha}\left[\frac{5}{16}B(\alpha,\ \alpha)+\frac{3}{16}B(\alpha,\ \alpha+1)\right]$$

$$= \frac{13\sqrt{\pi}}{128} = 0.1800148442.$$

$$\therefore\quad L = \left(\int_0^1 G_1(s,s)a(s)ds\right)^{-1} = 5.555097439.$$

Also, we have

$$\sigma\int_{\frac{1}{4}}^{\frac{3}{4}} m(s)G_1(s,s)a(s)ds$$

$$= \sigma\int_{\frac{1}{4}}^{\frac{3}{4}}\left(\frac{(\frac{3}{4})^{\alpha-1}\left[w(1-s)^{\alpha-2}+\gamma(1-s)^{\alpha-1}\right]-\mu\left(\frac{3}{4}-s\right)^{\alpha-1}}{\mu\Gamma\alpha}\cdot\frac{(1-s)}{2}\right)ds$$

$$= \frac{\sigma}{\Gamma\alpha}\left[\frac{5}{16}\cdot\left(\frac{3}{4}\right)^{0.5}\int_{\frac{1}{4}}^{\frac{3}{4}}(1-s)^{\alpha-1}ds\ +\ \frac{3}{16}\cdot\left(\frac{3}{4}\right)^{0.5}\int_{\frac{1}{4}}^{\frac{3}{4}}(1-s)^{\alpha}ds\right.$$

$$\left.-\frac{1}{2}\int_{\frac{1}{4}}^{\frac{3}{4}}\left(\frac{3}{4}-s\right)^{\alpha-1}(1-s)ds\right]ds,$$

$$= \frac{25}{11\sqrt{\pi}}(0.0946347551+0.0263207804-0.0648181215) = 0.0719821459.$$

$$\therefore\quad M = \left(\sigma\int_{\frac{1}{4}}^{\frac{3}{4}} m(s)G_1(s,s)a(s)ds\right)^{-1} = 13.892333821.$$

Choose $r^* = \frac{2}{17}$ and $R = 5$. Then for $(t,u)\in[0,1]\times[0,5]$, we obtain

$$f(t,u) = \frac{u^2}{3}+\frac{7t}{2}+1 \le 12.8333333333 \le LR = 27.775487195.$$

Also, for $(t,u)\in[\frac{1}{4},\frac{3}{4}]\times(0,\frac{2}{17}]$, we obtain

$$f(t,u) = \frac{u^2}{3}+\frac{7t}{2}+1 \ge 3.6296136101 \ge Mr^* = 1.6343922142.$$

Hence, all the hypotheses of Theorem 3.2 are satisfied and the BVP (4.2) has at least one positive solution $u\in K$ such that $\frac{2}{17}\le\|u\|\le 5$.

5. Conclusion

In this work, the existence of positive solutions to the BVP (1.4) was established by the application of Krasnosel'skii fixed-point theorem in a cone. We discussed two practical examples to support our theoretical results.

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