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WEIBULL-LOGISTIC WITH EXPONENTIAL QUANTILE FUNCTION DISTRIBUTION

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ABSTRACT. The T-R{Y} is a T-X method of using a quantile function to generate probability distributions. It is a generalization of the T-X, Beta-X and many other families. This paper developed a 4-parameter Weibull-Logistic distribution using the T-R{Y} framework. This was achieved by combining the flexibility of the Weibull distribution with the two-parameter logistic distribution that has a location parameter, using the standard quantile function of the exponential distribution. Properties of the resulting distribution are extensively investigated, viz; rth non-central moments, quantiles, mode, survival function and hazard function. Plots of its density and cumulative distribution functions were presented to show its various shapes such as skewness or normal-type for some parameters' values. The Logistic, Weibull, Weibull-logistic and skew logistic distributions are sub-models of the 4-parameter Weibull-Logistic distribution. The distribution is also found to relate with the Weibull distribution through its quantile function, a general feature of the T-R{Y} family. The maximum likelihood method was used to estimate the parameters of the distribution. Simulation study was carried out to show the consistency of its maximum likelihood parameters estimated, and it showed that the shape of the distribution approaches symmetry as the sample size increases. The applicability of the distribution was demonstrated using real life dataset and the likelihood ratio test showed that the location parameter is significant. The proposed distribution would be very useful in areas where Weibull and Logistic distributions are not good fit. The new generator can also be used to generate many other distributions in this family.

1. INTRODUCTION

The need to transform and generalize basic distributions arises in many applied problems in physics, biology, finance, engineering, survival and environmental studies (Mikolaj 1972; Shakil and Kibria 2006; Kibria and Nadarajah2007; Cordeiro et al., 2008; Akarawak et al., 2013; Famoye et al., 2018; Ekum et al., 2020a, 2020b, 2021; Ogunsanya et al., 2021a, 2021b). Generalized distributions have been extensively studied by researchers in the field of distribution theory (Mudholkar and Srivastava 1993; Nadarajah 2005; Famoye et al., 2005; Alzaatreh et al., 2013; Aljaarah et el., 2014; Alzaatreh et al., 2014 and Alzaatreh et al., 2016). In recent years, families of distributions that have been studied include: Beta-X family (Eugene

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et al., 2002), the Transformed-Transformer family (Alzaatreh et al., 2013), T-X{Y} family (Aljarrah et al., 2014) and the T-R{Y} by (Ajarrah el al., 2014 and Alzaatreh et al., 2014).

On the Weibull-X family of distributions proposed by Alzaatreh and Ghosh (2015), the Weibull-Logistic with three parameters was proposed using the T-X framework. The three parameters Weibull-Logistic distribution may not capture all the variations in some dataset, especially for the adjustment of the location parameter, which value may determine the shift of the distribution along the horizontal axis. So, a Weibull-Logistic distribution with a location parameter may be necessary. If this is the case, it is then necessary to propose a 4-parameter Weibull-Logistic (4WLD) using a related but different framework, T-R{Y}.

In this paper, the 4-parameter Weibul-Logistic distribution is proposed using the Transformed-Transformer with quantile function framework, T-R{Y}. Let $F_R(x)$ be the cumulative distribution function (cdf) of any random variable R and $Q_Y[.]$ the quantile function of a random variable Y, and $f_T(t)$ the probability density function (pdf) of a random variable T defined on $[0,\infty)$ and its corresponding cumulative distribution function (cdf) is $F_T(t)$. The cdf of a generalized family of distributions is therefore given as

$$F_X(x) = \int_0^{Q_Y[F_R(x)]} f_T(t)dt = P\left\{T \le Q_Y\left[F_R(x)\right]\right\} = F_T\left\{Q_Y\left[F_R(x)\right]\right\}, \tag{1.1}$$

where X is the newly formed random variable derived from the framework, with cdf $F_X(x)$ .The family of distribution defined by (1.1) is called the Transformed-Transformer with quantile function (T-R{Y}) family, where the random variable R is being generalized by another random variable T. The corresponding pdf of the generalized distribution in (1.1) is given by

$$f_X(x) = f_T\left\{Q_Y\left[F_R(x)\right]\right\} \times Q_Y{}'\left[F_R(x)\right] \times F_R(x), \tag{1.2}$$

with some differentiation and re-arrangement equation (1.2) becomes

$$f_X(x) = f_R(x) \times \frac{f_T\left\{Q_Y\left[F_R(x)\right]\right\}}{f_Y\left\{Q_Y\left[F_R(x)\right]\right\}}. \tag{1.3}$$

The survival and hazard functions from equation (1.1) and (1.3) of the random variable X is given by

$$S(x) = 1 - F_X(x) \tag{1.4}$$

and

$$h(x) = \frac{f_X(x)}{1 - F_X(x)} \tag{1.5}$$

respectively.

The remaining sections of this paper are organized as follows. Section 2 showcased the materials and method, in which the 4-parameter Weibull-Logistic distribution was derived and some of its properties were presented. Section 3 is the results and discussion, where the simulation study and the application of the distribution were carried out; while the conclusion was done in Section 4.

2. MATERIALS AND METHODS

2.1. **The 4-Parameter Weibull–Logistic Distribution (4WLD)**

Let *T* follows the Weibull distribution with shape parameter *a* and scale parameter *b*, having probability density function (pdf) $f_T(x) = \frac{a}{b}\left(\frac{x}{b}\right)^{a-1} e^{-\left(\frac{x}{b}\right)^a}; \quad x \geq 0; a, b > 0$, and *Y* follows an exponential distribution with a standard quantile function $Q_Y(x)$ given by $Q_Y(F_R(x)) = -\log[1 - F_R(x)]$. Substituting $f_T(t)$ and $Q_Y\left[F_R(x)\right]$ in equation (1.1), we have

$$F_X(x) = \frac{a}{b^a}\int_0^{-\log(1-F_R(x))} t^{a-1} e^{-\left(\frac{t}{b}\right)^a} dt\,, \tag{2.1}$$

where $F_R(x)$ is the cumulative distribution function of random variable *R*. Integrating (2.1) and with proper substitutions gives

$$F_X(x) = 1 - \exp\left\{-\left(-\frac{\log(1 - F_R(x))}{b}\right)^a\right\} \tag{2.2}$$

Differentiating the cdf in (2.2), the probability density function of the generalized distributions is given by

$$f_X(x) = \frac{a}{b}\frac{f_R(x)}{1 - F_R(x)}\left(-\frac{\log(1 - F_R(x))}{b}\right)^{a-1} \exp\left\{-\left(-\frac{\log[1 - F_R(x)]}{b}\right)^a\right\}. \tag{2.3}$$

Thus, equations (2.2) and (2.3) are the cdf and pdf of a generalized Weibull-R-Exponential family. The R can be any random variable that follows any distribution.

2.1.1. **The pdf of the 4-Parameter Weibull–Logistic Distribution**

Recall (2.3), when *R* is taking to be Logistic random variable with probability density function (pdf), $f_R(x)$ and cumulative distribution function (cdf), $F_R(x)$ as respectively given in (2.4), we derive the pdf of the 4-parameter Weibull-Logistic distribution as follows:

$$f_R(x) = \frac{e^{-\left(\frac{x-\mu}{\sigma}\right)}}{\sigma\left(1+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right)^2}\,;\, F_R(x) = \left[1+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right]^{-1}. \quad (2.4)$$

Substituting (2.4) into equation (2.3) gives the pdf of the 4-parameter Weibull-Logistic distribution (4WLD) and it is given by:

$$f_X(x) = \frac{ae^{\left(\frac{x-\mu}{\sigma}\right)}}{b\sigma\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a-1}\exp\left\{-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}\right\}; \quad (2.5)$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $a,b,\sigma > 0$. Parameter *a* is the shape parameter, σ is the scale parameter, μ is the location parameter and *b* is rate parameter. Each of these parameters are very important in distribution theory. So, the three parameter Weibull logistic distribution defined by Alzaatreh *et al.* (2013) is a special case of this distribution.

To show that equation (2.5) is a pdf, we do the following:

$$\int_{-\infty}^{\infty} f_X(x)dx = 1 \quad (2.6)$$

Let $u = x-\mu$,and $x = u+\mu$, so that, $dx = du$ (2.7)

So, substitute (2.7) into (2.5) to have

$$\int_{-\infty}^{\infty}\frac{aexp\left(\frac{u}{\sigma}\right)}{b\sigma\left[1+exp\left(\frac{u}{\sigma}\right)\right]}\left\{\frac{log\left[1+exp\left(\frac{u}{\sigma}\right)\right]}{b}\right\}^{a-1} exp\left(-\left\{\frac{log\left[1+exp\left(\frac{u}{\sigma}\right)\right]}{b}\right\}^{a}\right)du = \int_{-\infty}^{\infty} f(u)\,du = 1 \quad (2.8)$$

Equation (2.8) completes the proof, since $f(u)$ is the pdf of the Weibull-Logistic distribution with three parameters defined by Alzaatreh and Ghosh (2015).

Special cases of the 4WLD

Lemma 2.1. The 4-parameter Weibull-Logistic distribution reduces to Logistic distribution if $a = b = 1$.

Proof

Put $a = b = 1$ in (2.5), we have,

$$f_X(x) = \frac{e^{\left(\frac{x-\mu}{\sigma}\right)}}{\sigma\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)^2} = \frac{e^{-\left(\frac{x-\mu}{\sigma}\right)}}{\sigma\left(1+e^{-\left(\frac{x-\mu}{\sigma}\right)}\right)^2}, \tag{2.9}$$

Equation (2.9) is true and completes the proof, based on the symmetric property of Logistic distribution. Equation (2.9) is the same as the pdf of Logistic distribution given in (2.2). Therefore, the 4-parameter Weibull-Logistic distribution generalizes the Logistic distribution.

Additive Noise

A good way of describing the location families is through the concept of additive noise. The additive noise is described thus: if θ is a constant and W is random noise with probability density $f_W(w)$, then, $X = \theta + W$ has probability density $f_\theta(x) = f_W(x-\theta)$ and its distribution is therefore part of a location family, where θ is a location parameter. So, W can be a random variable that follows a three-parameter Weibull-distribution, while X follows 4-parameter Weibull distribution. Thus, the 4WLD is a member of the location family of distribution.

Lemma 2.2. If X is a continuous random variable that follows the 4-parameter Weibull-Logistic distribution, then a random variable $W = X - \mu$ follows the three-parameter Weibull-Logistic distribution, defined by Alzaatreh and Ghosh (2015), if $w = x-\mu$.

Proof

Put $w = x-\mu$ in (2.5), we have,

$$g(w) = \frac{ae^{w/\sigma}}{b\sigma\left(1+e^{w/\sigma}\right)}\left(\frac{\log\left(1+e^{w/\sigma}\right)}{b}\right)^{a-1}\exp\left\{-\left(\frac{\log\left(1+e^{w/\sigma}\right)}{b}\right)^{a}\right\}, -\infty \le y \le \infty, a, b, \sigma > 0\,. \tag{2.10}$$

Equation (2.10) is the pdf of the three-parameter Weibull-Logistic distribution defined by Alzaatreh and Ghosh (2015). Therefore, the 4-parameter Weibull-Logistic distribution generalizes the three-parameter Weibull-Logistic distribution.

Thus, the Logistic distribution and the three-parameter Weibull-Logistic distribution are special cases of the 4-parameter Weibull-Logistic distribution.

2.1.2. **The cdf of the 4WLD**

The cumulative distribution function (cdf) of the proposed 4-parameter Weibull-Logistic Distribution (4WLD) is obtained as follows:

Recall (2.2)

$$F_X(x) = 1 - \exp\left\{ - \left(- \frac{\log(1 - F_R(x))}{b} \right)^a \right\}$$

Substitute the cdf of random variable R, that is, $F_R(x)$ into (2.2) to have

$$F_X(x) = 1 - \exp\left\{ - \frac{1}{b^a} \left(- \log\left\{ 1 - \left[1 + e^{-\left(\frac{x-\mu}{\sigma}\right)} \right]^{-1} \right\} \right)^a \right\}. \quad (2.11)$$

Reduce (2.11) to have the cdf of the proposed 4WLD in (2.12).

$$F_R(x) = 1 - \exp\left\{ - \left(\frac{\log\left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b} \right)^a \right\} \quad (2.12)$$

2.1.3. **Survival and Hazard Functions of 4WLD**

From here henceforth, we will write the pdf and cdf of the proposed random variable X, as just f(x) and F(x) respectively.

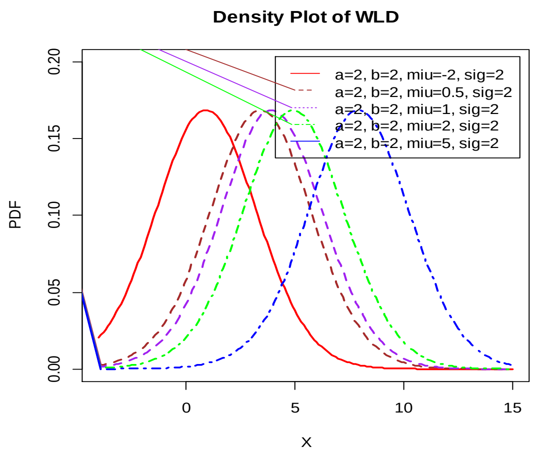
The survival function $S(x)$ of 4WLD is given by

$$S(x) = 1 - F(x) = \exp\left\{ - \left(\frac{\log\left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b} \right)^a \right\}, \quad (2.13)$$

and the hazard function h(x) is given by:

$$h(x) = \frac{f(x)}{S(x)} = \frac{a e^{\left(\frac{x-\mu}{\sigma}\right)}}{b\sigma\left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)}\right)} \left(\frac{\log\left(1 + e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b} \right)^{a-1}, \quad (2.14)$$

2.1.4. **Plots of 4WLD PDF, CDF, Survival and Hazard Functions**



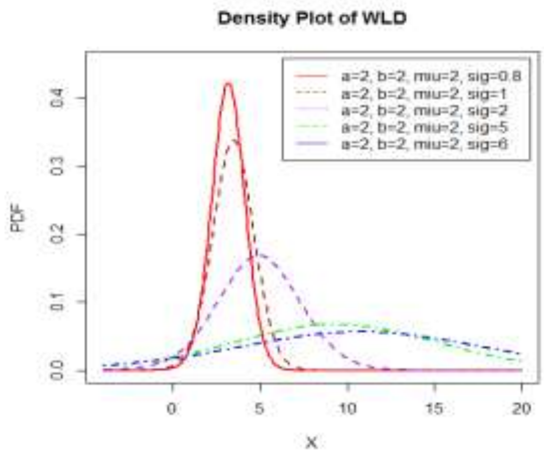
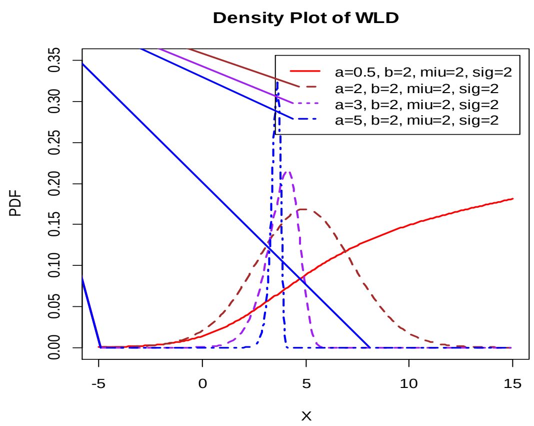


Figure 1. PDF plots of μ = -2,0.5,1,2,5

Figure 2. PDF plots of σ = 0.8,1,2,5,6



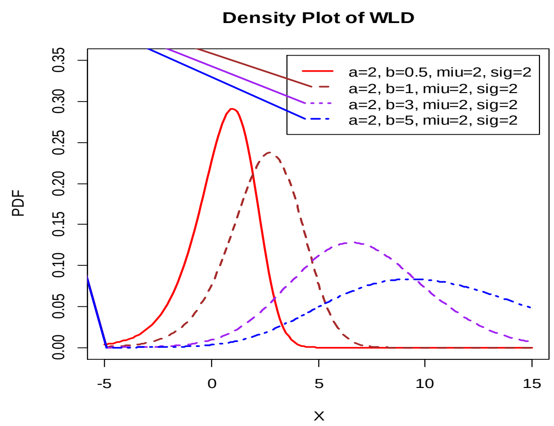
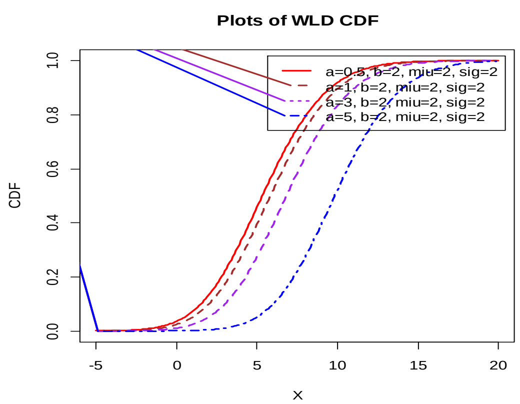


Figure 3. PDF plots of a = 0.5,2,3,5

Figure 4. PDF plots of b = 0.5,1,3,5



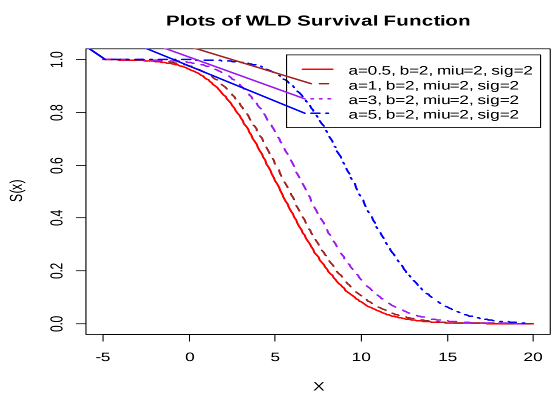


Figure 5. Plots of CDF

Figure 6. Plots of Survival Function

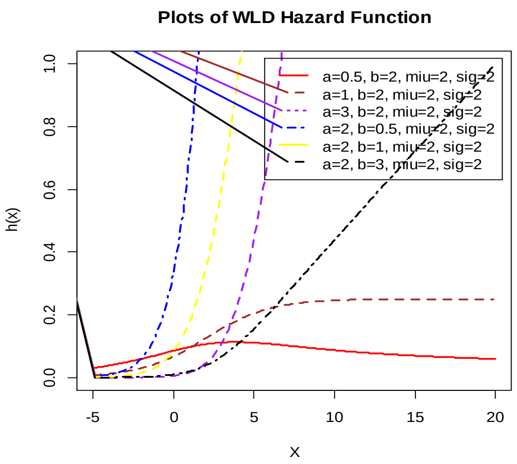


Figure 7. Plots of Hazard Function

The plots of the 4WLD pdf in Figures 1 to 4 clearly reveal the effect of the parameters on the distribution. It is observed that a affects the shape of the distribution, μ shifts the location while b and σ have effect on the scale. Hence, we infer that a is the shape parameter, μ is the location parameter while b and σ are the scale parameters.

The cdf plot presented in Figure 5 shows that as x increases the cdf increases and approaches 1. Similarly, the plot of survival function in Figure 6 is a decreasing function of x. The plots of hazard function (HF) as presented in Figure 7 reveal that for $a < 1$, HF increases to a maximum and then decreases, for $a = 1$, HF increases to a maximum and then remain constant, for $a > 1$, 4WLD exhibit increasing hazard rate.

2.2. **Some Properties of the 4WLD**

2.2.1. **The Limit of PDF and CDF of 4WLD**

The limiting behavior of 4WLD as values of X approaches $-\infty$ and ∞ is presented below:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{ae^{\left(\frac{x-\mu}{\sigma}\right)}}{b\sigma\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)} \left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a-1} \exp\left\{-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}\right\} = 0. \tag{2.15}$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{ae^{\left(\frac{x-\mu}{\sigma}\right)}}{b\sigma\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)} \left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a-1} \exp\left\{-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}\right\} = 0. \tag{2.16}$$

From (2.15) and (2.16), $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 0$. Therefore, it is expected that the pdf curve should rise to a maximum point and then drop.

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} 1 - \lim_{x\to\infty} \exp\left\{-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}\right\} = 1, \tag{2.17}$$

$$\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} 1 - \lim_{x\to-\infty} \exp\left\{-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}\right\} = 0, \tag{2.18}$$

Equations (2.17) and (2.18) shows that the cdf is a monotonic increasing function. The highest probability value is 1 and the lowest is 0. It follows that F(x) converges to one as $x \to \infty$, and converges to zero as $x \to -\infty$. Hence, $0 \le F(x) \le 1$.

2.2.2. **The r^{th} Non-Central Moment of 4WLD**

Theorem 2.1. The raw moment of the proposed 4-parameter Weibull-Logistic distribution is the weighted sum of the raw moment of the 3-parameter Weibull-distribution proposed by Alzaatreh and Gosh (2015).

Proof

The r^{th} raw moment of 4 WLD is given as:

$$E\left(X^{r}\right) = \int_{-\infty}^{\infty} \frac{x^{r} a e^{\left(\frac{x-\mu}{\sigma}\right)}}{b\sigma\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)} \left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a-1} \exp\left\{-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}\right\} dx, \tag{2.19}$$

Let $z = x - \mu$, dx = dz and $x = z + \mu$

$$E\left(X^r\right)=\int_{-\infty}^{\infty}\frac{(z+\mu)^r ae^{z/\sigma}}{b\sigma\left(1+e^{z/\sigma}\right)}\left(\frac{\log\left(1+e^{z/\sigma}\right)}{b}\right)^{a-1}\exp\left\{-\left(\frac{\log\left(1+e^{z/\sigma}\right)}{b}\right)^{a}\right\}dz \tag{2.20}$$

$$E\left(X^r\right)=\int_{-\infty}^{\infty}\sum_{i=0}^{r}\binom{r}{l}\mu^{r-l}\frac{z^l ae^{z/\sigma}}{b\sigma\left(1+e^{z/\sigma}\right)}\left(\frac{\log\left(1+e^{z/\sigma}\right)}{b}\right)^{a-1}\exp\left\{-\left(\frac{\log\left(1+e^{z/\sigma}\right)}{b}\right)^{a}\right\}dz \tag{2.21}$$

$$E\left(X^r\right)=\int_{-\infty}^{\infty}\sum_{i=0}^{r}\binom{r}{l}\mu^{r-l}z^l f(z)dz=\sum_{i=0}^{r}\binom{r}{l}\mu^{r-l}\int_{-\infty}^{\infty}z^l f(z)dz \tag{2.22}$$

$$E\left(X^r\right)=\sum_{i=0}^{r}\binom{r}{l}\mu^{r-l}E(Z^l) \tag{2.23}$$

Where Z is a 3-parameter Weibull-Logistic distribution proposed by Alzaatreh and Ghosh (2015), and E(z^l) is the l^{th} raw moment of the 3-parameter Weibull-Logistic distribution. Equation (2.23) completes the proof.

The l^{th} raw moment of the 3-parameter Weibull-Logistic distribution proposed by Alzaatreh and Ghosh (2015), is given by

$$E\left(Z^l\right)=ab^l\sum_{j}^{l}\sum_{k_1,k_2,\ldots k_j=1}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{l}{j}\frac{\Gamma(ia+a+l-j)}{i!l^{a(i+1)}\tau_j S_j^{ia+a+l-j}} \tag{2.24}$$

where $\tau_j = k_1, k_2, \ldots k_j$ and $S_j = k_1 + k_2 + \ldots + k_j$

See the proof of (2.24) on pages 176 and 177 by Alzaatreh and Ghosh (2015).

The rth raw moment of the 4-parameter Weibull-Logistic distribution is therefore given by substituting (2.24) into (2.23) to have

$$E\left(X^r\right)=\sum_{i=0}^{r}\binom{r}{l}\mu^{r-l}ab^l\sum_{j}^{l}\sum_{k_1,k_2,\ldots k_j=1}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{l}{j}\frac{\Gamma(ia+a+l-j)}{i!l^{a(i+1)}\tau_j S_j^{ia+a+l-j}} \tag{2.25}$$

$$E\left(X^r\right)=\sum_{i=0}^{r}\sum_{j}^{l}\sum_{k_1,k_2,\ldots k_j=1}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\binom{r}{l}\binom{l}{j}\mu^{r-l}ab^l\frac{\Gamma(ia+a+l-j)}{i!l^{a(i+1)}\tau_j S_j^{ia+a+l-j}} \tag{2.26}$$

So, (2.26) is the rth raw moment of the proposed 4-parameter Weibull-Logistic distribution. Hereafter, the 4-parameter Weibull-Logistic distribution logistic distribution will be referred to as 4WLD, while the 3-parameter Weibull-Logistic distribution by Alzaatreh and Ghosh (2015) will be referred to as WL(a, σ, b).

Note that from (2.26), if $r = 1$ and $l = 1$, we have the mean of 4WLD

$$E(X)=\binom{1}{0}\mu^{1-0}E(Z)+\binom{1}{1}\mu^{1-1}E(Z)=\mu E(Z)+E(Z)=(\mu+1)E(Z)$$

where E(Z) is the mean of three-parameter Weibull-Logistic distribution proposed by Alzaatreh and Ghosh (2015).

2.2.3. **Quantile Function**

Theorem 2.2. The quantile function of 4WLD exist and it is given by

$$x=\mu+\sigma\ln\left\{e^{b[-\ln(1-p)]^{1/a}}-1\right\}$$

Proof

Given the cdf of 4WLD defined in (2.12) as

$$F(x)=1-\exp\left\{-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}\right\}$$

We make x the subject of the formula. Let $F(x) = p$.

$$\exp\left\{-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}\right\}=1-p \tag{2.27}$$

Take the log of (2.27) to have

$$-\left(\frac{\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)}{b}\right)^{a}=\ln(1-p) \tag{2.28}$$

Multiply (2.28) by minus and solve further to have

$$\log\left(1+e^{\left(\frac{x-\mu}{\sigma}\right)}\right)=b\left[-\ln(1-p)\right]^{1/a} \tag{2.29}$$

Take exponential of (2.29) and solve to have

$$e^{\left(\frac{x-\mu}{\sigma}\right)} = e^{b[-\ln(1-p)]^{1/a}} - 1 \tag{2.30}$$

Take the log of (2.30) to have

$$\frac{x-\mu}{\sigma} = \ln\left\{e^{b[-\ln(1-p)]^{1/a}} - 1\right\} \tag{2.31}$$

Solve for *x* in (2.31) to have

$$x = \mu + \sigma \ln\left\{e^{b[-\ln(1-p)]^{1/a}} - 1\right\} \tag{2.32}$$

Thus, the quantile function of 4WLD exists and (3.18) completes the proof.

Median

The median of 4WLD is obtained by letting $p = 0.5$in (2.32)

$$x = \mu + \sigma \ln\left\{e^{b[-\ln(1-0.5)]^{1/a}} - 1\right\} \tag{2.33}$$

Solve (2.33) further to have (2.34)

$$x = \mu + \sigma \ln\left\{e^{b[-\ln(0.5)]^{1/a}} - 1\right\} \tag{2.34}$$

$$x = \mu + \sigma \ln\left\{e^{b[\ln(2)]^{1/a}} - 1\right\} \tag{2.35}$$

Thus, (2.35) is the median of 4WLD.

Let the pdf of Weibull distribution be

$$f(y) = \frac{a}{b}\left(\frac{y}{b}\right)^{a-1} exp\left[-\left(\frac{y}{b}\right)^{a}\right] \tag{2.36}$$

Remark 1. If Y is a Weibull distributed random variable, then from (2.36), $X = \mu + \sigma \log\left[e^{Y} - 1\right]$ is a 4-parameter Weibull-Logistic random variable, using a simple transformation technique.

The simple transformation technique $f(x) = f(y)|dx|$, where Y follows a Weibull distribution and *X* follows a 4WLD. It is obvious that $y = log\left[1 + exp\left(\frac{x-\mu}{\sigma}\right)\right]$ and $dx = \frac{\frac{1}{\sigma}exp\left(\frac{x-\mu}{\sigma}\right)}{1+exp\left(\frac{x-\mu}{\sigma}\right)}$. Substituting y and dx into (2.36) appropriately will produce the desired result.

2.2.4. **Modal Function**

The mode is that value of *x* which satisfies

$$\frac{df(x)}{dx} = 0 \text{ or} \frac{dlnf(x)}{dx} = 0. \quad (2.37)$$

Recall the pdf of 4WLD in (2.5) given as

$$f(x) = \frac{aexp\left(\frac{x-\mu}{\sigma}\right)}{b\sigma\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}\left\{\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{b}\right\}^{a-1} exp\left(-\left\{\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{b}\right\}^{a}\right)$$

Take the log of the pdf to have

$$lnf(x) = lna + \left(\frac{x-\mu}{\sigma}\right) - lnb - ln\sigma - ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right] + (a-1)ln\left\{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\} - (a-1)lnb - \left\{\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{b}\right\}^{a} \quad (2.38)$$

Take the first derivative of (2.39) and equate the result to zero to arrive at

$$0 = \frac{1}{\sigma} - \frac{exp\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]} + \frac{(a-1)exp\left(\frac{x-\mu}{\sigma}\right)}{\sigma ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]} + a\left\{\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{b}\right\}^{a-1}\left\{\frac{exp\left(\frac{x-\mu}{\sigma}\right)}{\sigma b\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}\right\} \quad (2.39)$$

Solve (2.39) further to arrive at (2.40)

$$\frac{x-\mu}{\sigma} = ln\left\{-\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{(a-1)+\frac{a}{b^a}\left\{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^a}\right\} \quad (2.40)$$

From (2.40) make *x* from the left hand side to be the subject of the formula to have

$$x = \mu + \sigma ln\left\{-\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{(a-1)+\frac{a}{b^a}\left\{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]\right\}^a}\right\} \quad (2.41)$$

Equation (2.41) is mode of the 4WLD but it is not in a closed form. We have to proffer an alternative solution by multiplying both sides of (2.41) by *n*.

$$nx = n\mu + \sigma\sum_{i=1}^{n} ln\left\{-\frac{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}{(a-1)+\frac{a}{b^a}\left\{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]\right\}^a}\right\} \quad (2.42)$$

So, dividing (2.42) by *n*, we arrive at (2.43)

$$x = \mu + \frac{\sigma}{n}\sum_{i=1}^{n} ln\left\{-\frac{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}{(a-1)+\frac{a}{b^a}\left\{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]\right\}^a}\right\} \tag{2.43}$$

where n is the number of observation, which can be gotten from the data, a, b, μ and σ are parameters estimated using the maximum likelihood estimator (MLE). So, the mode of 4WLD can be computed with (2.43).

2.2.5. **Likelihood Ratio Test**

In statistics, the likelihood-ratio test (LRT) measures the goodness of fit of two competing statistical models based on the ratio of their likelihoods, where one is found by maximization over the entire parameter space and another found after imposing some constraint. The 4WLD can be reduced to the three parameter Weibull-logistic distribution proposed by Alzaatreh and Ghosh (2015) if a constraint is imposed on the location parameter. The Likelihood Ratio Test (LRT) for the 4WLD and Weibull-logisticis derived thus:

Given the pdf in (2.5) as

$$f_1(x) = \frac{a exp\left(\frac{x-\mu}{\sigma}\right)}{b\sigma\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}\left\{\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{b}\right\}^{a-1} exp\left(-\left\{\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{b}\right\}^{a}\right);\ a, b, \sigma, \mu > 0, -\infty < x < \infty,$$

and this is reduced to the pdf of the Weibull-logistic developed by Alzaatreh and Ghosh (2015) if the constraint $\mu = 0$ then we have

$$f_0(x) = \frac{a exp\left(\frac{x}{\sigma}\right)}{b\sigma\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}\left\{\frac{ln\left[1+exp\left(\frac{x}{\sigma}\right)\right]}{b}\right\}^{a-1} exp\left(-\left\{\frac{ln\left[1+exp\left(\frac{x}{\sigma}\right)\right]}{b}\right\}^{a}\right);\ a, b, \sigma > 0, -\infty < x < \infty.$$

By definition, the LRT is given by

$$LRT = -2log\left(\frac{L_0}{L_1}\right)$$

where L_0 and L_1 are the likelihood functions of the Weibull-logistic distribution and the 4WLD respectively.

$$LRT = -2log\left(\frac{\frac{a^n exp\left(\sum\frac{x_i}{\sigma}\right)}{b^n\sigma^n\prod\left[1+exp\left(\frac{x_i}{\sigma}\right)\right]}\prod\left\{\frac{ln\left[1+exp\left(\frac{x_i}{\sigma}\right)\right]}{b}\right\}^{a-1} exp\left(-\sum\left\{\frac{ln\left[1+exp\left(\frac{x_i}{\sigma}\right)\right]}{b}\right\}^{a}\right)}{\frac{a^n exp\left(\sum\frac{x_i-\mu}{\sigma}\right)}{b^n\sigma^n\prod\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}\prod\left\{\frac{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}{b}\right\}^{a-1} exp\left(-\sum\left\{\frac{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}{b}\right\}^{a}\right)}\right) \quad (2.44)$$

$$LRT = -\frac{2n\mu}{\sigma} - \frac{2}{b^a}\sum\left\{log\left[1+e^{\frac{1}{\sigma}(\sum x_i - n\mu)}\right]\right\}^a + \frac{2}{b^a}\sum\left\{log\left[1+e^{\left(\frac{1}{\sigma}\sum x_i\right)}\right]\right\}^a - 2\sum log\frac{\left\{ln\left[1+e^{\left(\frac{1}{\sigma}\sum x_i\right)}\right]\right\}^{a-1}}{\left[1+e^{\left(\frac{1}{\sigma}\sum x_i\right)}\right]}$$
$$-2\sum log\frac{\left[1+e^{\frac{1}{\sigma}(\sum x_i - n\mu)}\right]}{\left\{ln\left[1+e^{\frac{1}{\sigma}(\sum x_i - n\mu)}\right]\right\}^{a-1}}$$

Equation (2.44) is asymptotically chi-square with $\nu = k_1 - k_0$, where k_1 and k_0 are the number of parameters of 4WLD and Weibull-logistic distribution, that is, $LRT \sim \chi^2_{\alpha,\nu}$. Note that α is the level of significance and ν is the degrees of freedom.

2.2.6. **Maximum Likelihood Estimation (MLE) for Parameters Estimates**

The maximum likelihood estimates of the 4WLD parameters is derived thus:

Given the pdf in (2.5) as

$$f(x) = \frac{aexp\left(\frac{x-\mu}{\sigma}\right)}{b\sigma\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}\left\{\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{b}\right\}^{a-1} exp\left(-\left\{\frac{ln\left[1+exp\left(\frac{x-\mu}{\sigma}\right)\right]}{b}\right\}^{a}\right)$$

Take the likelihood of the pdf in (2.5) to arrive at

$$L[f(x)] = \frac{a^n exp\left(\sum\frac{x_i-\mu}{\sigma}\right)}{b^n\sigma^n\prod\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}\prod\left\{\frac{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}{b}\right\}^{a-1} exp\left(-\sum\left\{\frac{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}{b}\right\}^{a}\right) \quad (2.45)$$

Take the log of the likelihood in (2.45) to arrive at the log likelihood function, l given by

$$l = nlna + \sum_{i=1}^{n}\left(\frac{x_i-\mu}{\sigma}\right) - nlnb - nln\sigma - \sum_{i=1}^{n} ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right] + (a-1)\sum_{i=1}^{n} ln\left\{\frac{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]}{b}\right\} - b^{-a}\sum_{i=1}^{n}\left\{ln\left[1+exp\left(\frac{x_i-\mu}{\sigma}\right)\right]\right\}^{a} \quad (2.46)$$

To estimate the values of a, b, μ and σ, we differentiate equation (2.46) partially with respect to each of the parameters and equate each to zero and solve for each parameter.

3. RESULTS

3.1. **Density Plots of WLD Simulated Data**

The values of the parameters are estimated using numerical method with R program (maxLik). The maxLik package requires initial value for each parameter. The initial value for the parameter σ can be obtained by assuming that X follows the logistic distribution with parameter σ and μ. Thus, the initial value for σ (Johnson *et al.*, 1994) is $\sigma_0 = \sqrt{3}s_x/\pi$, where s_x is the sample standard deviation of X. Using Remark 1, the initial values for the parameters a and b can be obtained by considering $x_i = \mu + \sigma log(e^{y_i} - 1), i = 1, 2, \cdots, n$ for a random sample of size n drawn from the Weibull distribution with parameters a and b. Thus, the initial values for the parameters a and b (Johnson *et al.*, 1994) are $a_0 = \frac{\pi}{\sqrt{6}S_{logy_i}}$ and $b_0 = exp\left(\bar{x}logy_i + {\delta}/{a_0}\right)$, where S_{logyi} and $\bar{x}$logyi are the sample standard deviation and the sample mean for logy_i, and δ is the Euler gamma constant which approximately equals 0.57722.

A simulation study is done for evaluating the performance of the MLE parameters of the 4WLD. We first simulated Weibull random variates, say y, then we simulated 4WLD variates x. It is easy to generate Weibull random variates using R inbuilt codes, so the 4WLD was then generated from the Weibull variates using the transformation in Remark 1. We considered several parameter choices for a, b, μ and σ. 4 different sample sizes n = 50, 100, 500 and 1000 were considered. For each parameter combination, we generate a random sample $y_1, y_2, \cdots, y_n$ from Weibull distribution with parameters a and b. The maxLik package in R was used to achieve the iterative process 500 times in order to find the means and standard deviations of the parameter estimates.

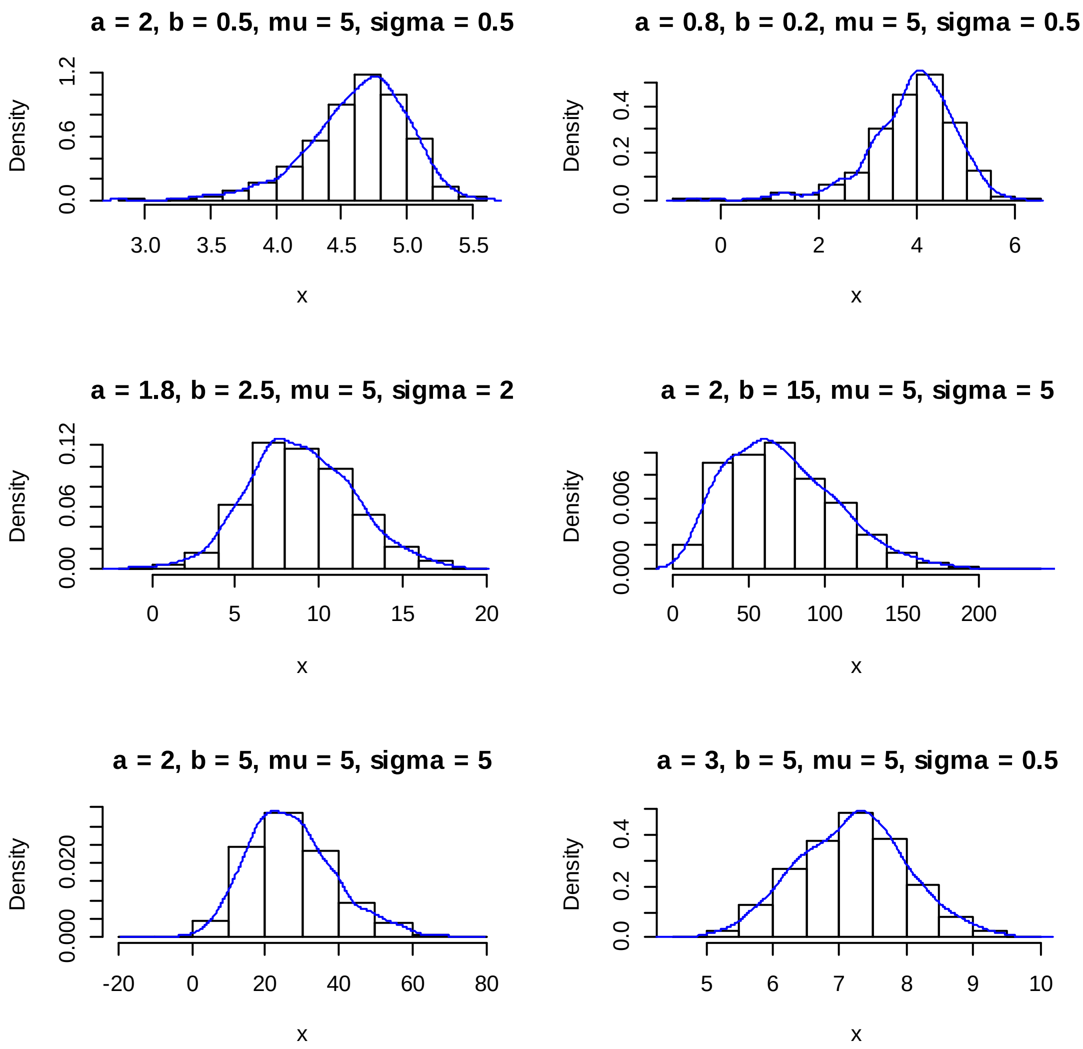


Figure 8. Histogram with Density for 4WLD Simulated Data

The histograms and density plots of the simulated data for different parameter values at $N = 1000$ are presented in Fig. 8. The histograms show that the distribution can be positively skewed (the two middle histograms), negatively skewed (the first two histograms) or asymptotically normal (the last two histograms), depending on the parameter values.

3.1.2. An Application to Data

To illustrate its applicability, the 4WLD is applied to a data set obtained from Smith and Naylor (1987) on the strengths of 1.5 cm glass fibres measured at the National Physical Laboratory in England. The 4WLD is compared with the distributions in Alzaatreh *et al.* (2015)

of which Weibull-logistic distribution emerged as the most favourable distribution. The Exploratory Data Analysis (EDA) of the data, its histogram and Q-Q plot are presented in Table 1; the maximum likelihood estimates, log-likelihood, Akaike Information Criterion (AIC), the Kolmogorov-Smirnov test statistic (K-S), and the K-S p-value for 4WLD, Weibull-logistic, skew logistic with location parameter, Weibull and logistic distributions are given in Table 2. The AIC = $2k - 2\log(L)$, where k is the number of estimated parameters in the model and L maximum value of the likelihood function for the model. The LRT result is given in Table 3.

Table 1 shows the exploratory data analysis of the 51 measured strengths of 1.5 cm glass fibres. Table 1 shows that the data has a mean of 1.4418 with a standard deviation of 0.3269, resulting to a coefficient of variation of 0.2267. The data is negatively skewed and has a kurtosis of 3.8025.

Table 1. Data Exploration

Statistics	Values	Histogram
N	51	
Mean	1.4418	
Variance	0.1068	
Standard Dev.	0.3269	
Skewness	-0.6440	
Kurtosis	3.8025	
C.V	0.2267	

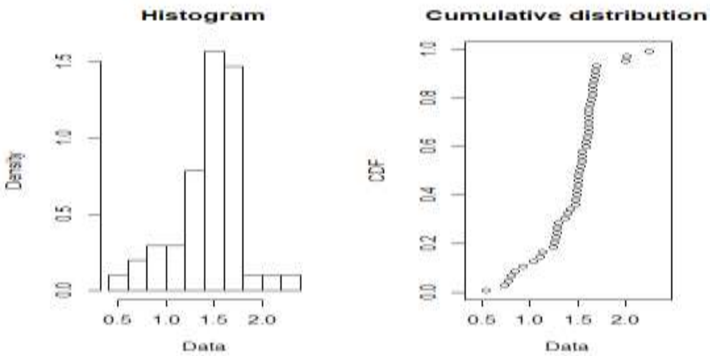


Table 2. Parameter Estimates for the Glass Fibre Data

Distribution	Parameter Estimates	Log likelihood	AIC	K-S	K-S p-value
4WLD	$\hat{\mu}$ = 1.440 $\hat{\sigma}$ = 0.3280 $\hat{a}$ =0.1444 $\hat{b}$ = 0.000081	-0.1286	7.7428	0.0321	1.0000
Weibull-logistic	$\hat{c}$ = 10.9867 $\hat{\gamma}$ = 1.2397 $\hat{\theta}$ = 1.3909	-2.0928	10.1856	0.0417	1.0000
Skew logistic with location	$\hat{\alpha}$ = -2.8240 $\hat{\beta}$ = 0.2210 $\hat{m}$ = 1.4170	-2.2925	10.5850	0.0516	0.9997
Weibull	$\hat{c}$ = 5.234792 $\hat{\gamma}$ = 1.564321	-13.7431	31.4862	0.2353	0.1188
Logistic	$\hat{\theta}$ = 0.7425	-75.4164	152.8328	0.6544	0.0000

Table 3. Log-Likelihood Ratio Test Result Using the Glass Fibre Data

Distribution	Log likelihood	AIC	LRT	$\chi^2_{0.05,1}$
4WLD	-0.1286	7.7428	3.9284	3.841
Weibull-logistic	-2.0928	10.1856		

4. DISCUSSION

In this study, the T-R{Y} framework was used to develop a novel four-parameter distribution called Weibull-logistic with exponential quantile function (4WLD) distribution. Different properties of the proposed distributions were derived such as its cdf, pdf, survival, and hazard rates with their respective curves at different parameter values. The relationship between the proposed distribution with logistic and three-parameter Weibull-logistic distributions. The proof to show that the PDF is a proper one was established through proofs and can also be seen from the PDF and CDF plots. Other properties were derived, such as, the limit properties of the PDF and CDF, the non-central moments, quantile function, modal function, likelihood ratio test. The Maximum Likelihood Estimation (MLE) method was used to estimate the parameters of the new distribution. Histograms with density plots were used to show various shapes of the distribution, which can be negatively skewed (a = 2, b = 0.5, μ = 5 and σ = 0.5) or positively skewed (a = 2, b = 15, μ = 5 and σ = 5), asymptotically normal (a = 2, b = 5, μ = 5 and σ = 5).

The applicability of the new 4WLD was applied to the data obtained from Smith and Naylor (1987) on the strengths of 1.5 cm glass fibres measured at the National Physical Laboratory in England, and the results were compared with the distributions in Alzaatreh *et al.* (2015) of which 4WLD emerged as the most favourable distribution. The EDA using the histogram and Q-Q plots were presented in Table 1; the maximum likelihood estimates, log-likelihood, Akaike Information Criterion (AIC), the Kolmogorov-Smirnov test statistic (K-S), and the K-S p-value for 4WLD, Weibull-logistic, skew logistic with location parameter, Weibull and logistic distributions were presented in Table 2, and the LRT result was presented in Table 3. Table 1 showed 51 measured strengths of 1.5 cm glass fibres were observed with mean of 1.4418 with a standard deviation of 0.3269, resulting to a coefficient of variation of 0.2267. The data is negatively skewed and has a kurtosis of 3.8025.

Table 2 showed that all the competing distributions show give adequate fit to the glass fibre data, except for logistic distribution, as shown in the K-S p-value. Using the selection criteria, log-likelihood and AIC, it is seen that the 4WLD outperformed the other competing distributions. Table 2 showed that the location parameterμ significantly improves the Weibull-logistic distribution fits for the strength of 1.5 glass fibres data. Table 3 showed that the LRT statistic is greater than the Chi-square critical value. This showed that the 4WLD is significantly better than the Weibull-logistic distribution, meaning that the location parameter has effect on the distribution base on the data of interest (strength of 1.5 glass fibres data).

5. CONCLUSION

The 4-parameter Weibull-Logistic Distribution (4WRD) has been proposed in this paper using the T-R{Y} framework proposed by Aljarrah *et al.* (2014). The proposed distribution is an improvement to the Weibull-logistic distribution by adding a location parameter. The importance of adding a location parameter is demonstrated using the likelihood ratio test with the glass fibres data. Several plots have been presented to show the effects of the parameters and it was found that the distribution is unimodal, skewed and normal-type for some values of the parameters. Expressions for some statistical properties of the proposed distribution have also been derived. We showed that the proposed distribution is a member of the location family of distribution, with some advantages. Simulation results show that as the sample size increases the shape of 4WRD approaches symmetry, this is backed up by the central limit theorem. Furthermore, the study shows that 4WRD has a relationship with the Logistic distribution, Weibull distribution and as well as the three parameter Weibull-Logistic distribution.

Finally, we illustrated the applicability of the 4WLD to data and compared the result with Weibull-logistic, skew logistic with location parameter, Weibull and logistic distributions using log-likelihood, AIC and K-S statistic criteria. The performance of the proposed 4WLD is better when compared with the performance of the competing distributions using the glass fibres data. Also, the LRT showed that the 4WLD performed better at 5% level of significance using the glass fibres data. Thus, the 4WRD can be used to model data that are not well fitted by Weibull-logistic, skew logistic with location parameter, logistics and Weibull distributions. Also, it should be noted that all the competing distributions are sub-model of the proposed 4WRD.

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