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SOLVING PARTIAL INTEGRO-DIFFERENTIAL EQUATIONS USING LAPLACE-ADOMIAN DECOMPOSITION METHOD

ABDULWAHAB EBOSETALE SALIH*, TIMOTHY TERFA ASHEZUA AND TERSOO LUGA

ABSTRACT. In this paper, the Laplace-Adomian decomposition method (LADM) is applied in obtaining numerical solutions to partial integro-differential equations. In order to verify the accuracy and efficiency of the LADM, the numerical results generated are compared with the variational iteration method and differential transform method. The comparison with the above methods shows that the LADM achieves better accuracy than the others.

1. INTRODUCTION

Various physical phenomena can be modeled by ordinary/partial differential equations. The differential operator (ordinary or partial) does not help in modeling some properties such as memory and hereditary properties due to the local nature of the differential operator. Amongst the best remedies to overcome such drawback is to introduce integral term to the model. The differential equation along with the integral of unknown function gives rise to an integro-differential equation (IDE) or a partial integro-differential equation (PIDE) [1-4].

Literature Review. Partial integro-differential equation has been applied in various fields. In [5] the variational iteration method (VIM) was applied to solve PIDEs arising heat conduction of materials with memory and viscoelasticity. In [6] the two-dimensional transform method was also applied to obtain the approximate solution of PIDEs with convolution kernel which occur naturally in various field of science and engineering.

The Laplace-Adomian decomposition method (LADM) has shown to be an efficient method, due to the fact that it is a hybrid technique developed by the combination of two well-known methods namely Laplace transform method and Adomian decomposition method. It has been successfully applied to approximate the third order Dispersive Fractional Partial Differential Equation, General fisher's Equation and so on [7-10].

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In this research, we apply the Laplace–Adomian decomposition method to Partial integro-differential equations and obtain their approximate exact solutions. Since the method has shown to be very effective and powerful for solving various kinds of linear and nonlinear ordinary and partial differential equations.

2. MATERIALS AND METHODS

2.1 Partial Differential Equation using LADM

In this section we will discuss the basic concept of Laplace-Adomian Decomposition Method (LADM) as developed in [11]. We consider the following general differential equation

$$Lu(x,t) + Nu(x,t) + Ru(x,t) = g \tag{2.1}$$

with initial condition

$$\mathrm{u}(\mathrm{x},\mathrm{t}) = \mathrm{f}(\mathrm{x}) \tag{2.2}$$

where L is the linear deferential operator of higher order which is easily invertible, u is an unknown function, N is the nonlinear operator, R is the remaining linear part and g is any given function.

The method consists of first applying the Laplace transform to both sides of equation (2.1)

$$\mathcal{L}[Lu(x,t)] + \mathcal{L}[Nu(x,t)] + \mathcal{L}[Ru(x,t)] = \mathcal{L}[g]. \tag{2.3}$$

Using the differentiation property of Laplace transform and applying the initial condition in equation (2.2) we get :

$$\mathcal{L}[u] = \frac{f(x)}{s} + \frac{1}{s}\mathcal{L}[g] - \frac{1}{s}\mathcal{L}[Nu] - \frac{1}{s}\mathcal{L}[Ru]. \tag{2.4}$$

The LADM defines the solution $\mathrm{u}(\mathrm{x},\mathrm{t})$ by the infinite series

$$u(x,t) = \sum_{n=0}^{\infty} u_n \tag{2.5}$$

The nonlinear term N is usually represented by an infinite series of the Adomian polynomials

$$N(x,t) = \sum_{n=0}^{\infty} A_n \tag{2.6}$$

Where A_n is defined thus

$$A_n = \frac{1}{n!}\left[\frac{d^n}{d\lambda^n}\left[N\sum_{i=0}^{\infty}(\lambda^i u_i)\right]\right]_{\lambda=0} \tag{2.7}$$

Substituting (5) and (6) into (4), gives

$$\mathcal{L}[\sum_{n=0}^{\infty} u_n] = \frac{f(x)}{s} + \frac{1}{s}\mathcal{L}[g] - \frac{1}{s}\mathcal{L}[\sum_{n=0}^{\infty} A_n] - \frac{1}{s}\mathcal{L}[R(\sum_{n=0}^{\infty} u_n)] \quad (2.8)$$

Applying the linearity of the Laplace transform, we can define the following recursive formula:

$$\mathcal{L}[u_0] = \frac{f(x)}{s} + \frac{1}{s}\mathcal{L}[g] \quad (2.9)$$

$$\mathcal{L}[u_1] = -\frac{1}{s}\mathcal{L}[R(u_0) - \frac{1}{s}\mathcal{L}[A_0] \quad (2.10)$$

$$\mathcal{L}[u_2] = -\frac{1}{s}\mathcal{L}[R(u_1) - \frac{1}{s}\mathcal{L}[A_1] \quad (2.11)$$

$$\mathcal{L}[u_3] = -\frac{1}{s}\mathcal{L}[R(u_2) - \frac{1}{s}\mathcal{L}[A_2] \quad (2.12)$$

In general, for $n \geq 0$, the recursive relations are given by

$$\mathcal{L}[u_{n+1}] = -\frac{1}{s}\mathcal{L}[R(u_n)] - \frac{1}{s}\mathcal{L}[A_n] \quad (2.13)$$

Applying the inverse Laplace transform, we can evaluate u_n and get the solution of the infinite series as

$$u(x,t) = \sum_{n=0}^{k} u_n\,(x,t) \quad (2.14)$$

where

$$\lim_{k\to\infty} \sum_{n=0}^{k} u_n\,(x,0) = u(x,t)$$

2.2 Partial Integro Differential equation (PIDE) [6]

The general form of PIDE with convolution kernel is,

$$\sum_{i=1}^{m} a_i \left(\frac{\partial^i u}{\partial x^i}\right) + \sum_{i=1}^{n} b_i \left(\frac{\partial^i u}{\partial t^i}\right) + cu(x,t) + \sum_{i=1}^{r} d_i \int_0^t K_i\,(t-y) \left(\frac{\partial^i u}{\partial x^i}\right) dy + f(x,t) = 0$$

(With prescribed conditions)

where a_i, b_i, c and d_i are constant or the function of x alone. And $f(x,t), K_i(t-y)$ are known functions.

2.3 Solution to Partial Integro Differential equations using LADM

Consider the following partial integro differential equations of the second kind given by

$$u(x) = f(x) + \lambda \int_a^x k(s,t)\left[L\big(u(s)\big) + N\big(u(s)\big)\right]ds, \lambda \neq 0 \qquad (2.15)$$

Where $f(x)$ is a given function, λ is a parameter, $k(x,t)$ is the Kernel, $L\big(u(x)\big)$ and $N\big(u(x)\big)$ are linear and nonlinear operator respectively. Assume that the solution

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (2.16)$$

and

$$N\big(u(x)\big) = \sum_{n=0}^{\infty} A_n \qquad (2.17)$$

Substitute equation (2.16) and (2.17) into (2.15) we get

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^x k(s,t)\left[L\left(\sum_{n=0}^{\infty} u_n(x)\right) + \sum_{n=0}^{\infty} A_n\right] ds \qquad (2.18)$$

This gives the following scheme

$$u_0 = f(x)$$

$$u_1 = \lambda \int_a^x k(s,t)\,[L(u_0(s)) + A_0]\,ds$$

$$u_2 = \lambda \int_a^x k(s,t)\,[L(u_1(s)) + A_1]\,ds$$

$$u_{n+1} = \lambda \int_a^x k(s,t)\,[L(u_n(s)) + A_n]\,ds\,, n = 0,1.2, \dots$$

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$$u_{n+1} = \lambda \int_{a}^{x} k(s,t)\,[L(u_n(s)) + A_n]\, ds\ , n = 0,1.2, \dots$$

3. RESULTS

In this section the Laplace-Adomian decomposition method (LADM) is applied to solve some linear partial integro-differential equations.

Example 1

Consider the linear partial integro-differential equation

$$u_{tt} = u_x + 2\int_0^t (t-s)\,u(x,s)ds - 2e^x \quad (3.1)$$

With initial conditions

$$u(x,0) = e^x\,, u_t(x,0) = 0 \quad (3.2)$$

Taking the Laplace transform of equation (3.1) we have

$$s^2U(x,t) - su(x,0) - u_t(x,0)$$
$$= \mathcal{L}(u_x) + \mathcal{L}\left(2t\int_0^t u(x,s)ds - 2\int_0^t su(x,s)ds - 2e^x\right) \quad (3.3)$$

Applying initial condition in equation (3.2) to (3.3) respectively we have

$$\mathcal{L}(U) = \frac{e^x}{s} + \frac{1}{s^2}\mathcal{L}(u_x) + \frac{1}{s^2}\mathcal{L}\left(2t\int_0^t u(x,s)ds - 2\int_0^t su(x,s)ds\right) - \frac{2e^x}{s^3} \quad (3.4)$$

Taking the Laplace inverse of equation (3.4) we get

$$u(x,t) = e^x - e^xt^2 + \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}(u_x)\right)$$
$$+ \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(2t\int_0^t u(x,s)ds - 2\int_0^t su(x,s)ds\right)\right) \quad (3.5)$$

Taking the finite series of equation (3.5) where

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,t)$$

Therefore,

$$\sum_{n=0}^{\infty} u_n(x,t) = e^x - e^xt^2 \quad + \; \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(\sum_{n=0}^{\infty} u_{n_x}\right)\right)$$
$$+ \; \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(2t\int_0^t \sum_{n=0}^{\infty} u(x,s)ds - 2\int_0^t s\sum_{n=0}^{\infty} u(x,s)ds\right)\right) \quad (3.6)$$

From equation (3.6) above we deduced the following recursive formula

$$u_0 = e^x - e^xt^2$$

$$u_{n+1} = \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(u_{n_x}\right)\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2}\mathcal{L}\left(2t\int_0^t u_n(x,s)ds - 2\int_0^t s u_n(x,s)ds\right)\right),$$

$n \geq 0$ (3.7)

where

$u_{n_x} = \frac{\partial}{\partial x}(u_n).$

Now we express the above recursive formula as follow:

$$u_0 = e^x - e^x t^2$$

$$u_1 = \frac{e^x t^2}{2!} - \frac{e^x t^6}{180}$$

$$u_2 = \frac{e^x t^4}{4!} - \frac{4e^x t^8}{8!} + \frac{e^x t^6}{360} - \frac{8e^x t^{10}}{10!}$$

$$u_3 = \frac{e^x t^6}{6!} + \frac{4e^x t^8}{8!} - \frac{16e^x t^{14}}{14!}$$

Similarly, we can find other components. Using equation (3.7), the series solution is therefore given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

$$u(x,t) = e^x - e^x t^2 + \frac{e^x t^2}{2!} - \frac{e^x t^6}{180} + \frac{e^x t^4}{4!} - \frac{4e^x t^8}{8!} + \frac{e^x t^6}{360} - \frac{8e^x t^{10}}{10!} + \frac{e^x t^6}{6!} + \frac{4e^x t^8}{8!} - \frac{16e^x t^{14}}{14!} + \cdots$$

$$= e^x - \frac{e^x t^2}{2!} + \frac{e^x t^4}{4!} - \frac{e^x t^6}{6!} + \cdots$$

Which converges to

$u(x,t) = e^x cost$

Example 2

Consider the linear partial integro-differential equation

$$u_t = -x^2 t + \int_0^t (yt + u)dy \tag{3.8}$$

With initial conditions

$$u(x,0) = 1 \tag{3.9}$$

Taking the Laplace transform of equation (3.8) we have

$$su(x,t) - u(x,0) = \frac{-x^2}{s^2} + \mathcal{L}\left(\frac{x^2 t}{2}\right) + \mathcal{L}\left(\int_0^x u\,dy\right) \tag{3.10}$$

Applying initial condition in equation (3.9) to (3.10) we have

$$\mathcal{L}(u) = \frac{1}{s} - \frac{x^2}{s^3} + \frac{x^2}{2s^3} + \frac{1}{s}\mathcal{L}\left(\int_0^x u\,dy\right) \tag{3.11}$$

Taking the Laplace inverse of equation (3.11) we get

$$u(x,t) = 1 - \frac{x^2t^2}{2!} + \frac{x^2t^2}{4} + \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left(\int_0^x u\,dy\right)\right) \tag{3.12}$$

Taking the finite series of equation (3.12) where

$$\sum_{n=0}^{\infty} u_n(x,t) = u(x,t)$$

Therefore,

$$\sum_{n=0}^{\infty} u_n(x,t) = 1 - \frac{x^2t^2}{2!} + \frac{x^2t^2}{4} + \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left(\int_0^x \sum_{n=0}^{\infty} u\,dy\right)\right) \tag{3.13}$$

From equation (3.13) above we deduced the following recursive formula

$$u_0 = 1 - \frac{x^2t^2}{4}$$

$$u_{n+1} = \mathcal{L}^{-1}\left(\frac{1}{s}\mathcal{L}\left(\int_0^x u_n\,dy\right)\right) n \geq 0 \tag{3.14}$$

Now we express the above recursive formula as follow:

$$u_0 = 1 - \frac{x^2t^2}{4}$$

$$u_1 = xt - \frac{x^3t^3}{36}$$

$$u_2 = \frac{x^2t^2}{4} - \frac{x^4t^4}{576}$$

$$u_3 = \frac{x^3t^3}{36} - \frac{x^5t^5}{14400}$$

...

Similarly, we can find other components. Using equation (3.14), the series solution is therefore given by

$$u(x,t) = \sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \cdots$$

$$u(x,t) = 1 - \frac{x^2t^2}{4} + xt - \frac{x^3t^3}{36} + \frac{x^2t^2}{4} - \frac{x^4t^4}{576} + \frac{x^3t^3}{36} - \frac{x^5t^5}{14400} + \cdots$$

Which converges to

$$u(x,t) = 1 + xt$$

4. DISCUSSION

Partial integro-differential equations (PIDE) are used in modelling physical phenomena in sciences and engineering. In this work, the Laplace-Adomian Decomposition Method (LADM) was used in obtaining the series solution of partial integro-differential equations and comparisons were carried out as follows

Table 1. Comparison of numerical solutions for Example 1

X / t	0.2		0.4	
	LADM	DTM	LADM	DTM
0.01	2.0000×10^{-9}	4.1276×10^{-4}	2.0000×10^{-9}	4.1757×10^{-4}
0.02	1.1000×10^{-7}	1.6001×10^{-3}	1.6400×10^{-7}	1.6168×10^{-3}
0.03	1.2460×10^{-6}	3.4096×10^{-3}	1.5210×10^{-6}	3.4348×10^{-3}
0.04	7.0480×10^{-6}	5.5897×10^{-3}	8.6090×10^{-6}	5.6028×10^{-3}
0.05	2.7100×10^{-5}	7.7918×10^{-3}	3.3100×10^{-5}	7.7500×10^{-3}

Table 2. Comparison of numerical solutions for Example 2

X / t	0.25		0.50	
	LADM	VIM	LADM	VIM
0.01	1.0000×10^{-9}	3.1267×10^{-2}	1.1000×10^{-8}	6.2570×10^{-2}
0.02	1.1000×10^{-8}	6.2570×10^{-2}	1.7500×10^{-7}	1.2528×10^{-1}
0.03	5.5000×10^{-8}	9.3907×10^{-2}	8.8400×10^{-7}	1.8813×10^{-1}
0.04	1.7500×10^{-7}	1.2528×10^{-1}	2.8000×10^{-6}	2.5113×10^{-1}
0.05	4.2600×10^{-7}	1.5669×10^{-1}	6.8500×10^{-6}	3.1426×10^{-1}

5. CONCLUSION

The Laplace Adomian decomposition method was applied to two examples and it presents a more useful, accurate and efficient way to develop a semi-analytical solution when compared to some other semi-analytical methods. In addition, the LADM does not involve perturbation and linearization. The method can be applied to other types of PIDEs with initial condition.

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Authors' Conflicts of interest. The authors declare that there are no conflicts of interest regarding the publication of this paper

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ABDULWAHAB EBOSETALE SALIH *
Department of Mathematics, College of Science University of Agriculture Makurdi, Benue State, Nigeria.
E-mail address: abdulwahabsalih55@gmail.com

TIMOTHY TERFA ASHEZUA
Department of Mathematics, College of Science University of Agriculture Makurdi, Benue State, Nigeria.
E-mail address: ttashezua@gmail.com.

TERSOO LUGA
Department of Mathematics, College of Science University of Agriculture Makurdi, Benue State, Nigeria.
E-mail address: tersooluga2000@gmail.com