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EXISTENCE OF WEAK SOLUTIONS FOR THE INCOMPRESSIBLE NONLINEAR PARABOLIC SYSTEM WITH DAMPING

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ABSTRACT. This work concerns the existence of weak solutions associated with the incompressible parabolic system with damping. We prove the existence of the solution with initial data in the Lesbegue space, L^2.

1. Introduction

We consider the following 3D parabolic system with damping term $\beta|u|^2u$.

$$(PS)\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \frac{1}{2}u\text{div u} - \frac{1}{\epsilon}\nabla\text{div u} + \beta|u|^2u = 0 & (t,x)\in[0,T]\times\mathbb{R}^3 \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^3 \\ |u| \longrightarrow 0, \quad \text{as } |x|\longrightarrow\infty \end{cases} \tag{1.1}$$

where $u(t,x):[0,T]\times\mathbb{R}^3\to\mathbb{R}^3$ is the velocity, $u_0:\mathbb{R}^3\to\mathbb{R}^3$ is the given divergence-free initial data. The constant $\nu>0$ is viscosity and $\beta>0$ is a positive constant. The damping term describes the fluid's resistance to motion. It describes various physical situations such as porous media flow and so on. The damping term will make the solutions of this class of nonlinear parabolic system better. In this paper we intend to understand the influence of the damping term $\beta|u|^2u$ on the well-posedness of the system. The equation (1.1) is a dissipative nonlinear equation modelling certain features of the Navier-Stokes equations (NSE). The NSE describes the evolution of a homogeneous viscous incompressible newtonian fluid. The (PS) shares a number of features with the (NSE), both of them have same scaling properties and energy estimate: if $u(x;t)$ is a solution to

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(1.1) then $\lambda u(\lambda^2 t, \lambda x)$ is also a solution for $\lambda > 0$

$$(NSE) \begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= 0 \qquad (t,x) \in [0,T]\times\mathbb{R}^3 \\ \nabla\cdot u &= 0 \\ u(0,x) &= u_0(x) \quad \text{in } \mathbb{R}^3 \end{cases} \tag{1.2}$$

Leray[5] and Hopf[2] established the global existence of weak solutions of NSE. Since then, many researchers have been working on uniqueness and regularity (see [7,8] and their references). In [3], the authors proved the existence of global solution of Navier-Stokes equation with the damping term $f(u) = \beta|u|^{r-1}u$ on a 3D periodic domain, for values of exponent $r > 1$. In addition they proved that global, regular solutions exist also for the critical value of exponent $r = 3$ provided both the viscosity of a fluid and the porosity of a porous medium are large enough. In [9], authors showed that the Cauchy problem of the incompressible NavierStokes equations with the damping term $f(u) = \alpha|u|^{\beta-1}$ $(\alpha > 0)$ has global strong solution for any $\beta > 3$ and the strong solution is unique when $3 < \beta \leq 5$. In (1.2), $u(t,x) : [0,T]\times\mathbb{R}^3 \to \mathbb{R}^3$ is the unknown velocity and $p(t,x) : [0,T]\times\mathbb{R}^3 \to \mathbb{R}$ is the corresponding pressure. The choice of the Lesbegue space $L^2(\mathbb{R}^3)$ was made because of energy estimate. It is observed that the energy inequality estimates only $\frac{1}{\sqrt{\epsilon}}\text{div}u$ in $L^2_t L^2_x$ or $L^2(0,T;H^1)$. To have control over $\frac{1}{\epsilon}\nabla\text{div}u$ we need to see $\partial_t u$ as an element of the space dual to the space of divergence-free vector fields in H^1. In this case, the term that is constituting problem will be equal to zero. (PS) has an advantage over (NSE) due its non-local action as well as the absence of the pressure term. We believe that the study of system (1.1) can enhance our understanding of (NSE) from the regularity theory point of view . In [6] the authors studied a dissipative nonlinear equation modeling certain features of the NSE. They proved that singularities do not occur in dimensions $n \leq 4$ for the evolution of radially symmetric compactly supported initial data. For dimensions $n > 4$, they proved the existence of blow-up of solutions numerically.

We introduce some function spaces and notations that will be used in this paper. $L^q(\mathbb{R}^3)$ denotes the Lebesgue space of order q, and the L^q-norm of a measurable function f is denoted by $||f||_q$, $\hat{u}$ denotes the Fourier transform of u. Given a Banach space Y with norm $||\cdot||_Y$, we denote by $L^q(0,T;Y)$, $1 \leq q \leq \infty$, the set of functions $f(t)$ defined on $(0,T)$ with values in Y such that $\int_0^T ||f(t)||_Y^q dt \leq \infty$. We use C to express an absolute constant. We apply the Galerkin method to construct the approximate solutions and make a prior estimates to precede compactness arguments. The remaining sections are planned as follows: In section 2, we give some basic definitions and state a lemma. We give a brief description of Galerkin method in section 3. Section 4 is devoted to establishing the existence of weak solution to (1.1) when the initial condition is in Lesbegue space $L^2(\mathbb{R}^3)$.

2. Materials and Methods

The following operators are written as follows: $\text{div}u = \sum_{j=1}^3 \partial_j u^j$, $\quad u\cdot\nabla = \sum_{j=1}^3 u^j\partial_j$, $\quad \Delta = \sum_{j=1}^3 \partial_j^2$,
$u\cdot\nabla u = \text{div}(u\otimes u)$ and $\text{div}(u\otimes u)^j = \sum_{k=1}^3 \partial_k(u^j u^k)$. If the scalar product

of (1.1) is taken in the L^2-space with the solution vector field u, we obtain the following for each term using integration by parts, we have

$$(u\nabla u \mid u)_{L^2} = \sum_{1\leq j\leq d}\int_{\mathbb{R}^3} u^j(\partial_j u^k)u^k dx = \frac{1}{2}\sum_{1\leq j\leq d}\int_{\mathbb{R}^3} u^j\partial_j(|u|^2)dx = -\frac{1}{2}\sum_{1\leq j\leq d}\int_{\mathbb{R}^3}(\mathrm{div}|u|^2)dx = 0$$

and $-\nu(\Delta u \mid u)_{L^2} = \nu||\nabla u||^2_{L^2}$

Definition 2.1: Given any $T > 0$, the function $u(x,t)$ is said to be a weak solution to (1.1), when the following conditions are satisfied:

1 $u \in L^\infty(0,T;L^2(\mathbb{R}^3)\cap L^2(0,T;H^1(\mathbb{R}^3))\cap L^4(0,T;L^4(\mathbb{R}^3))$
2 for any $\phi \in C_0^\infty([0,T]\times\mathbb{R}^3))$ with $\mathrm{div}\phi = 0$, we have

$$\int_0^T\int_{\mathbb{R}^3}\left(\nabla u\cdot\nabla\phi + (u\cdot\nabla u + \beta|u|^2u)\cdot\phi + \frac{1}{2}u\cdot\phi\mathrm{div}u - u\cdot\partial_t\phi\right)dxdt = \int_{\mathbb{R}^3}u_0\cdot\phi(.,0)dx$$

Definition 2.2: The function $u(t,x)$ is said to be a strong solution to (1.1) on $(0,T)\times\mathbb{R}^3$ if it is a weak solution and satisfies $u \in L^\infty(0,T;H^1(\mathbb{R}^3))\cap L^2(0,T;H^2(\mathbb{R}^3))\cap L^\infty(0,T;L^4(\mathbb{R}^3))$.
The weak solution u to (1.1) is global if for all $T > 0$, it is a weak solution.
We state the following lemma which will be used in the proof of our theorem

Lemma 2.1 (Xiaojing Cai and Quansen Jiu, 2004)
Given that Y_0 and Y are Hilbert spaces that satisfy compact embedding $Y \hookrightarrow Y$. Let $0 < \alpha < 1$ and $\{v_j\}_{j=1}^\infty \subset L^2(\mathbb{R};Y_0)$ with $sup_j(\int_{-\infty}^\infty ||v_j||^2_{X_0}dt) < \infty$ and $sup_j(\int_{-\infty}^\infty |\tau|^{2\alpha}||\hat{v}_j||^2_X dt) < \infty$. Then there exists a subsequence of $(v_j)_{j=1}^\infty$ which converges strongly in $L^2(\mathbb{R};X)$ to some $v \in L^2(\mathbb{R};X)$.

2.1. **Galerkin method.** Galerkin method was invented by a Russian mathematician, Boris Grigoryevich Galerkin. The idea of approximating infinite-dimensional by finite-dimensional problems is known as Galerkin method. It is a well known device for doing numerical calculations by converting a continuous operator problem (such as ode or pde) to a discrete problem. It is equally useful as a theoretical tool (as it is used in this article). The following steps are taken to show existence of weak solutions to a particular pde using Galerkin approximations.

i **Galerkin approximations**
We build a weak solution of a pde say

$$\begin{cases} u_t + Lu = f & \text{in } \Omega_T \\ u = 0 & \text{on } \partial\Omega\times[0,T] \\ u = g & \text{on } \Omega\times\{t=0\} \end{cases} \tag{2.1}$$

We first construct solutions of certain finite-dimensional approximations to (2.1) and then the solutions tend to some limits, the functions $w_k = w_k(x)(k = 1, ...)$ are assumed to be smooth. $\{w_k\}_{k=1,...}$. $\{w_k\}_{k=1}^\infty$ is an orthogonal basis of $H_0^1(\Omega)$ and orthonormal basis of $L^2(\Omega)$ respectively.

The positive integer n is fixed. We look for a function $u_n : [0,T] \rightarrow H_0^1(\Omega)$ of the form

$$u_n(t) = \sum_{k=1}^{n} d_n^k(t) w_k \quad (2.2)$$

where $d_n^k(t) \in \mathbb{R} \quad (0 \leq t \leq T, k = 1, ..., n)$. So that

$$d_n^k(0) = (g, w_k)(k = 1, .., n) \quad (2.3)$$

and

$$(u_n', w_k) + B[u_n, w_k; t] = (f, w_k)(0 \leq t \leq T, k = 1, .., n) \quad (2.4)$$

We seek a function of the form (2.2) that satisfies (2.4) spanned by $\{w_k\}_{k=1}^n$ and an approximate solution is constructed.

ii **Energy estimates**
As $m \longrightarrow \infty$, we show that a subsequence of u_n converges to a weak solution of (2.1). For this, some uniform estimates are needed. There exists a constant C, depending on Ω and T such that

$$max_{0 \leq t \leq T} ||u_n(t)||_{L^2(\Omega)} + ||u_n||_{L^2(0,T;H_0^1(\Omega))} + ||u_n'||_{L^2(0,T;H^{-1}(\Omega))} \leq C(||f||_{L^2(0,T;L^2(\Omega))} + ||g||_{L^2}) \quad (2.5)$$

for $n = 1, 2...$

iii As $n \longrightarrow \infty$, we build a weak solution of the problem and pass to the limits.

3. Result

The existence result is stated in the following:

Theorem 3.1:
Supposed $u_0 \in L^2(\mathbb{R}^3)$. Then given $T > 0$, a weak solution $u : [0,T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of (1.1) in the sense of Definition 2.1 exists such that $u \in L^\infty([0,T]; L^2(\mathbb{R}^3)) \cap L^2([0,T]; H^1(\mathbb{R}^3)) \cap L^4([0,T], L^4(\mathbb{R}^3))$ and $sup_{0 \leq t \leq T} ||u||_{L^2}^2 + 2\nu \int_0^T ||\nabla u(t)||_{L^2}^2 ds + \frac{1}{\epsilon} \int_0^T ||\text{div} u||^2 ds + 2\beta \int_0^T ||u(t)||_{L^4}^4 dt \leq ||u_0||_{L^2}^2$

Proof:
The proof is established in sequence as follows:

Step 1
We construct a weak solution u_n of finite-dimensional approximation of (1.1) and the passing to the limits. Since H^1 is separable and C_0^∞ is dense in H^1, there exists a sequence $\omega_1, \omega_2, \omega_3, ..., \omega_n$ of members of C_0^∞, in H^1. For each n, an approximate solution which satisfies the equation is defined as follows:

$$u_n(t) = \sum_{i=1}^{n} g_{in}(t) \omega_i(x), \quad (3.1)$$

and by multiplying the equation by a test function $w_j \in C_0^\infty$ and integrating, we obtain the following

$$(u_n'(t),\omega_j)+\nu(\nabla u_n(t),\nabla\omega_j)+(u_n(t)\cdot\nabla u_n(t),\omega_j)+(\frac{1}{2}u_n\nabla\cdot u_n,w_j)$$
$$+\frac{1}{\epsilon}(\nabla \text{div} u_n,w_j)+(\beta|u_n|^2u_n,\omega_j)=0 \quad (3.2)$$

$$t\in[0,T], j=1,2,...,n. \text{ and } u_{0n}\longrightarrow u_0\in\dot{H}^s,\ as\ n\longrightarrow\infty.$$

Step 2

To show that a subsequence of the solutions u_n of the approximate problems converges to a weak solution of (1.1), uniform estimates are needed on the approximate solutions and this follows from the following Lemma.

Lemma 3.1:

Let $u_0 \in L^2$. Then given any $T>0$, we have

$$sup_{0\le t\le T}||u_n||_{L^2}+||u_n||_{L^2(0,T;\dot{H}^1)}+||\text{div}u_n||_{L^2(0,T;\dot{H}^1)}+||u_n||^4_{L^4(0,T;L^4)}\le C,$$

Proof

Multiply both sides of (3.2) by $g_{jn}(t)$ and summing over $j=1,...,n,$. By integration by parts, we obtain the following for each term

$$(u_n'(t),\omega_j)\cdot g_{jn}(t)=\sum_{j=1}^{3}\int u_n'g_{jn}w_j=\sum_{j=1}^{3}\int u_n'u_ndx=\sum_{j=1}^{3}\frac{1}{2}\int\frac{d}{dt}(u_n)^2dx\le\frac{1}{2}\frac{d}{dt}||u_n||^2$$

and

$$\nu(\nabla u_n(t),\nabla\omega_j)\cdot g_{in}=\nu\sum_{j=1}^{3}\int(\nabla u_n\cdot\nabla g_{jm}w_j)dx\le\nu||\nabla u_n||^2.$$

Similarly,

$$(u_n(t)\cdot\nabla u_n(t),w_j)\cdot g_{im}=0$$
$$\frac{1}{\epsilon}(\nabla\text{div}u_n,w_j)\cdot g_{in}\le\frac{1}{\epsilon}||\text{div}u_n||^2$$

and

$$\beta(|u_n|^2u_n,w_j)\cdot g_{im}\le\beta||u_n||^4.$$

While we have used $((u\cdot\nabla)v,v)=0$, after getting the bound on each term, we have

$$\frac{1}{2}\frac{d}{dt}||u_n||^2_{L^2}+\nu||\nabla u_n||^2_{L^2}+\frac{1}{\epsilon}||\text{div}u_n||^2+\beta||u_n||^4_{L^4}dt\le C$$

Integrating on time t over $(0,T)$, we obtain

$$sup_{0\le t\le T}||u_n||^2_{L^2}+2\nu\int_0^T||\nabla u_n||^2_{L^2}dt+\frac{1}{\epsilon}\int_0^T||\text{div}u_n||^2+2\beta\int_0^T||u_n||^4_{L^4}dt\le||u_0||^2_{L^2}$$

□

Step 3

Next we pass to limits as $n\to\infty$ to build a solution of (1.1). Involking Lemma 3.1, the existence of approximate solutions is obtained: $u_n\in L^\infty(0,T;L^2(\mathbb{R}^3))\cap$

$L^2(0,T;H^1(\mathbb{R}^3))\cap L^4(0,T;L^4(\mathbb{R}^3))$. By using Lemma 2.1, we prove that u_n (or its subsequence) convergences strongly in $L^2\cap L^4([0,T]\times\mathbb{R}^3)$. $\tilde{u}_n$ is denoted as a function from $\mathbb{R}$ into H^1 and has the same value as u_n on $[0,T]$ and zero on it's complement. In the same vein, $g_{in}(t)$ is extended to $\mathbb{R}$ by giving the definition $\tilde{g}_{in}(t)=0$ for $t\in\mathbb{R}\backslash[0,T]$. The Fourier transform on variable t of $\tilde{u}_n$ and $\tilde{g}_{in}$ is given by $\hat{\tilde{u}}_n$ and $\hat{\tilde{g}}_{in}$ respectively. The solutions $\tilde{u}_n$ satisfy

$$\frac{d}{dt}(\tilde{u}_n,\omega_j)=\nu(\nabla\tilde{u}_n(t),\nabla\omega_j)+(\tilde{u}_n(t)\cdot\nabla\tilde{u}_n(t),\omega_j)+\frac{1}{2}(\tilde{u}_n(t)\nabla\tilde{u}_m(t),\omega_j)+\frac{1}{\epsilon}(\nabla\text{div}\tilde{u}_n(t),\omega_j)$$
$$+(\beta|\tilde{u}_n|^2\tilde{u}_n(t),\omega_j)\equiv(\tilde{f}_n,\omega_j)+(\beta|\tilde{u}_n|^2\tilde{u}_n,\omega_j)\ j=1,2,...,n. \quad (3.3)$$

where

$$(\tilde{f}_n,\omega_j)=\nu(\nabla\tilde{u}_n(t),\nabla\omega_j)+(\tilde{u}_n(t)\cdot\nabla\tilde{u}_n(t),\omega_j)+\frac{1}{2}(\tilde{u}_n(t)\nabla\tilde{u_n}(t),w_j)+\frac{1}{\epsilon}(\nabla\text{div}\tilde{u}_n(t),w_j).$$

If the Fourier transform is taken about the time variable, (3.3) becomes

$$2\pi i\tau(\widehat{\tilde{u}}_n,\omega_j)=(\widehat{\tilde{f}}_n(\tau),\omega_j)+\beta(\widehat{|\tilde{u}_n|^2\tilde{u_n}(\tau)}_n,\omega_j)+(u_{0n},\omega_j)-(u_n(T),\omega_j)exp(-2\pi iT\tau). \quad (3.4)$$

where $\hat{\tilde{f}}_m$ is the Fourier transforms of $\tilde{f}_n$.
Using $\hat{\tilde{g}}_{jn}(\tau)$ to multiply (3.4) and add for $j=1,...,n$ to get:

$$2\pi i\tau||(\hat{\tilde{u}}_n(\tau)||_2^2=(\hat{\tilde{f}}_n(\tau),\hat{\tilde{u}}_n)+\beta(\widehat{|\tilde{u}_n|^2\tilde{u_n}}(\tau),\hat{\tilde{u}}_n))+(u_{0n},\hat{\tilde{u}}_n)-(u_n(T),\hat{\tilde{u}}_n)exp(-2\pi iT\tau). \quad (3.5)$$

For any $v\in L^2(0,T;H^1)\cap L^4(0,T;L^4)$ we have

$$(f_m(t),v)=(\nabla u_n,\nabla v)+(u_n\cdot\nabla u_n,v)+\frac{1}{2}(\nabla u_n\nabla\cdot u_n,v)+\frac{1}{\epsilon}(\nabla\text{div}u_n,v)$$
$$\leq C(||\nabla u_n||_2^2+||\nabla u_n||_2+||\nabla u_n||_2^2+||\text{div}u||^2)||v||_{H^1}.$$

Given any $T>0$, it follows that

$$\int_0^t||f_n(t)||_{H^{-1}}dt\leq\int_0^T C(||\nabla u_n||_2^2+||\nabla u_n(t)+||\nabla u_n||_2^2+||\text{div}u||_2^2)dt\leq C \quad (3.6)$$

and hence

$$sup_{\tau\in\mathbb{R}}||\hat{\tilde{f}}_n(\tau)||_{H^{-1}}dt\leq\int_0^T||f_n(t)||_{H^{-1}}dt\leq C \quad (3.7)$$

We have from Lemma 3.1 that

$$\int_0^T||u_n||^2_{L^{\frac{4}{3}}}dt\leq\int||u_n||^3_{L^4}dt\leq C$$

which implies that

$$Sup_{\tau\in\mathbb{R}}|||u|^2u||_{L^{\frac{4}{3}}}(\tau)\leq C \quad (3.8)$$

From Lemma 3.1, we have

$$||u_n(0)||\leq C,\ \ ||u_n(T)||\leq C \quad (3.9)$$

We deduce from (3.5) - (3.9) that

$$|\tau|||\hat{\tilde{u}}_n(\tau)||_2^2 \leq C(||\hat{\tilde{u}}_n(\tau)||_{H^1} + ||\text{div}\hat{\tilde{u}}||_{L^2} + ||\hat{\tilde{u}}_n(\tau)||_{L^4})$$

For $0 < \alpha < \frac{1}{4}$, it is noted that

$$|\tau|^{2\alpha} \leq C\frac{1+|\tau|}{1+|\tau|^{1-2\alpha}},\ \forall \tau \in \mathbb{R}$$

Thus

$$\int_{-\infty}^{\infty} |\tau|^{2\alpha}||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau \leq C\int_{-\infty}^{\infty} \frac{1+|\tau|}{1+|\tau|^{1-2\alpha}}||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau$$

$$\leq \int_{-\infty}^{\infty} ||\hat{\tilde{u}}_m(\tau)||_{L^2}^2 d\tau + C\int_{-\infty}^{\infty} \frac{||\hat{\tilde{u}}_n(\tau)||_{H^I}}{1+|\tau|^{1-2\alpha}d\tau} + C\int_{-\infty}^{\infty} \frac{||\text{div}\hat{\tilde{u}}_n(\tau)||_{L^2}}{1+|\tau|^{1-2\alpha}}d\tau + C\int_{-\infty}^{\infty} \frac{||\hat{\tilde{u}}_n(\tau)||_{L^4}}{1+|\tau|^{1-2\alpha}}d\tau \tag{3.10}$$

By Lemma 3.1 and Perseval equality, the first integral on the rhs of (3.10) is uniformly bounded on m.
By the Parseval equality, the Schwarz inequality and Lemma 3.1, we have

$$\int_{-\infty}^{+\infty} \frac{||\hat{\tilde{u}}_n(\tau)||_{H^1}}{1+|\tau|^{1-2\alpha}}d\tau \leq \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}} \left(\int_0^T ||u_n(t)||_{H^1}^2 dt\right)^{\frac{1}{2}} \leq C, \tag{3.11}$$

for $0 < \alpha < \frac{1}{4}$
Also, we have

$$\int_{-\infty}^{+\infty} \frac{||\text{div}\hat{\tilde{u}}_n(\tau)||_{L^2}}{1+|\tau|^{1-2\alpha}}d\tau \leq \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\alpha})^2}\right)^{\frac{1}{2}} \left(\int_0^T ||\text{div}\hat{\tilde{u}}_n(\tau)||_{L^2}^2(\tau)d\tau\right)^{\frac{1}{2}}$$

$$\leq C\int_{-\infty}^{+\infty} (||\text{div}u_n(\tau)||_{L^2}^2 d\tau)^{\frac{1}{2}} \leq C \tag{3.12}$$

and for $0 < \alpha < \frac{1}{4}$, we have

$$\int_{-\infty}^{+\infty} \frac{||\hat{\tilde{u}}_n(\tau)||_{L^4}}{1+|\tau|^{1-2\alpha}}d\tau \leq \left(\int_{-\infty}^{+\infty} \frac{d\tau}{(1+|\tau|^{1-2\alpha})^{\frac{4}{3}}}\right)^{\frac{3}{4}} \left(\int_0^T ||\hat{\tilde{u}}_n(\tau)||_{L^4}^4(\tau)d\tau\right)^{\frac{1}{4}}$$

$$\leq C\int_{-\infty}^{+\infty} (||\hat{\tilde{u}}_n(\tau)||_{L^4}^{\frac{4}{3}} d\tau)^{\frac{3}{4}} \leq C(\int_0^T ||u_n||_{L^4}^4 dt)^{\frac{1}{4}} \leq C \tag{3.13}$$

It follows from (3.10)

$$\int_{-\infty}^{+\infty} |\tau|^{2\alpha}||\hat{\tilde{u}}||_m(\tau)||_2^2 \leq C \tag{3.14}$$

From Lemma 2.1, Lemma 3.1 and (3.14) we obtain that there exists a subsequence of u_n given by u_n such that $u_n \longrightarrow u$ strongly in $L^2(0,T;L^2)$ and $\nabla u_n \rightharpoonup \nabla u$ converges weakly in $L^2(0,T;H^1)$, $\text{div}u_n \rightharpoonup \text{div}u$ converges weakly in $L^2(0,T;H^1)$. $u_n \longrightarrow u$ converges strongly in $L^4(0,T;L^4)$ as $\int_0^T \int_{\mathbb{R}^3} |u|^4 dxdt \leq C$. These convergences show that $u(x,t)$ is a weak solution of (1.1). □

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