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ITERATIVE INTERVAL FORMULAS FOR SYSTEM OF EQUATION IN TOPOLOGICAL SPACE

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ABSTRACT. Given a map $F: X \to Y$ acting between two topological spaces X and Y, it is pertinent to ask if the path from a point X to a point Y is a closed path, and under what conditions can the topological space from X to a topological space Y be said to be contractible to a point? We give answers to this poised question using the concept of hemi-continuity for F-differentiable function and the Banach Fixed point theorem. Furthermore, solving the resulting linear system of equation for the map $F: R^n \to R^n$ using either Guassian or LU decomposition, we again ask under what condition can we say that Guassian elimination method or LU Factorization cannot compute exactly the inverse of the matrix. .In this paper, we give such error bounds for the LU Factorization and the resulting residual error estimate for system of equation. We realized our solutions to systems of nonlinear equations using the interval LU Factorization, the interval Gauss-Siedel iteration and the Krawczyk's interval method with guaranteed error bounds. A ray tracing implicit surface for the obtained solution is described and a normalized distance between imaging and distortion of a ray tracing implicit surface in the obtained solutions from the nonlinear system is computed.

1. INTRODUCTION

The first aim of this paper is to show that there are convergent and monotone iterative formulas for the multivariate equations relating to Newton methods and their variants Ortega and Rheinboldt (2000) in topological spaces.

We extend the presentation using circular interval arithmetic in the sense of Uwamusi $(2009, 2010_a)$. It was showed in the affirmative that there exists a normalized distance between imaging and distortion of a ray tracing implicit surface Brunet et al (2017) in the solution to the interval nonlinear system of equations using interval LU Factorization, interval Gauss- Siedel method and Krawczyk's algorithm .

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1.1 Preliminaries/Literature Review

Let $F : D \subset R^n \to R^n$ be defined and continuous at the point $x_0 \in D$. Given any $\varepsilon > 0$ and a $\delta(\varepsilon) > 0$ there is a ball $S\left((x^{(0)}, \delta\right) \subset D$ such that the $F-$ derivative exists and is given by

$$\left\|F(x^{(m)}) - F(x^{(n)}) - F^{/}(x^{(0)})(x^{(m)} - x^{(n)})\right\| \le \varepsilon \left\|x^{(m)} - x^{(n)})\right\|, \qquad \forall x^{(m)}, x^{(n)} \in S\left((x^{(0)}, \delta\right) \quad (1.1.1)$$

We discuss the presentation in the context of topological spaces and give conditions under which a map from one topological space X to another topological space Y be a contraction map, that is, a fixed point operator in the sense of F-differentiable function and the well-known Banach Fixed point theorem Ortega and Rheinboldt (2000). We relate that a Baire space $\Rightarrow$Polish$\Rightarrow$Suslin$\Rightarrow$ K-Suslin $\Rightarrow$quasi-Suslin spaces and that every Frechet space is strongly hyper complete Iyahen (1998) and Karlova (2016) which also holds true for the dual of a reflexive Frechet space in the strict sense. We rely Uwamusi (2004) on the existence of a strong form of Meanvalue theorem and continuity of F-derivatives of the map.

We explain what role each of the open and closed mapping theorems strive to achieve in the discussion of continuity of two metrics in topological vector space.It is stressed that a topological space is a connected space which cannot be represented as the union of two disjoint non- empty open sets. The importance of Lebesgue number for the cover of each $D \subset X$ with a diameter $< \delta$ an important topic in topology was also stressed. Motivation of the paper is as follows refreshing the ideas as given in Ortega and Rheinboldt (2000):

Let $f : D \subset R^n \to R^n$ be defined and continuous in the domain $D \subset R^n$. A point $x^* \in D$ is a local minimizer of f if there bean open neighborhood S of x^* such that for all $x \in S \bigcap D$, then

$$f(x) \ge f(x^*), \qquad (1.1.2)$$

x^* is a proper local minimizer of f if strict inequality holds in equation (1.1.1) for all $x \in S \bigcap D, x \ne x^*$

To buttress interest in the presentation, firstly, a well-known univariate functional map $f : D \subset R^n \to R^1$ connected on D, is assumed by the readers given that every level set of f is path connected.

Fundamentally in this direction, are the basic tenets of path connectedness of a functional in the sense of Ortega and Rheinboldt (2000) as follows:

Abstractly, the map $f : D^n \to R^n$ is a convex functional on $D \subset R^n$ if given that $L(\eta)$ is any level set of f and $x^{(m)}, x^{(n)} \in L(\eta)$ for all $t \in (0,1)$ for which holds the inequality:

$$f\left(tx^{(n)} + (1-t)x^{(m)}\right) \le tf(x^{(n)}) + (1-t)f(x^{(m))}) \le t\eta + (1-t)\eta .$$
(1.1.3)

Since $L(\eta)$ is convex and path connected, the functional $f : D_0 \subset R^n$ $(x \in D_0)$ shall be represented by the equation:

$$f(x) = \int_0^1 \left(x^{(m)} - x^{(n)}\right)^T g(x^{(0)} + t(x - x^{(0)}))dt$$
$$+ \int_0^1 (x - x^{(0)})^T [g\left((x^{(0)} + t\left(x - x^{(0)}\right)\right)\right) - f\left(x^{(0)} + t(x^{(m)} - x^{(0)})\right)]dt \qquad \textbf{(1.1.4)}$$

The concept of criticality which is an important aspect in the theory of global minimum in the F-differentiable function for the nonlinear system of equations which very important is hereby brought to bear in the analysis. To this end, we state the following theorem for a useful purpose.

Theorem 1.1, Ortega and Rheinboldt (2000). Assuming $f : D \subset R^n \to R^n$ has a G-derivative on an open, bounded set $D_0 \subset D$ and that f be continuous on D_0. Let there exists an $x^{(0)} \in D_0$ such that $f\left(x^{(0)}\right) < f(x)$ for all x on the boundary of D_0, then f has a critical point in D_0. In addition, it holds that

$$\left[f^{/}\left(x^{(m)}\right) - f^{/}\left(x^{(n)}\right)\right]\left[x^{(m)} - x^{(n)}\right] \geq \left\|x^{(m)} - x^{(n)}\right\|_2^2, \quad \forall x^{(m)}, x^{(n)} \in S\left(x^{(0)}, r\right) \subset D_0, \qquad (1.1.5)$$

Moreover, $\left\|f^{/}\left(x^{(0)}\right)\right\| \leq \frac{1}{2} r$ where, r is the radius of the Riemann sphere. This means that f thus have, a critical point in $S\left(x^{(0)}, r\right)$.

The concept of a hemivariate function as a prelude to the presentation of multivariate functions is important for our analysis to the criticality of global minimization of a continuously differentiable functional. Thus, a functional $f : D \subset R^n \to R^n$ is hemivariate on a set $D_0 \subset R^n$ if it is not constant on any line segment of D_0, and if there exists no distinct points $x^{(0)}, x^{(1)} \subset D_0$ such that

$(1-t)x^{(0)} + tx^{(1)} \in D_0$, where for instance, $f((1-t)x^{(0)} + tx^{(1)}) = f(x^{(0)}), \ \forall t \in (0,1)$.

Thus, for a given a map $f : D \subset R^n \to R^n$, a sequence $\left\{x^{(k)}\right\}$ in some subset $D_0 \subset D$ is strongly downward (downhill) $\downarrow$ in D_0 if given $t \in [0,1]$, then there holds the estimate

$$f\left(x^{(k)}\right) \geq f\left((1-t)x^{(k)} + tx^{(k+1)}\right) \geq f\left(x^{(k+1)}\right), \forall t \in [0,1]. \qquad (1.1.6)$$

From practical experiences, it is known that for a convex functional F, the strong form of Meanvalue theorem for the function defined on the domain $D \subset R^n$ shows that if $g(t) = F\left(x^{(0)} + t(x - x^{(0)})\right)$ is a continuously differentiable function in the interval $[0,1]$, then

$$F(x)-F(x^{(0)})=g(1)-g(0)=\int_0^1 g^{/}(t)dt=\int_0^1 F^{/}(x^{(0)}+t(x-x^{(0)}))(x-x^{(0)})dt$$
$$=F[x,x(0)](x-x^{(0)}). \tag{1.1.7}$$

Thus in the sense of Ortega and Rheinboldt (2000), there follows:

The Gateaux differentiable (or mildly, F -differentiable) F on a convex set $D_0 \subset R^n$ is that, given any $x_n, x_m \in D_0$, we have that $\|F(x_m)-F(x_n)\| \le \sup_{0\le t\le 1}\|F^{/}(x_n+t(x_m-x_n))\|\|x_m-x_n\|$.

Therefore, a function F defined at the point vector $x_0 \in R^n$ is said to have a strong G- Gateaux derivative or respectively, an F - derivative if given any $\varepsilon>0$ there is a $\delta(\varepsilon)>0$ such that $S(x_0,\delta)\subset D$ there holds $\| F(x_m)-F(x_n)-F^{/}(x_0)(x_m-x_n) \| < \varepsilon\|x_m-x_n\|, \forall x_m, x_n \in S(x_0,\delta)$.

It follows that by definition of hemi-continuity for the Gateaux differentiability F the inequality holds:

$$\|F(x_n+\eta(x_m-x_n))-F(x)\| \le M\eta\|x_m-x_n\|+\varepsilon\eta\|x_m-x_n\|, \tag{1.1.8}$$

for small enough ε and $\eta<1$ with $M=\sup_{0\le t\le 1}\|F^{/}(x_n+t(x_m-x_n))\|<\infty$.

The remaining part in the paper is categorized as follows. Section 2 in the paper discusses the question of abstraction of iterated map and compactness in topology. The notion of Baire space in the contexts of Banach ultrbornological space is discussed. It is stated that a topological space X with its set of all subsets is completely uniformable and Hausdorff, if it is Polish, metrizable, complete and separable. Furthermore, in the sense of Borsik et al (2011) and Bourles (2011), it is showed that Polish space implies Baire space and that inexhaustible density of F imply a Baire's Second Category theorem. It is stated that a Gauss-Siedel iteration has an effective covering property which connotes effective inclusion property.

It is further showed that a functional iterative Newton's method forms the basis of existences of many iterative operators for system of equations based on the adoption of Kantorovich theorem. Here, it is showed that Newton's iteration will halt to an end if certain conditions are met. We give the question of weak and strong regular splitting which induces the Gauss-Siedel and Successive overrelaxation iterative methods Uwamusi (2004) in this class. In section 3, the backward error bounds occurring in the perturbed system of linear equations is presented in the senses of Bjorck (2009), Varga (2000). We give conditions under which the Gaussian elimination method as well as LU Factorization method cannot compute exactly the inverse of a matrix in the ideas of Golub and Van-Loan(1983) . Later in the section 3, it is given that computation of imaging and distortion with a ray tracing implicit surface maybe obtained based on the knowledge of results obtained from these interval operators of Newton's method and Krawczyk's algorithm. We give basic Circular interval arithmetic properties and numerical example demonstrated with modified interval Gauss-Siedel method, Krawczyk's method in the sense of Uwamusi (2009, 2011), a version of chainable maps. Thereafter, normalized distance for the imaging and distortion for a ray tracing implicit surface is calculated. Discussion of results in the paper is effected in section 4.In section 5 we make conclusion based on the strengths of our findings.

2. MATERIALS AND METHODS

We borrow the following two definitions from the work of Bourles (2011), namely:

2.1 The Question of Abstraction of Iterated Map and Compactness in Topology: The **Fundamental Baire Space.**

Definition 2.1. Let X, Y be two topological spaces and $F : X \to Y$ be a map. The map F is continuous (respectively sequentially continuous) if , and only if for any point $x \in X$, whenever $\{x_i\}$ is a net (respectively a sequence) in X converging to x_i, the net (respectively the sequence) $(F\{x_i\})$ converges to $F(x)$.

Definition 2.2. The graph $G_r(F)$ of F is said to be closed (respectively, sequentially closed) if, and only if whenever $(x_i, F(x_i))$ is a net (respectively a sequence) in $X \times Y$ converging to (x, y), necessarily $(x, y) \in G_r(F)$, that is, $y = F(x)$.

As is well known, a topological space is a Baire space provided that we take into consideration the countable collections of dense open subsets which haveBorsik et al (2011) a dense intersection of second Baire Category. The Baire Category is considered in the context of dense function in our presentation see for example Karlova (2016)**.** A space which contains a dense Baire subspace is Baire.To begin with as an illustration to this, let X be a complete metric space, that is, the space equipped with bounded and convergent Cauchy sequence. Let U_i be a sequence of dense open sets in X and assume that $B_0 \in D_0$ .Then, there could be found a nested sequence of closed balls $B_0 \supset B_1 \supset B_2 \supset ....\supset...$ in the domain $X \subset D_0$ which induces:

$B_{k+1} \subset B_k \bigcap U_k \neq \phi$,with diameter $\|B_{k+1} - B_k\| \leq \frac{1}{k+1} \to 0 \; as \; k \to \infty$ wherein, $B(x, \frac{1}{k})$ forms a ball at x in the metric space. From this, it follows that the generated U_k coincide with the interior of B_k that stabilizes for the centers of the balls B_k by inductive Cauchy sequence $\ni x \in B_0 \cap \bigcap U_i \neq \phi$ where , $x \in X$.

Being inspired in this direction, we take a note of the following points which are assumed to be familiar to the readers.

A subset F of a metric space will be called a closed set if it contains each of its limiting points. Thus if X be a metric space and $x \in X$ with $F(x) \subset x$, a point x is called an interior point of F if it is the Centre of some open sphere contained in F .Practically, it follows that for a metric space (X, ρ) with $x \in X$ and for $\varepsilon > 0$, the $u_\rho(x, \varepsilon) = \{v \in X \,|\, \rho(x, v) < \varepsilon\}$ is an ε -disc.

Via inclusion theorem, for two metric spaces (X, ρ) and (Y, δ) , a function $F : X \to Y$ will be continuous at $x \in X$ if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $F(u_\rho(x, \delta)) \subset u_\delta(F(x), \varepsilon)$.

If the sequence has a convergent net, what then is the nature of the shrinking base? To answer this question, let (X,ρ) be a metric space. A subset $X^{(k)} \subset X$ is called δ-net if $x \in X$, $d\left(x, X^{(k)}\right) < \delta$ so that the metric space (X,ρ) is totally bounded if and only if for $1 > \delta > 0$ one can find a finite δ-net for which the above assertions hold. There is a contracting map from the metric space $X \to X$ given a real number α, $(0 \le \alpha \le 1)$, such that for all points $m, n \in X$,one can find that $d\left(f(m), f(n)\right) \le \alpha d(m,n) < d(m,n)$.

This means that every contracting space is Hausdorff, particularly a category T_2-space, given that $x \in X$ implying $f(x) = x$ (by fixed point theorem).We thus have erected a structure of contraction as follows:

Given a point $x_0 \in X$ and the iterated map $F : D \subset X \to Y$.By the well-known Banach fixed point theory in place, it follows that $x_0 = f(x_0),\ x_1 = f(x_1) = f^2(x_0),\ldots,x_k = f(x_{k-1}) = f^k(x_0),\ldots$

We thus have created a set of points (vectors) $\{x_1, x_2, x_3, \ldots\}$ that is Cauchy with the metric topology

$d\left(f^{m+n}(x_0), f^m(x_0)\right) \le \alpha^n d\left(f(x_0), x_0\right)$, (where $\alpha < 1$, $m, n \in N$) and with the sum of the sequence

$$\left(1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^m\right) \le \frac{|\alpha|^m}{1-\alpha}.$$

Having noted what a complete metric space topology connotes, and turning our attention again to a Baire space, we in the vicinity of compactness, noted that a dense subspace of a convex Baire space is convex-Baire. However, a convex Baire space needs not be complete, Iyahen (1998),Chidume (1995).This raises a fundamental question what actually a Baire in topological vector space aims at – the density of F ?

A Baire in Topological vector space is a set which is convex, balanced, absorbing and closed. Thus, a barreled space is a locally convex space in which every barrel is a neighbourhood of origin O.A Banach space is barreled, just as an ultrabornological space is barreled see e.g.,Iyahen (1998). To wit, it follows that a Hausdorff locally convex space isultralbornological if it is an inductive limit of Banach spaces. A family B of subset of a topological space X is called a base for a mapping $f : X \to Y$ if the preimage $f^{-1}(v)$ of an arbitrary open set $v \in Y$ is a union of sets from B. Thus if X is a Baire Space and Y is a metric Space, the point-wise limit of $f : X \to Y$ is a sequence from $G(X,Y)$ which has a set $C(f)$ dense G_r-set. Note that for every function $f : X \to Y$, the graph of f is defined by $G_r f = \{(x, f(x)) : x \in X\}$.

The open mapping theorem shows that a bounded linear surjection of a Banach space X into a Banach space Y is an open mapping. We give a stronger version of open mapping theorem – the Banach-Steinhaus theorem in the sense of Chidume (1995) as follows:

Let F be a bounded linear map from a Banach space X into a normed linear space Y. If the image $F[X]$ is non meagrein Y, then F is subjective, i.e., $F[X]=Y$ which is dense in topology.

In the realm of computable analysis and topology we include uniform continuity, compactness, for an operator functional equation. The notions of absorbent map, fundamental sequence of bounded sets are important.

The closed graph theorem, open mapping theorem, hyper- complete metric space, reflective Frechet space, B-complete linear topological space are all webbed. As pointed out in Iyahen (1998), it is stated that the link between the closed graph theorem, B-complete and B_r-complete locally convex spaces can be obtained from the two assertions, namely:

(i) Every closed linear nearly open one-to-one map from E onto any Hausdorff locally convex space is open. E being the Euclidean space.

(ii) Every closed linear nearly continuous map from any Hausdorff locally convex space into E is continuous.

Therefore, it is instructive to note that a Hausdorff space is called ultrabornological if it is an inductive limit of Banach spaces.

We henceforth give such categorizations of a Hausdorff space as follows: Let X be a topological space and $\mathrm{B}(X)$ (resp. $\Re(X)$) be the set of all subsets (resp. of all compact subsets of X),the topological space X is

(i) Completely regular if it is uniformable and Hausdorff;

(ii) Polish, if it is metrizable, complete and separable;

(iii) Suslin, if it is Hausdorff and there exist a Polish space and a continuous surjection $P \to X$;

(iv) Lindelof, if, from any open covering, one can extract a countable covering.

A topological space is Lindelof if every open cover has a countable sub- cover. By local compactness in a topological space X, we mean an open set U whose closure $\bar{U}$ compact forms a neighborhood base for the topology.

A completely regular topological space is K-Suslin if, and only if, it is K-analytic. In the sense of Bourles (2011),a Polish space, Suslin space, K-Suslin space and the Quasi—Suslin space are webbed in the sense that:

(a) Polish $\Rightarrow$ Suslin $\Rightarrow$ K-Suslin $\Rightarrow$ quasi-Suslin;

(b) Polish $\Rightarrow$ Baire (Baire's theorem) $\Rightarrow$ non meagre in itself $\Rightarrow$ inexhaustible $\Rightarrow$ of Second category;

(c) K Suslin $\Rightarrow$ Lindelof $\Rightarrow$ para-compact;

(d) 2^{nd} countable $\Rightarrow$ Lindelof and 2^{nd} countable $\Rightarrow$ 1^{st} countable separable.

It is known that every Frechet space is strongly hyper complete and the dual of a reflexive Frechet space is strictly hyper complete.

Therefore, Gausss-Siedel iterative type method as well as the Successive Over-relaxation method (SOR) given ahead in section2 in the paper fits into this advantage of above descriptions and admits a linear space with a countable dimension of a linear topology under which it is $B_{\delta/2}$ - complete and its linear subspace, the Jacobi iterative method is closed and converges on the entire dense subset of the Gauss-Siedel map. Notably, in abstraction, we mean a generic measure preserving homeomorphism of the square with a dense orbit. Motivated by the above enumerations we are led by the following facts.

Definition 2.3, Collins (2005). A computable topological space (X, τ, β, v) is a computable Hausdorff space if (X, τ) is a locally compact separable Hausdorff space, and B is a base for τ such that $I \in B$ is pre-compact.

Under this condition, the Gauss-Siedel and SOR operators are required to have effective covering properties which connote the effective inclusion property with the pullback diagrams not undermined. Let U be an open cover and $f(x) \subset x$ where, $x \subset X$ .A sequence $x^{(0)}, x^{(1)}, .. x^{(n)}$ is a U -chain Uwamusi (2016) for F if there exist points $z_1, z_2, ..., z_k \in X$ and open sets $U_1, U_2, ..., U_n \in U$ such that $z^{(i+1)} \in F(x^{(i)})$ and $x^{(i+1)}, z^{(i+1)} \in U_{i+1}$ for $i = 0,1,2,...,n-1$.

For a given reachable set Reach$\left(F, X^{(0)}, \mu\right) = \left\{x \in X : \exists \mu - chain\, x^{(0)}, x^{(1)}, x^{(2)},\ x^{(n)}\right\}$ for F such that given $x^{(0)} \in X^{(0)}$ and $x^{(n)} = x$, the set of points reachable from $X^{(0)}$ by a μ -chain, the chain-reachable set for F from $X^{(0)}$ is given by

Chain Reach $\left(F, X^{((0))}\right) = \bigcap_{\mu} \mathrm{Re}\, ach\left(F, X^{(0)}, \mu\right)$ where μ runs over open covers of X .

The above notion of a multivalued map will be a useful a tool for our work in this paper. It will hold that a multivalued map is (weakly) continuous if it is both lower-semi-continuous and (weakly) upper –semi-continuous. This means that, the map F is lower-semi-continuous whenever $F(cl(X)) \subset cl(F(X))$ for any set X with $cl(G \circ F(x)) = cl(G(F(x)))$.

The main idea in the above is motivated by the following facts, namely:

Assuming that $F = \lim_{k \to \infty} F_k$ is a strict countable inductive limit,

(i) The topology induced in F_k by that of F coincides with that of F which by implication, is the topology of F which is Hausdorff.

(ii) If F_k is closed in F_{k+1} for every k , then F is complete.

(iii) The topology of F is the finest topology among all topologies computable with the vector space structure of F (locally convex or not) which induces in F_k a coarser topology than the given topology $\Im_k$.

2.2 The Functional Iterative Methods

Specifically, Newton's method cuts across the fundamental basis for presenting and analyzing results of both univariate and multivariate systems of equations. In abstraction, Newton's method is given by

$$x^{(k+1)} = x^{(k)} - \frac{f\left(x^{(k)}\right)}{f^{/}\left(x^{(k)}\right)};(k = 0,1,2,\ldots) \tag{2.2.1}$$

Let $D \subseteq R^n$ be open. Let a sequence $f_1, f_2, \ldots$ from $F(D)$ converges locally uniformly in D to a function f so that $f \in F(D)$, $D \subset R^n$. Then in the sense of Berg (2012), the sequence $f_1^{/}, f_2^{/}, \ldots$ of derivatives converges locally uniformly in D to $f^{/}$ if $F(D) \subset D$ holds true. This is one of the basis in which various modifications of Newton's methods are derived.

We say that a sequence $\left\{x^{(k)}\right\}, (k = 0,1,2,..)$ is said to be (at least) linearly convergent to x^* if there is a number $q \in (0,1)$ and a number $c > 0$ such that

$$\left|x^{(k+1)} - x^*\right| \leq cq^i \ , \ \forall i \geq 1. \tag{2.2.2}$$

Within the vicinity of expositions as a prelude to obtaining results, we give more facts pertaining existence solutions to the map $F : D \subset R^n \to R^n$ based on the assertions of Kantorovich theorem given below.

Theorem (2.2.1),Ortega and Rheinboldt (2000).Let $F : D \subset R^n \to R^n$ be an F -differentiable function on a convex set $D_0 \subset D$ and assume that

$$\left\| F^{/}\left(x^{(k1)}\right) - F^{/}\left(x^{(k)}\right)\right\| \leq \eta \left\| x^{(k+1)} - x^{(k)}\right\|, \ (\forall k+1, k \in D_0) \tag{2.2.3}$$

Assumin**g** further that there exists $x^{(0)} \in D_0$ such that

$$\left\|F^{/}\left(x^{(0)}\right)^{-1}\right\| \leq \beta, \text{and, } \alpha = \beta\gamma\eta \leq \frac{1}{2} \text{ where } \gamma \geq \left\| F^{/}\left(x^{(0)}\right)^{-1} F\left(x^{(0)}\right)\right\|.$$

By setting as

$$t^* = (\beta\gamma)^{-1}\left[1 - (1 - 2\alpha)^{\frac{1}{2}}\right], \tag{2.2.3}$$

$$t^{**} = (\beta\gamma)^{-1}\left[1 + (1 - 2\alpha)^{\frac{1}{2}}\right], \tag{2.2.5}$$

for $\bar{B}_{\frac{\delta}{2}}\left(x^{(0)},t^*\right)\subset D_0$, for which the iterates are well defined and, remain in $\bar{B}_{\frac{\delta}{2}}\left(x^{(0)},t^*\right)$ converging to a solution x^* of $F(x)=0$ which is unique in $B_\delta\left(x^{(0)},t^{**}\right)\bigcap D_0$. Via Kantorovich theorem, the 2

Newton operator remains in $\bar{B}_\delta\left(x^{(0)},t^*\right)$ and converges to the solution x^* of $F(x)=0$ which is unique in $B_\delta\left(x^{(0)},t^{**}\right)\bigcap D_0$ whenever holds the error estimate

$$\left\|x^*-x^{(k)}\right\|\leq\left(\beta\gamma 2^k\right)^{-1}(2\alpha)^{2^k}\ ,(k=0,1,\ldots). \tag{2.2.6}$$

Thus for $p-$ order iterative process converging to x^* whose multiplicity of the roots is $m\geq 2$ for the system of equation $F(x)=0$ in the sense of Lagouanelle's limiting formula Farmer and Loizou (1997),Petkovic and Trikovic (1995), Uwamusi (1999) assuming that $p\leq m$,the inequality

$$\left\|x^{(k+1)}-x^*\right\|\leq\left(1-\frac{p-1}{(m+p-2)}\right)\left\|x^{(k)}-x^*\right\|, \tag{2.2.7}$$

and converges to the desired solution x^* monotonically. Therefore, the iterative process will come to an end when $x^{(k+1)}$ has more significant $\frac{s}{m}$ correct digits than does $x^{(k)}$. This is usually estimated Petkovic and Trikovic (1995),Uwamusi (1999),2004) in the form :

$$\frac{\left\|x^{(k)}-x^*\right\|}{\left\|x^{(0)}-x^*\right\|}=t^{-\frac{s}{m}} \tag{2.2.8}$$

Here, t is the base arithmetic and s is machine accuracy. Equations (2.2.7) and (2.2.8) hold verbatim also for a univariate function.

2.3 The Question of Weak and Strong Regular Splitting

We use notation $A\left(x^{(k)}\right)$ as representing Jacobian matrices at the vector $x^{(k)}$. Then, the splitting matrix is given in the form:

$$A\left(x^{(k)}\right)=B\left(x^{(k)}\right)-C\left(x^{(k)}\right),\ (k=0,1,\ldots) \tag{2.3.1}$$

Following Varga (2000) and letting A denotes $A(x^{(k)})$, $B=B(x^{(k)}),C=C(x^{(k)})$ then it holds that $A=B-C$ is a weak regular splitting of $A\in L(R^n)$ if $B^{-1}C\geq O,B^{-1}\geq O$ and $CB^{-1}\geq O$ .Therefore, the matrix $B^{-1}C=(A+C)^{-1}C$ exists.

Using the above as stated, further analysis leads to $B^{-1}C=(I+G)^{-1}G$ for $G=A^{-1}C$. Therefore, the eigenvalue λ of the matrix G has the corresponding eigenvector μ for the matrix eigenvalue problem in the form $G\mu=\lambda\mu$ so that $(I+G)^{-1}G\mu=\frac{\lambda}{1+\lambda}$. Because $\frac{\lambda}{1+\lambda}$ is a decreasing function of λ in order of magnitudes of λ then, $\rho\left(B^{-1}C\right)<1$ which again coincides with the assertions that

$\rho\left(B^{-1}C\right)=\frac{\rho\left(A^{-1}C\right)}{1+\rho\left(A^{-1}C\right)}<1$ for $A^{-1}\geq O$ .Thus the matrix regular splitting of equation (2.3.1) is feasible.

This leads us to the concept of Successive Over- relaxation method (SOR)in the sense expressed in Hageman et al (1980) and Wasilkowski (1980) with further details in the form.

The classical SOR iterative method is in the form

$$x^{(k+1)}=\Im_{\omega}x^{(k)}+C_{\omega} \tag{2.3.2}$$

Wherein, defined that

$$\Im_{\omega}=\left(D-\omega L\right)^{-1}\left[(1-\omega)D+\omega U\right] \text{ (2.3.3)}$$

$$C_{\omega}=\omega\left(D-\omega L\right)^{-1}b,\ \ (\omega\in R) \text{ (2.3.4)}$$

The term ω appearing in the equation (2.3.2) is the relaxation parameter. The optimum relaxation parameter $(0<\omega<2)$ in the Jacobi iteration matrix B, see e.g.,Young (1971) that minimizes $\rho\left(\Im_{\omega}\right)$ for the SOR method in the form equation (2.3.3) is the expression:

$$\omega_{b}=\omega_{b}(\beta)=\frac{2}{1+\sqrt{1-\beta^{2}}}=1+\left(\frac{\beta}{1+\sqrt{1-\beta^{2}}}\right)^{2} \tag{2.3.5}$$

Wherefrom, $1>\rho\left(\Im_{\omega}\right)>\rho\left(\Im_{\omega_b}\right)=\omega_{b}-1$ for all $0<\omega<2$ and $\omega\neq\omega_{b}$. Let us take notice that for $\omega=1$ the SOR iteration formula degenerates to Gauss-Siedel iterative method.

The asymptotic decay in the norms of the error vectors e_k corresponding to the sequence of vectors $\{x_k\}_{k=0}^{\infty}$ obtained from SOR method is the quantity

$$K\left(\Im_{k},p\right)=\lim_{k\to\infty}\sup_{e_0\neq 0}\left(\frac{\|e_k\|}{\|e_0\|}\right)^{\frac{1}{2}} \tag{2.3.6}$$

In equation (2.3.6), the term p is the order of the polynomial or a matrix.

Empirical estimate has it that 8] we link the Young’s functional Young (1971) relationship

$$\left(\lambda+\omega-1\right)^2=\lambda\omega^2\mu^2, \quad (2.3.7)$$

between the eigenvalues λ of the SOR iteration matrix $\Im_\omega$ with eigenvalues μ of Jacobi iteration matrix B in the form:

$$\delta\left(\Im_\omega\right)\subset\Omega_{\omega,\beta}=\begin{cases}\left[\lambda_2,\lambda_1\right]\subset\left(1,\infty\right),\left(-\infty<\omega<0\right)\\ \left[\lambda_1,\lambda_2\right]\subset\left(0,1\right),\quad\left(0<\omega<1\right)\end{cases} \quad (2.3.8)$$

Where,

$$\lambda_1=\lambda_1\left(\omega,\beta\right)=\left[\frac{\omega\beta-\sqrt{\omega^2\beta^2-4\left(\omega-1\right)}}{2}\right]^2 \quad (2.3.9)$$

$$\lambda_2=\lambda_2\left(\omega,\beta\right)=\left[\frac{\omega\beta+\sqrt{\omega^2\beta^2-4\left(\omega-1\right)}}{2}\right]^2 \quad (2.3.10)$$

This guarantees that $\lambda_1,\lambda_2\in\delta\left(\Im_\omega\right)$ and, $1\in\Omega_{\omega,\beta}$.

We measure average rate of convergence Varga (2000) according to the following definition.

Definition 2.3.1,Varga (2000). Let A and B be two $n\times n$ real matrices. If for some positive integer k, $\left\|A^k\right\|<1$, and $R\left(A^k\right)=-\ln\left[\left(\left\|A^k\right\|\right)\right]^{\frac{1}{k}}=\frac{-\ln\left\|A^k\right\|}{k}$ will be the average rate of convergence for k iterations for the matrix A .If $R\left(A^k\right)<R\left(B^k\right)$, B is iteratively faster, for k iterations than A.

The error estimate for the iterations Uwamusi (2016) and Varga (2000) is obtained as follows. Let $\varepsilon^{(k)}=x^{(k)}-x$ be the error vector for the vector iterates. Then the quantity in equation (2.3.6) for the k iterations is given by $\delta=\left(\frac{\left\|\varepsilon^{(k)}\right\|}{\left\|\varepsilon^{(0)}\right\|}\right)^{\frac{1}{k}}$ and it is then the average reduction factor per iteration, for k iterations for successive error. Moreover, it is true that $\delta<\left(\left\|A^k\right\|\right)^{\frac{1}{k}}=e^{-R\left(A^k\right)}$, where, e is the base of the natural logarithm. Thus by further setting as $N_k=\left(R\left(A^k\right)\right)^{-1}$ and letting $\delta^{N_k}<\frac{1}{e}$, we have succeeded in obtaining N_k as a measure of the number of iterations required to reduce the norm of the initial error by a factor e.

2. THE BACKWARD ERROR BOUNDS

The frequently occurring perturbed linear system with backward error is in the form:

$$((A+\Delta A)x=b+\Delta b \quad (3.1.1)$$

The normwise backward error of x Bjorck (2009), Rump (1999),Uwamusi (2010 $_a$, 2016), Walden and Kartson (1995) is

$$\eta(x) = \min \left\{\varepsilon \left| (A + \Delta A)x = b + \Delta b, \|\Delta A\| \le \varepsilon \|A\|, \Delta b \le \varepsilon \|b\| \right.\right\} \tag{3.1.2}$$

The normwise backward error will be small if the residual $b - Ax$ is reasonably small enough.

Theorem 3.1, Bjorck (2009), Walden and Kartson(1995). The norm-wise backward error is given by

$$\eta(x) = \frac{\|r\|}{\|A\|\|x\| + \|b\|} \quad \text{for } r = b - Ax \tag{3.1.3}$$

where, ΔA is defined by the equation

$$\Delta A = \frac{r\left(\bar{x}\right)^T}{\left\|\bar{x}\right\|_2^2} \quad (3.1.4)$$

The $\bar{x}$ satisfies the linear system $(A + \Delta A)\bar{x} = b$ and has the smallest energy norm (l_2 -norm) defined as

$$\|\Delta A\|_2 = \frac{\|r\|_2}{\left\|\bar{x}\right\|_2} \tag{3.1.5}$$

for any such ΔA. Equation (3.1.4) is the optimal value anyone hopes to get from the perturbed system $A\bar{x} = b$.If we decide to use either LU Factorization or pure Guassian elimination methods for the above analysis, then such Gaussian algorithm is given in the form:

Guassian elimination Algorithm:

Given the matrix $A \in R^{n \times n}, b \in R^n$.

for $k = 1 : n - 1$

for $i = k + 1 : n$

$l_{ik} = \frac{a_{ik}}{a_{kk}}$;

for $j = k + 1 : n$

$a_{ij} = a_{ij} - l_{ik} a_{kj}$;

End

$b_i = b_i - l_{ik} b_k$;

end

end

Now turning our attention to the following LU Factorization

$$\left(L + \Delta \bar{L} \right) \bar{d} = b, \quad (U + \Delta U) \bar{x} = \bar{d} \tag{3.1.6}$$

wherein,

$|\Delta L| \leq \eta_n \left| \bar{L} \right|, \ |\Delta U| \leq \eta_n \|U\|$, one immediately forms the product

$$\left(\bar{L} + \Delta \bar{L} \right) \left(U + \Delta \bar{U} \right) \bar{x} = b \tag{3.1.7}$$

The backward error $|\Delta A|$, see e.g., Golub and Van-Loan (1983) is bounded by the error estimate:

$$|\Delta A| \leq \eta_n (3 + \eta_n) |L| |U| \tag{3.1.8}$$

The elements in $\bar{U}$ satisfy $|u_{ij}| \leq \rho_n \|A\|_\infty$ with partial pivoting $|l_{ij}| \leq 1$ and polynomially bounded by the factor

$$\left\| \left| \bar{L} \right| |U| \right\|_\infty \leq \frac{1}{2} n(n+1) \rho_n \tag{3.1.9}$$

By further neglecting terms of order $O\left((n\mu)^2\right)$ in equation (5.8), the best possible bound for $\|\Delta A\|_\infty$ is given by the inequality

$$\|\Delta A\|_\infty = 1.5n(n+1)\eta_n \rho_n \|A\|_\infty \tag{3.1.10}$$

Following equation (3.1.10), it holds that

$$\|Ax - b\|_\infty \leq 1.5n(n+1)\eta_n \rho_n \|A\|_\infty \left\| \bar{x} \right\|_\infty . \tag{3.1.11}$$

This means that the residual is bounded by the quantity

$$\|r\|_\infty \leq 1.5n(n+1)\eta_n \rho_n \|A\|_\infty \left\| \bar{x} \right\|_\infty \tag{3.1.12}$$

We conclude this section by saying that the Gaussian elimination gives small relative residual error for ill-conditioned system so long the growth factor is large whose quantity is $\frac{\left\|b - A\bar{x}\right\|_\infty}{\left(\|A\|_\infty \left\|\bar{x}\right\|_\infty\right)}$ and in general is of order ns, where s is the machine precision number.

3.2 Computation of Imaging and distortion: Ray tracing implicit surface.

In this section, we discuss the image quality assessment measures Brunet et al (2017). In this sense, we define the universal image quality index whose original image signal is $x = \{x_i \mid i = 1,2,\ldots,n\}$ and, its distorted version is $y = \{y_i \mid i = 1,2,\ldots,n\}$ by the equation

$$Q(x,y) = \left(\frac{2\mu_x\mu_y}{\mu_x^2 + \mu_y^2}\right)\left(\frac{2\delta_x\delta_y}{\delta_x^2 + \delta_y^2}\right)\left(\frac{\delta_{xy}}{\delta_x\delta_y}\right) \qquad (3.2.1)$$

In equation (3.2.1), the term μ_r denotes the mean image signal whilst δ_r is the variance with δ_{xy} being the covariance between the original and distorted version of the image signal.

In particular, the first term in equation (3.2.1) is that it measures the luminance similarity between the images, the second term measures contrast similarity and the third term measures correlation or structural similarity between the images.

If we are able to collect such N different vector images, the overall image quality is the average sum vector $Q = \frac{1}{N}\sum_{j=1}^{N} Q_j$, where the jth term denotes the image patch and, N is the number of patches.

By addition of constants c_1, c_2, c_3 called the stabilizing constants to the terms above, and because the structural similarity index (SSIN) depends on the weighted means, variance, covariance and stabilizing constant, we therefore, give the SSIN by the equation**:**

$$SSIN = \left(\frac{2\mu_x\mu_y + c_1}{\mu_x^2 + \mu_y^2 + c_1}\right)\left(\frac{2\delta_x\delta_y + c_2}{\delta_x^2 + \delta_y^2 + c_2}\right)\left(\frac{\delta_{xy} + c_3}{\delta_x\delta_y + c_3}\right) \qquad \textbf{(3.2.2)}$$

Given two point vectors we compute the normalized metric distance by the equation

$$d(x,y) = \left(\frac{\|x-y\|_2^2}{\|x\|_2^2 + \|y\|_2^2 + c}\right)^{\frac{1}{2}} \text{ (for some positive constant } \mathbf{c}\text{)} \qquad \textbf{(3.2.3)}$$

Thus the Standard statistical estimate for illuminating set is computed in the form

$$SSIN(x,y)=\left(\frac{2\mu_x\mu_y+c_1}{\mu_x^2+\mu_y^2+c_1}\right)\left(\frac{2\delta_x\delta_y+c_2}{\delta_x^2+\delta_y^2+c_2}\right)=S_1(x,y)\,S_2(x,y) \qquad \textbf{(3.2.4)}$$

To decompose an image patch into structural and nonstructural parts, we use the formula

$x=\mu_x e+(x-\mu_x e)$ (for $e=1,1,\dots,1$) in the direction of mean and, $x-\mu_x e$ being the zero mean**.**

3.3. BASIC INTERVAL SOLVERS/ NUMERICAL RESULTS

The aim of interval methods is to deliver good quality results in a computing time not too distance from a pure numerical algorithm and also give proof of existence (and possibly uniqueness) of a solution Petkovic and Trikovic (1995), Rump (1999), Uwamusi (1999, 2009, 2010 a ,2010 $_b$,2011). Basic interval arithmetic operations are the $+,-,x,/$.

We give the following notations:

A set of real numbers is denoted by R^n. A real point interval $a=[a_1,a_2]=\{a\in R: a_1\le a_2\}$ is a segment of the real line.

IR , the set of real point intervals may be represented by its end points or by the midpoint and radius.

We say that a linear system is a parameterized system if

$$A(\varepsilon)x=-b(\varepsilon), \qquad (3.3.1)$$

where, $A(\varepsilon)\in R^{n\times n}$ and $b(\varepsilon)\in R^n$ respectively depend affine linearly on a parameter vector $\varepsilon\in R^n$ for which the parametric solution set is defined in the form:

$$\sum^{\varepsilon}=\sum(A(\varepsilon),b(\varepsilon),[\varepsilon])=\{x\in R^n\,|A(\varepsilon).x=b(\varepsilon)\} \qquad (3.3.2)$$

for some $\varepsilon\in[\varepsilon]$. Various basic interval operations are to be found in Uwamusi (2010 $_a$, 2010 $_b$, 2011).

Thus from equation (7.1), and in the midpoint radius form, we have that

$$A=(A_c-\Delta A, A_c+\Delta A)$$

$$b=(b_c-\delta, b_c+\delta)$$

so that

$$A_c=\frac{1}{2}\left(\underline{A}+\bar{A}\right), x_c=\frac{1}{2}\left(\underline{x}+\bar{x}\right),\ b_c=\frac{1}{2}\left(\underline{b}+\bar{b}\right).$$

The following interval operators are well known:

Modified Interval Gauss-Siedel method Uwamusi (2004, 2010_a, 2011):

$$x_i^{(k+1)} = x_i^{(k)} + d_i^{(m)}$$
$$d_i^{(m+\frac{s+1}{2})} = \frac{1}{a_{ii}}\left(-b_i - \sum_{j=1}^{i-1} a_{ij} d_j^{\left(m+\frac{s+1}{2}\right)} - \sum_{j=i+1}^{n} a_{ij} d_j^{\left(m+\frac{s}{2}\right)}\right), (s = 0,1,\ldots,i = 1,2,\ldots,n) \quad (3.3.3)$$

Modifield Krawczyk's method [19]:

$$K[x]^{\left(k+\frac{v+1}{q}\right)} = y^{(k)} - H^{(k)} f\left(y^{(k+\frac{v+1}{q})}\right) + \left(I - H^{(k)} F'\left([x]^{(k)}\right)\right)\left(\hat{X}^{\left(k+\frac{v+1}{q}\right)} - y^{\left(k+\frac{v+1}{q}\right)}\right) \quad (3.3.4)$$

Where,

$$\hat{X}^{(k+\frac{v+1}{q})} = y^{\left(k+\frac{v}{q}\right)} - H^{(k)} f\left(y^{\left(k+\frac{v}{q}\right)}\right) + \left(I - H^{(k)} F'(x)\right)^{(k)}\left(X^{\left(k+\frac{v}{q}\right)} - y^{(k+\frac{v}{q})}\right) \quad (3.3.5)$$

$(k = 0,1,\ldots,v = 0,1,\ldots,t-1, q = 2,3,\ldots,t)$ for t a positive integer.

We present the following Example1 with corresponding solutions of their operators.

Example 1.

$$F(x) = \begin{cases} 6x_1 - 2\cos(x_2 x_3) - 1 = 0 \\ 9x_2 + \sqrt{\left(x_1^2 + \sin x_3 + 1 - 06\right) + 0.9} = 0 \\ 60x_3 + 3e^{-x_1 x_2} + 10\pi - 3 = 0 \end{cases}$$

$$x^{(0)} = \begin{pmatrix} 0.1 \\ -0.1 \\ 0.1 \end{pmatrix}.\text{Take } \varepsilon = 10^{-2}.$$

The following results were obtained earlier as represented in Tables 1 and 2.

Table 1: Showing results computed by Modified Krawczyk'smethod of Equation (3.3.4)

Iteration k	Results $[x_k]$
1	[0.498901717, 0.498901710] [-0.199872459, -0.199873833] [-0.530073299, -0.530083371]
2	[0.498144782, 0.498144782] [-0.199605179, -0.199605179] [-0.528826126, -0.528826126]

Table2: Showing Results computed by Modified Interval Gauss-Siedel method (3.3.3)

Iteration k	Modified Interval Gauss-Siedel Method (7.3) , x_k	$\|F(m(x^{(k)})\|$
1	[0.499714714, 0.500295002] [[-0.1992852586, -0.191648811] [-0.527055653, -0.52654115]	0.010277487 0.068207109 0.123361608
2	[0.498146418, 0.498147691] [-0.199594945, -0.199594528] [-0.529037166, -0.529036769]	0.000021842 0.000000148 0.000031327
3	[0.498143417, 0.498143420] [-0.1995945558, -0.199594558] [-0.529036401, -0.529036401]	0.000000011 0.000000002 0.000000065

Table 3: Showing Results computed by interval LU Factorization.

Iteration k	Results, $[x_k]$	$\|F(m(x^{(k)}))\|$
1	[0.499712049, 0.500298881] [-0.199983564, -0.196435131] [-0.529487889, -0.529058968]	0.011028171 0.013388645 0.015284972
2	[0.49815262, 0.498153567] [-0.199740354, -0.19973938] [-0.529063563, -0.529061228]	0.00000075344 0.001314920 0.001313437
3	[0.49814342, 0.498814342] [-0.199594562, 0.199594552] [-0.529036524, -0.52903634]	0.000000002 0.000000003 0.000001925

We compute the average incident ray vectors for the sequence of iterates as $\frac{MX_1 + MX_2 + MX_3}{3}$ for each operator as displayed below where, $MX_i = \frac{1}{2}[X_i]$.

Table 4

Method 3.3.3 (Average iterated Gauss-Siedel method)	Method 3.3.4 (Average iterated Modified Krawczyk's method)	Average iterated LU Decomposition Method
$\begin{pmatrix} 0.498765110333333 \\ -0.19821877643333 \\ -0.52829059 \end{pmatrix}$	$\begin{pmatrix} 0.49852324775 \\ -0.197962468 \\ -0.5294522305 \end{pmatrix}$	$\begin{pmatrix} 0.4988791465 \\ -0.19915711516666 \\ -0.52911560933333 \end{pmatrix}$

The imaging and distortion can now be discussed. We compute the normalized distance

$d(x,y) = \left(\frac{\|x-y\|_2^2}{\|x\|_2^2 + \|y\|_2^2 + c} \right)^{\frac{1}{2}}$ for $c = 0.5$ to be 0.760644028145487 using for example, the Modified Interval Gauss-Siedel method for $y = \frac{1}{n}\sum_{k=1}^{n} F(m(x_k))$.We hope to dwell further on this in subsequent papers.

4. DISCUSSION

The basic circular interval arithmetic initiated in Rump (1999) as modified in Uwamusi (2010 a ,2010 $_b$) were motivated in the implementation of interval Gauss-Siedel method as well as interval LU Factorization for the given problem. Also discussed in the implementation was the interval Kracwzyky's method as modified in Uwamusi (2009) which uses Moore's interval arithmetic see e.g., Uwamusi (1999)whose speed of convergence was twice as fast as the original Krawczyk's method adopted. In the sense of continuity and effective covering properties of these methods, it is assumed that multivalued maps arising from the modifications of these interval operators are both lower and weakly upper semi-continuous with $U-$chains reachable domains where $U \cup D \subset R^n$. We also give the backward error analysis for the data problem for the linear system and showed that the Gaussian elimination method respectively, LU Factorization cannot calculate effectively the inverse of a matrix under certain conditions. It is known Uwamusi (2009) that interval LU Factorization is feasible only when interval Gaussian elimination method exists. In so doing we therefore gave error bounds for both Gaussian elimination method and LU Factorization in the sense of Golub and Van-Loan (1983).Using Brunet et al (2017), standard statistical estimate for illuminating set is described with a normalized distance for the imaging and distortion of a ray tracing implicit surface for the solution set of nonlinear system of equation computed. We hope to dwell more on this in subsequent works. All results computed for the solution set in the interval forms are displayed in Tables 1-4 with guaranteed self-validating error bounds. In particular, Table 4 gives the average of sequences of iterations for the operators which were used to compute the normalized distance for the imaging and distortion arising from a ray tracing implicit surface.

5. CONCLUSION

The paper presented functional iterative methods for both univariate and multivariate functions in topological space. In particular, we paid special attentions to the solution to interval nonlinear system of equations. We explained in details conditions under which a solution to the given problem is contractible to a desired solution. For example, using Banach fixed point theory, our discussion on a Baire space showed in the sense ofBourles (2011), that a completely regular topological space is K-Suslin if, and only if, it is K-analytic. In addition it is pointed out that a Polish space, the Suslin space, K-Suslin space , the Quasi—Suslin space are webbed which also coincides with Iyahen(1998).This formed the basis of completely regular uniformable computable topological for the chainable maps with effective covering property. In this sense, the three main iterative methods of Jacobi, Gauss-Siedel and Successive Over-relaxation techniques were presented and analyzed. The average rates of convergence of these methods were compared. It was also discussed under what conditions will Gaussian elimination and LU decomposition methods fail to compute exactly, the inverse of a regular matrix.

We took numerical example from the earlier works of Uwamusi (2004,2009, 2010_a, 2010_b, 2011) to describe interval implementation of these algorithms bearing in mind that distributive law does not hold in general for interval arithmetic operations. In particular, the modified Fast Krawczyk' algorithm due to Uwamusi (2009) formed the basis of comparison of results obtained with circular interval arithmetic operations with guaranteed error bounds. This is one main advantage enjoyed by interval arithmetic operations since they are self-validating numeric. We again observed that the interval counterpart of Successive over-relaxation technique is not profitable in the interval libraries Uwamusi (2009, 2011) due to dependency problems of interval arithmetic operations. But we have to bear in mind that in real floating point calculations where interval arithmetic operations are completely absent, the SOR method has higher computational advantage over Gauss-Siedel method due to its high speed of convergence to the desired solutions.

We extended our views of application areas to imaging and distortion as they pertain to a ray tracing implicit surfaces with results computed from these aforementioned fast interval solvers. It is computed for the normalized distance between imaging and distortion of a ray tracing implicit surface. We hope to dwell more of such research in our future endeavors.

REFERENCES

[1] C. Berg. Complex analysis, Department of Mathematical Sciences, Universitetsparken 5, Copenhagen, (2012).

[2] A. Bjorck. Numerical methods in Scientific Computing, Vol. 2 , SIAM, (2009).

[3] I. J. Borsik, L. Hola, D. Holy.Baire Spaces and quasicontinuous mappings. Filomat 25:3,(2011),69-83,

[4] H. Bourles. On the Closed Graph theorem and Open mapping theorem , arXiv.org>math>arXiv1411-5500, (2011).

[5] D. Brunet, S. S. Channappayya, Z. Wang, E. R.Vrscary, A. C.Bovik. Chapter in Hand book of Convex optimization methods in Imaging Science, Springer International Publishing, (2017.

[6] C. E .Chidume. Functional Analysis: Fundamental Theorems and Applications. International Centre for Theoretical Physics,(1995).

[7] P. Collins. Continuity and Computability of reachable sets. Theoretical Computer Science 341,(2005), 162-195.

[8] M. Eiermann, R. S. Varga. Is the Optimal ω best for the SOR method? Algebra and its applications 182,(1993), 257-277.

[9] M. R. Farmer, G. Loizou. An algorithm factorization of a polynomial, Math. Proc. Cambridge Philos. Soc. 82 ,(1977), 427-437.

[10] G. H. Golub, C. F. van-Loan. Matrix Computations, The Johns Hopkins University Press, Baltimore,Maryland, USA, (1983).

[11] L.A. Hageman, F.T.Luk, D.M. Young.On the equivalence of certain iterative acceleration methods, SIAM Journal of Numerical Analysis, 17(60) ,(1980), 852-873.

[12] S. O. Iyahen. The Closed Graph Theorem, Osaruwa Educational Publications, First Edition, Benin City, (1998).

[13] O. Karlova. On BaireClassification of mappings with values in Connected Spaces. European Journalof Mathematics, Vol.2, Issue 2, (2016), 526-538.

[14] J. M. Ortega, W. C. Rheinboldt. Iterative Solution of Nonlinear equations in several variables. Classics in Applied Mathematics, SIAM Philadelphia, USA, (2000).

[15] M. Petkovic, S. Trikovic. On Zero finding method of the fourth order. J. Computational Appl. Math. Vol. 64 ,(1995), 291-294

[16] S. M. Rump. Fast and parallel interval arithmetic. BIT, 39(3), (1999) 534-554.

[17] S. E. Uwamusi. On a Family of Methods for Simultaneous Inclusion of Zeros of Polynomial and Analytic Functions inside a Simple Smooth closed Contour in the Complex Plane. Journal of Nig. Association of Mathematical Physics, Vol.3, (1999) 331-322.

[18] S. E. Uwamusi. “Q Step Methods for Newton-Jacobi Operator Equation” Journal of the Nig. Association of Math. Physics Vol. 8 No.1, (2004), 237-240.

[19] S. E. Uwamusi. Modified accelerated Krawczyk’s algorithm.International Journal of Physical sciences Vol. 4(2), (2009), 47-052. Available online at http://academicjournals.org/IJPS

[20] S. E. Uwamusi. Parallel iterations for Cholesky and Rohn's methods for solving over-determined linear interval systems. Research Journal of Mathematics and Statistics 2(3): (2010_a),86-90.

[21] S. E. Uwamusi. A Remark on the solution set of interval Nonlinear systems of equations via Markov's method. Research Journal of Mathematics and Statistics 2(3):(2010_b),97-100.

[22] S.E. Uwamusi. Contracting interval methods for Nonlinear systems of equations in R^n. ABACUS, Vol. 38, No. 2, (2011)186-194.

[23] S.E. Uwamusi. Computable Chain-reachable Mapping of Linear and Quadratic Backward SOR Iteration for Newton Operator. Journal of Applied Science and Research, 4(3):(2016),18-27.ISSN 2348-0416,USA CODEN: JASRHB, Available online at www.scientiaresearchlibrary.com

[24] R. Varga.Matrix iterative Analysis, Classics in Applied Mathematics, SIAM, Philadelphia, USA, (2000).

[25] B. Walden, R. Karlson, J.G. Sun. Optimal backward perturbation bounds for the linear least squares problem. Numer. Linear Algebra Appl., 2(3),(1995),271-286.

[26] W. Wasilkowski. Can any stationary iteration using linear information be globally convergent? J. Assoc. Computing Machinery 27, (1980), 263-269.

[27] D.M. Young. Iterative solution of large linear systems, Academic Press, New York, Chapter 5-6, (1971).

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