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WEIBULL-EXTENDED PRANAV DISTRIBUTION: APPLICATION TO LIFETIME DATA SETS

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In this article, Weibull link function initiated by Jones (2004) and extended Pranav distribution written by Uwaeme *et al*. (2018) are convolute to introduced Weibull-extended Pranav distribution using the idea of Tahir *et al*. (2016). The motivation is to develop a robust and flexible distribution that will have better fit to any skewed data set. Different properties of the proposed distribution are obtained. Estimation of model parameters with the method of maximum likelihood estimates are presented. Then, the new distribution is compared with some existing distributions by illustrating two different life time data sets. Hence, the results through estimation and goodness of fit criteria reveal that Weibull-extended Pranav distribution has better fit to the data sets than other distributions considered in the study.

1. INTRODUCTION

The Weibull distribution is a well-known, widely used and flexible distribution. While, Pranav distribution is a mixture of two different univariate continuous distributions known as exponential with a scale parameter and gamma with a shape parameter 4 and scale parameter θ. Shukla (2018). Therefore, in literature, different methods of convolution (link functions) have been developed and used by several authors and researchers. Some of the methods of convolute are discussed briefly as follows: generator approach initiated by (Eugene et al., 2002) and used by Lemonte, (2014) in his work. Beta link function was introduced by Jones (2004) also used by many authors in their studies, such as: Barreto-Souza and Cordeiro (2010), Badmus and Bamiduro (2014), (Badmus et al., 2015 and 2017) and many more.
However, exponentiated method was used by Bakoush (2012) where he introduced an extended Lindley distribution, the exponentiated inverted Weibull distribution by (Flaih et al., 2012) and exponentiated power Lindley distribution by Ashour and Eltehiwy (2015). While, exponentiated T-X family of distribution introduced by (Alzaghal et al., 2013), which was used by (Alzaatreh et al., 2013), (Carl et al., 2013) and Handy and Mahmoud (2014) in their works. While, Weibull link function was studied and written by Tahir et al. (2016) which (leren et al., 2018) followed their ideas in their paper.
The main aim of this work is to develop a new continuous distribution called the Weibull-extended Pranav distribution using Weibull link function in conjunction with extended Pranav distribution which was introduced by (Uwaeme et al., 2018). Therefore, the work is arranged and presented in sections as follows:

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2. MATERIALS AND METHODS

2.1 The Weibull-Extended Pranav Distribution

Recently, Uwaeme *et al*. (2018) discussed an extended Pranav distribution with two parameters and was used for modelling real life data set. The distribution and density functions are given respectively:

$$Q(y) = \left[1 - \left[1 + \frac{\alpha^3 y^3 + 3\alpha^2 y^2 + 6\alpha y}{\alpha^4 + 6}\right] e^{-\alpha y}\right]^{\beta}, y > 0,\ \alpha, \beta > 0. \qquad (2.1.1)$$

and

$$q(y) = \beta \frac{\alpha^4}{\alpha^4 + 6}(\alpha + y^3) e^{-\alpha y} \left[1 - \left[1 + \frac{\alpha^3 y^3 + 3\alpha^2 y^2 + 6\alpha y}{\alpha^4 + 6}\right] e^{-\alpha y}\right]^{\beta - 1};\ y > 0, \alpha, \beta > 0 \quad (2.1.2)$$

where, α is the scale parameter.

Tahir *et al*. (2016) proposed a Weibull link function for obtaining pdf and cdf of any Weibull based distribution and was used by leren *et al*. (2018). The pdf and cdf is defined as:

$$f(y) = \frac{\lambda\theta g(y)}{G(y)}\left[-log[G(y)]\right]^{\theta - 1} e^{-\lambda[-log[G(y)]]^{\theta}} \qquad (2.1.3)$$

and

$$F(y) = \int_0^{-logG(y)} \lambda\theta y^{\theta - 1} e^{-\lambda y^{\theta}} = e^{-\lambda[-log[G(y)]]^{\theta}} \qquad (2.1.4)$$

where, $\lambda, \theta, g(y)$ and $G(y)$ are shape parameters, pdf and cdf of any Weibull based distribution. Fortunately, we obtain the pdf and cdf of the Weibull-Extended Pranav distribution by simplifying and substituting equations (2.1.1 and 2.1.2) into (2.1.3 and 2.1.4) respectively:

$$f_{WEP}\ (y) = \frac{\lambda\theta\alpha^4(\alpha + y^3)e^{-\alpha y}\left\{1 - \left[1 + \frac{\alpha^3y^3 + 3\alpha^2y^2 + 6\alpha y}{\alpha^4 + 6}\right]e^{-\alpha y}\right\}^{\beta - 1}}{(\alpha^4 + 6)\left\{1 - \left[1 + \frac{\alpha^3y^3 + 3\alpha^2y^2 + 6\alpha y}{\alpha^4 + 6}\right]e^{-\alpha y}\right\}^{\beta}}$$

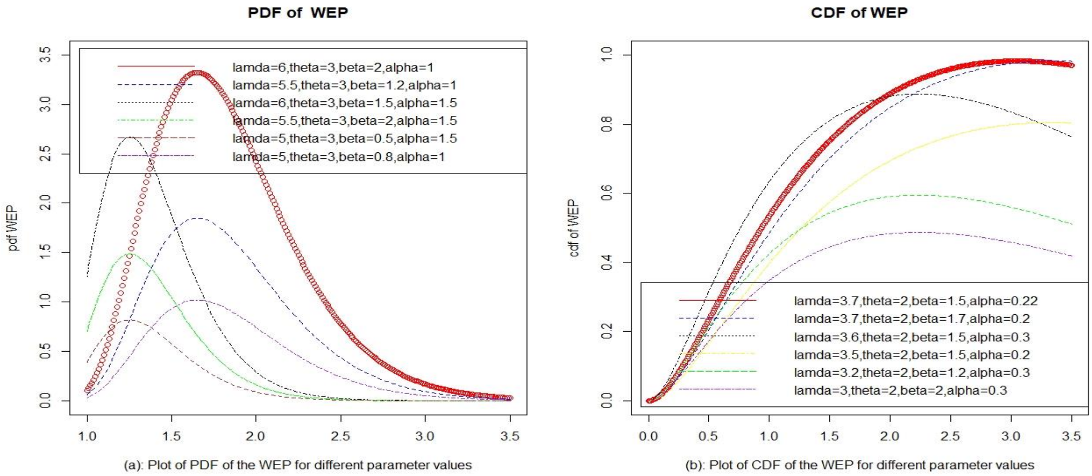
$$\left[-log\left[1 - \left[1 + \frac{\alpha^3 y^3 + 3\alpha^2 y^2 + 6\alpha y}{\alpha^4 + 6}\right] e^{-\alpha y}\right]^{\beta}\right]^{\theta - 1}$$

$$e^{-\lambda\left[-log\left[1 - \left[1 + \frac{\alpha^3y^3 + 3\alpha^2y^2 + 6\alpha y}{\alpha^4 + 6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}}\ y;\ \lambda, \theta, \beta\ and\ \alpha > 0 \quad (2.1.5)$$

and

$$F_{WEP}\ (y) = e^{-\lambda\left[-log\left[1 - \left[1 + \frac{\alpha^3y^3 + 3\alpha^2y^2 + 6\alpha y}{\alpha^4 + 6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} \qquad (2.1.6)$$

where, (λ, θ, β) and (α) the first-two parameter are the two additional shape parameters, the existing shape parameter and scale parameter of the Weibull-extended Pranav distribution. The usefulness of the shape parameter is to control the tails weights and the possible plots of the pdf and cdf of the Weibull-Pranav distribution for various values of parameters from (2.1.5) and (2.1.6) as shown in figure 1 below:



(a): Plot of PDF of the WEP for different parameter values (b): Plot of CDF of the WEP for different parameter values

FIGURE 1. The shape and plots of the PDF in (a) and CDF in (b) of the WEP distribution with various parameter values

Some reliability functions such as the survival, hazard rate and reverse hazard rate functions corresponding to WEP distribution were presented as follows:

2.2 The Survival Function

The survival function is known as reliability function is a function that gives the likelihood and probability that an individual, a system or object of interest will survive after some time. The mathematical expression is given by

$$SUR_{WEP}(y) = P(Y < y) = 1 - F(y) \qquad (2.2.1)$$

where $F(y)$ is the cdf of the Weibull-Extended Pranav distribution in (2.16). Then, substituting (2.16) in (2.21) and simplifying (2.21) yields (2.22) which is the survival function of WEP distribution.

$$SUR_{WEP}(y) = 1 - e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} \qquad (2.2.2)$$

2.3 The Hazard Rate Function

The hazard function is also called risk function or failure rate function that gives the probability that an object or device will die or fail within an interval of time. It is used to model data distribution in survival analysis. Therefore, its mathematical expression is given by

$$Hz_{WEP}(y) = \frac{f(y)}{1-F(y)} = \frac{\frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}[U]^{\beta-1}\left[-log[U]^{\beta}\right]^{\theta-1}e^{-\lambda\left[-log[U]^{\beta}\right]^{\theta}}}{(\alpha^4+6)[U]^{\beta}}}{1-e^{-\lambda\left[-log[U]^{\beta}\right]^{\theta}}} \qquad (2..3.1)$$

where $U = 1 - \left[1 + \frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}$

We obtain the WEP distribution hazard rate function by putting $f(y)$ and $F(y)$ as the pdf and cdf of the WEP which gives

$$Hz_{WEP}(y) = \frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\{U\}^{\beta-1}\left[-log[U]^{\beta}\right]^{\theta-1}e^{-\lambda\left[-log[U]^{\beta}\right]^{\theta}}}{(\alpha^4+6)\{U\}^{\beta}\left[1-e^{-\lambda\left[-log[U]^{\beta}\right]^{\theta}}\right]} \tag{2.3.2}$$

2.4 The Reversed Hazard Function

The reversed hazard function is the probability that defined the ration between the probability density to its distribution function. It's expressed mathematically below as:

$$RHz_{WEP}(y) = \frac{f(y)}{F(y)} = \frac{\dfrac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\{U\}^{\beta-1}\left[-log[U]^{\beta}\right]^{\theta-1}e^{-\lambda\left[-log[U]^{\beta}\right]^{\theta}}}{(\alpha^4+6)\{U\}^{\beta}}}{e^{-\lambda\left[-log[U]^{\beta}\right]^{\theta}}} \tag{2.4.1}$$

Hence, the reversed hazard function of the WEP is also obtain by taking $f(y)$ and $F(y)$ as the pdf and cdf of the WEP and its define by

$$RHz_{WEP}(y) = \frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\{U\}^{\beta-1}\left[-log[U]^{\beta}\right]^{\theta-1}e^{-\lambda\left[-log[U]^{\beta}\right]^{\theta}}}{(\alpha^4+6)\{U\}^{\beta}\left[e^{-\lambda\left[-log[U]^{\beta}\right]^{\theta}}\right]} \tag{2.4.2}$$

Then, the plots of the survival and hazard function of the Weibull Extended Pranav distribution are shown with different parameter values. In figure 2 below, plot (a) revealed that the WEP distribution can be used to model random variable whose survival rate decreases as they grow old due to the probability of survival for any random variable following a WEP distribution decreases as the values of the random variable increases, therefore, probability of life decreases as life moves on. While, plot (b) depict the characteristic of the hazard function of the WEP distribution. This equally means that the probability of risk for any WEP random variable increases as year or time of the variable continues.

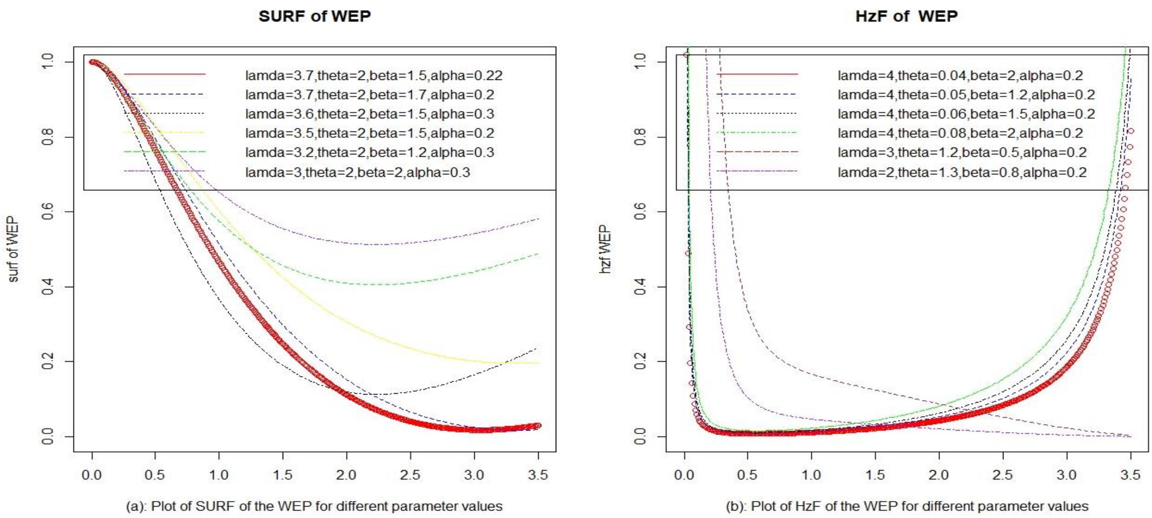


FIGURE 2. The plots of Survival and Hazard functions of the WEP distribution with different parameter values

2.5 Moments

We derive and obtain the s-th moment of the WEP distribution as the s-th moment of a random variable Y is given by

$$E(Y^s) = \int_0^y y^s f_{WEP}(y)dy \quad (2.5.1)$$

$$= \int_0^\infty y^s \frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta-1}$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} dy \quad (2.5.2)$$

$$= \int_0^\infty \frac{\lambda\theta\alpha^5 y^s e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta-1}$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} dy +$$

$$= \int_0^\infty \frac{\lambda\theta\alpha^4 y^{s+3} e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta-1}$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} dy \quad (2.5.3)$$

The purpose of the equations is to obtain the s-th order moment about the origin of the new distribution. Other equations are presented in appendix below

Hence, the s-th order moment about the origin, $E(Y^s)$ of the WEP distribution is given by

$$E(Y^s) = \frac{A_{i,j,q,k}\frac{(s+3j-q-k)!}{[i+1]^{s+3j-q-k+1}}+B_{i,j,q,k}\frac{(s+3j-q-k+3)!}{[i+1]^{s+3j-q-k+4}}}{(\alpha^4+6)^{j+q+k+1}C_{i,j,q,k}\frac{(s+3j-q-k+3)!}{[i+1]^{s+3j-q-k+3}}}.D_{i,j,q,k}\frac{(s+3j-q-k+3)!}{[i+1]^{s+3j-q-k+3}}E_{i,j,q,k}\frac{(s+3j-q-k)!}{[i+1]^{s+3j-q-k+1}} \quad (2.5.4)$$

where, A, B, C, D and E are stated in equations (i, ii, iii, iv and v) in the appendix.

2.6 Moment Generating Function

Here, we propose the moment generating function (mgf) as a useful tool for computing any distribution's moment. Wherefore, the mgf of WEP distribution is obtain as follows:

$$M_{WEP(y)}(t) = E(e^{ty}) = \int_0^\infty e^{ty} f_{WEP}(y)dy \quad (2.6.1)$$

$$\int_0^\infty e^{ty} \frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta-1}$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} dy$$

$$= \int_0^\infty \frac{\lambda\theta\alpha^5 y^s e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta-1}$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} dy +$$

$$\int_0^\infty \frac{\lambda\theta\alpha^4 y^{s+3} e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta-1}$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} dy$$

From the expressions above in the s-th moment, the mgf of the WEP distribution is therefore given by

$$M_{WEP(y)}(t) = \sum_{z=0}^{\infty}\left(\frac{t}{\alpha}\right)^z \left\{\frac{A_{i,j,q,z}\frac{(s+3j-q-z)!}{[i+1]^{s+3j-q-z+1}}+B_{i,j,q,z}\frac{(s+3j-q-z+3)!}{[i+1]^{s+3j-q-z+4}}}{(\alpha^4+6)^{j+q+z+1}C_{i,j,q,z}\frac{(s+3j-q-z+3)!}{[i+1]^{s+3j-q-z+3}}}\right. .$$

$$\left. D_{i,j,q,z}\frac{(s+3j-q-z+3)!}{[i+1]^{s+3j-q-z+3}} . E_{i,j,q,z}\frac{(s+3j-q-z)!}{[i+1]^{s+3j-q-z+1}}\right\} \quad (2.6.2)$$

2.7 Order Statistics

Order statistics can be defined as a tool widely used in statistical theory for solving complex problems namely: detection of outliers, goodness of fit tests, entropy estimation and so on Tahir *et al*. (2016) and leren *et al*. (2018). Let $Y_1,\dots,Y_n$ be a random sample from a distribution with density function $f_{WEP}(y)$ and $Y_{1:n} <,\dots,< Y_{k:n}$ denote the associating order statistics from the random sample. Then, the density function $f_{k:n}(y)$ of the kth order statistic is expressed as

$$f_{k:n}(y) = \frac{n!}{(k-1)!(n-k)!}\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i} f_{WEP}(y)F_{WEP}(y)^{\,i+k-1} \quad (2.7.1)$$

where, $f_{WEP}(y)$ and $F_{WEP}(y)$ are both the density and distribution functions of Weibull-Extended Pranav distribution. By substituting equations (2.1.5) and (2.1.6) into (2.7.1) yields the density function of the kth order statistics $Y_{k:n}$ as given below

$$f_{k:n}(y) = \frac{n!}{(k-1)!\,(n-k)!}\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\left[\frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\right]$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}} \quad (2.7.2)$$

Therefore, the density function of both the minimum and maximum ($Y_{(1)}$) and ($Y_{(n)}$) order statistic of the WEP distribution are expressed by

$$f_{1:n}(y) = n\sum_{i=0}^{n-1}(-1)^i\binom{n-1}{i}\left[\frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\right]$$

$$\left[e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}}\right]^{i} \tag{2.7.3}$$

also

$$f_{n:n}(y) = n\left[\frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}\right]$$

$$\left[e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}}\right]^{n-1} \tag{2.7.4}$$

2.8 Estimation

Maximum likelihood estimation MLE is mostly and widely used in estimating model parameters of any develop model in literature. Let $Y_1, \dots, Y_n$ be a random sample of size n from the Weibull-Extended Pranav distribution with unknown parameters λ, θ, β and α defined in (2.1.6) above. Then, the log-likelihood function can be expressed as

$$f_{WEP}\ (y) = \frac{\lambda\theta\alpha^4(\alpha+y^3)e^{-\alpha y}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}}{(\alpha^4+6)\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}}$$

$$\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta-1}$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}}$$

Then, the likelihood function (LF) is obtain as follows:

$$LL(Y|\lambda,\theta,\beta,\alpha) = \frac{(\lambda\theta\beta\alpha^4)^n\prod_{i=1}^{n}\{(\alpha+y_i^3)e^{-\alpha y_i}\}\prod_{i=1}^{n}\left\{1-\left[1+\frac{\alpha^3y_i^3+3\alpha^2y_i^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right\}^{\beta-1}\prod_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]^{\theta-1}}{(\alpha^4+6)^n\prod_{i=1}^{n}\left\{1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y_i}\right\}^{\beta}e^{\lambda\sum_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]^{\theta}}} \tag{2.8.1}$$

$$LL = nlog\lambda + nlog\theta + nlog\beta + 4nlog\alpha - nlog(\alpha^4+6) + \sum_{i=1}^{n}log\left(\alpha+y_i^3\right) - \alpha\sum_{i=1}^{n}y_i +$$
$$(\beta-1)\sum_{i=1}^{n}\left\{1-\left[1+\frac{\alpha^3y_i^3+3\alpha^2y_i^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right\} + (\theta-1)\sum_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]$$

$$-\lambda\sum_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]^{\theta} \tag{2.8.2}$$

Taking the partial derivatives with respect to parameters λ, θ, β and α, yields the following

$$\frac{\partial LL}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]^{\theta} \tag{2.8.3}$$

$$\frac{\partial LL}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right] - \lambda\sum_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]^{\theta}$$
$$\left[-log\left[1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right] \tag{2.8.4}$$

$$\frac{\partial LL}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\left\{1-\left[1+\frac{\alpha^3y_i^3+3\alpha^2y_i^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right\} + (\theta-1)\sum_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]$$
$$-\lambda\sum_{i=1}^{n}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]\left\{1-\left[1+\frac{\alpha^3y_i^3+3\alpha^2y_i^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right\}^{\theta} \tag{2.8.5}$$

$$\frac{\partial LL}{\partial \alpha} = \frac{4n}{\alpha} - \frac{n}{\alpha^4+6} - \sum_{i=1}^{n}y_i + (\beta-1)\sum_{i=1}^{n}\left[\frac{y_{ie^{-\alpha y_i}}\left\{\frac{1}{(\alpha^4+6)^2}-\frac{\alpha y_i+\alpha^4+6}{\alpha^4+6}\right\}}{\prod_{i=1}^{n}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}}\right]$$
$$-\sum_{i=1}^{n}\left[\frac{y_{ie^{-\alpha y_i}}\left\{\frac{1}{(\alpha^4+6)^2}-\frac{\alpha y_i+\alpha^4+6}{\alpha^4+6}\right\}}{\left\{1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y_i}\right\}^{\beta}}\right] + (\theta-1)\sum_{i=1}^{n}\left[\frac{y_{ie^{-\alpha y_i}}\left\{\frac{1}{(\alpha^4+6)^2}-\frac{\alpha y_i+\alpha^4+6}{\alpha^4+6}\right\}}{\prod_{i=1}^{n}\left\{1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y_i}\right\}^{\beta}\left\{1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y_i}\right\}^{\beta}}\right]$$
$$-\sum_{i=1}^{n}\left[\frac{y_{ie^{-\alpha y_i}}\left\{\frac{1}{(\alpha^4+6)^2}-\frac{\alpha y_i+\alpha^4+6}{\alpha^4+6}\right\}}{\left[\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]^{\theta}}\right]$$
$$+\lambda\theta\beta\sum_{i=1}^{n}\left\{\frac{y_{ie}-\alpha y_i\left\{\frac{1}{(\alpha^4+6)^2}-\frac{\alpha y_i+\alpha^4+6}{\alpha^4+6}\right\}\left\{1-\left[1+\frac{\alpha^3y_i^3+3\alpha^2y_i^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right\}^{\beta-1}\left[-log\left[1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}\right]^{\theta-1}}{\left\{1-\left[1+\frac{\alpha^3y_i^3+3\alpha^2y_i^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right\}\left[1-\left[1+\frac{\alpha^3{y_i}^3+3\alpha^2{y_i}^2+6\alpha y_i}{\alpha^4+6}\right]e^{-\alpha y_i}\right]^{\beta}}\right\} \tag{2.8.6}$$

Equations (2.8.3 to 2.8.6) can be solved by using Newton Raphson method to obtain the $(\hat{\lambda}, \hat{\theta}, \hat{\beta}, \hat{\alpha})$ the MLE of $(\lambda, \theta, \beta, \alpha)$, respectively. Hence, since the equations cannot be solved manually but numerically by implore statistical software like R, SAS and so on once the data sets are readily available. Then, we use the Newton Raphson method from maxLik package in R software.

2.9 Applications to Real-Life Data Sets

In this section, we show numerically the robustness and flexibility of the propose WEP distribution to various existing distributions in literature using two different real-life data sets. We also achieve this through the goodness-of-fit of the propose distribution comparing with distributions haven common characteristics like: Extended Pranav (EP) distribution by Uwaeme, *et al*. (2018), Pranav distribution by Shukla (2018), Weibull-Lindley (WLn) distribution by leren *et al*. (2018), Two-parameter Pranav (TPP) distribution by Umeh and Ibenegba (2019), The Akash distribution (AD) by Shanker (2015), A Discrete Pranav (DP) distribution by Abebe and Shukla (2019) and a new distribution called Weibull-Pranav distribution.

We based the comparison on some selected criteria of a distribution like parameter estimate, Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Haman-Quinn Information Criterion (HQIC) and Bayesian Information Criterion (BIC).

Dataset One
The data set one contains the strength of data of glass of the aircraft window as reported by Fuller *et al*. (1994) and used by Uwaeme *et al*. (2018). The summary of the data is shown in Table 1 below:

Dataset Two
Data set two represents the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm as reported by Bader and Priest (1982) and used by Shukla (2019). The summary of the data is shown in Table 2 below:

The maximum likelihood estimates, -2 *LogL*, *AIC*, *CAIC*, *HQIC* and *BIC* statistics of the fitted distributions are shown in Table 3 for data set one and two respectively. Likewise, we provide the histogram and the estimated pdf of the WEPD, EPD, TPPD, WLnD, WPD, PD, AD and DPD for the two data sets in figure 3 below.

3. RESULT

Here, we are able to showcase the efficiency and robustness of the new distribution over other distribution considered in the study by using two different real data sets. Results from the analysis are stated as follows: Table 1 and 2 showed the descriptive statistics summary of each data set, while Table 3 reflects the maximum likelihood estimated values and the model selection criteria i. e *LogL*, *AIC*, *CAIC*, *HQIC* and *BIC.* From all indications, the propose distribution has smaller values which made it to be more efficient and robust than others. Also, it is very easy to note from figure 3 (estimated pdfs for both datasets) below that the WEP distribution (in red colour line) has good and better representation of the two data sets than any other distributions considered in the study.

4. DISCUSSION

A new distribution called Weibull-Extended Pranav (WEP) distribution has been introduced using Weibull link model by Tahir *et al*. (2016) which has its baseline distributions from Extended Pranav and Pranav distributions by Uwaeme *et a*l. (2018) and Shukla (2018). Some of its statistical properties which include its reliability, hazard rate and reversed hazard function, moments, moment generating function, order statistics were derived and discussed. The method of maximum likelihood estimation for estimating the distribution parameter is presented as well. Two examples of real-life data sets were presented to illustrate the applications model selection criteria of the Weibull-Extended Pranav, Extended Pranav, Two-Parameter Pranav, Weibull-Lindley, Weibull Pranav, Pranav, Akash and Discrete Pranav distributions.

Table 1: Descriptive (summary) Statistics for strength of data of glass of the aircraft window

Min	Median	Mean	Max	Variance	Skewness	Kurtosis
18.83	29.90	30.81	45.38	52.61154	0.4053965	2.286637

Table 2: Descriptive (summary) Statistics for the tensile strength, measured in GPa, of 69 carbon fibers

Min	Median	Mean	Max	Variance	Skewness	Kurtosis
1.312	2.478	2.451	3.585	0.2451677	-0.028216	2.940733

5. CONCLUSION

Conclusively, result from the analysis in Table 3 indicates that the Weibull-Extended Pranav has the lowest values of *LogL, AIC, CAIC, HQIC* and *BIC* and this implies that the smaller the *LogL, AIC, CAIC, HQIC* and *BIC* values, the better the distribution. Therefore, Weibull-Extended Pranav can be used in modelling such skewed lifetime data sets.

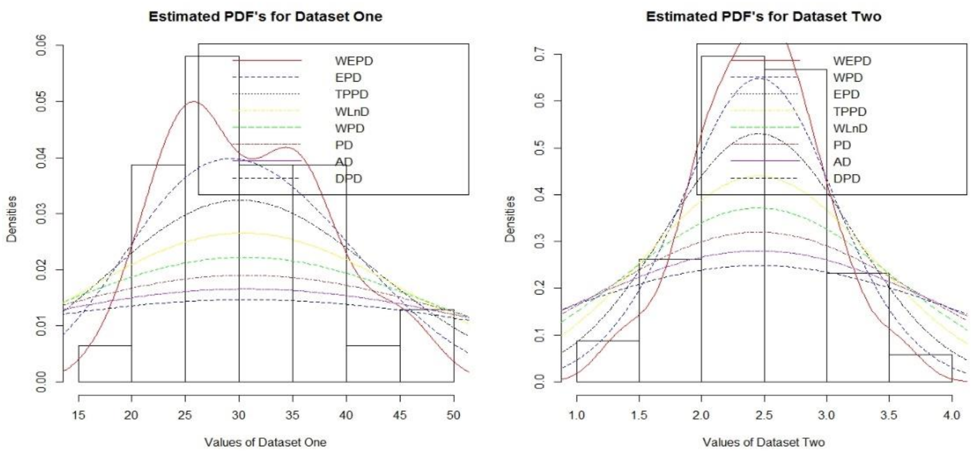


FIGURE 3. The plots of Histogram and Estimated Densities of the WEPD, EPD, TPPD, WLnD, WPD, PD, AD and DPD for the two data sets. Among the distributions presented in the graphs above the propose distribution has better representation and captured the two data sets well than other distributions

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Table 3 MLE's, *-2ln L, AIC, CAIC, HQIC* and *BIC* Statistics of the fitted distributions of data-sets 1 and 2.

Data	**Distr**	**Estimate**	***-2ln L***	***AIC***	***CAIC***	***HQIC***	***BIC***
Set 1	***WEP***	$\boldsymbol{\hat{\lambda} = 3.0000}$ $\boldsymbol{\hat{\theta} = 3.1000}$ $\boldsymbol{\hat{\beta} = 1.1000}$ $\boldsymbol{\hat{\alpha} = 1.5000}$	**303.1**	**614.29**	**615.83**	**616.16**	**620.03**
	EPD	$\hat{\beta} = 2.1000$ $\hat{\alpha} = 0.5000$	445.4	896.84	897.73	898.25	901.15
	TPP	$\hat{\theta} = 3.0000$ $\hat{\alpha} = 0.5000$	460.1	924.21	924.64	925.14	927.07
	WLnD	$\hat{\lambda} = 3.0000$ $\hat{\theta} = 5.5000$ $\hat{\alpha} = 3.5000$	460.1	926.21	927.10	927.61	930.51
	WPD	$\hat{\lambda} = 3.0000$ $\hat{\theta} = 3.5000$ $\hat{\alpha} = 2.5000$	475.5	956.93	957.82	958.33	961.24
	PD	$\hat{\alpha} = 5.0000$	700.7	1403.5	1403.6	1403.9	1404.9
	AD	$\hat{\alpha} = 3.5000$	700.7	1403.5	1403.6	1403.9	1404.9
	DPD	$\hat{\alpha} = 3.5000$	960.8	1923.7	1923.8	1924.2	1925.1
	WEPD	$\boldsymbol{\hat{\lambda} = 3.0000}$ $\boldsymbol{\hat{\theta} = 3.1000}$ $\boldsymbol{\hat{\beta} = 3.5000}$ $\boldsymbol{\hat{\alpha} = 2.5000}$	**357.8**	**723.65**	**724.28**	**727.20**	**732.59**
		$\hat{\lambda} = 7.5722$					

Set 2	*WPD*	$\hat{\theta} = 2.2387$ $\hat{\alpha} = 6.3420$	439.2	884.39	884.76	887.05	891.09
	EPD	$\hat{\beta} = 3.5000$ $\hat{\alpha} = 2.5000$	460.1	926.21	926.58	928.87	932.91
	TPPD	$\hat{\theta} = 2.5000$ $\hat{\alpha} = 3.5000$	460.1	924.21	924.39	925.98	928.68
	WLnD	$\hat{\lambda} = 2.0594$ $\hat{\theta} = 3.0487$ $\hat{\alpha} = 0.1011$	475.9	957.86	958.23	960.52	964.56
	PD	$\hat{\alpha} = 2.4984$	630.7	1263.5	1263.5	1264.4	1265.7
	AD	$\hat{\alpha} = 0.0198$	700.7	1411.6	1411.6	1412.4	1413.8
	DPD	$\hat{\alpha} = 0.2000$	862.992	1727.984	1728.044	1728.871	1730.219

Appendix
Here, we follow (Tahir *et al*., 2016), (Uwaeme *et al*., 2018) and (Ieren *et al*., 2018); and also used binomial expansion to derive the useful equations under moment. We have

$$\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]^{i+j+k}e^{-i,j,k\alpha y}$$

also the binomial expansion of $\left[1+\frac{\alpha y(\alpha y+2)}{\alpha^2+2}\right]^{i+j+k}$ given by

$$\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]^{i+j+k}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i}{p}\binom{j}{q}\binom{k}{r}\left[\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]^{p+q+r}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i}{p}\binom{j}{q}\binom{k}{r}\frac{\alpha y^{(p+q+r)}}{(\alpha^4+6)^{p+q+r}}\sum_{q=0}^{p}\binom{p}{q}\sum_{r=0}^{q}\binom{q}{r}3^q\,.\,2^r\,.\,\alpha^{2p-q-r}\,.\,y^{2p-q-r}$$

Therefore,

$$\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta-1}=\sum_{i=0}^{\infty}\binom{\beta-1}{i}\sum_{j=0}^{\infty}\binom{\lambda}{j}\sum_{k=0}^{\infty}\binom{\theta}{k}(-1)^{i+j+k}\sum_{p=0}^{\infty}\binom{i}{p}$$
$$\sum_{q=0}^{\infty}\binom{j}{q}\sum_{r=0}^{\infty}\binom{k}{r}3^q\,.\,2^r\,.\,\alpha^{2p-q-r}\,.\,y^{2p-q-r}$$

$$=\sum_{i=0}^{\infty}\binom{\beta-1}{i}\sum_{j=0}^{\infty}\binom{\lambda}{j}\sum_{k=0}^{\infty}\binom{\theta}{k}(-1)^{i+j+k}\sum_{p=0}^{\infty}\binom{i}{p}\sum_{q=0}^{\infty}\binom{j}{q}\sum_{r=0}^{\infty}\binom{k}{r}\frac{\lambda.\beta.\theta.3^q\,.\,2^r\,.\,\alpha^{3j-q-k+5}}{(\alpha^4+6)^{j+q+k+1}}$$
$$\int_0^{\infty}y^{s+3j-q-k}\,e^{-\alpha y(i+j+k+1)}dy \qquad \text{(o)}$$

$$\left\{1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right\}^{\beta}=\sum_{i=0}^{\infty}\binom{\beta}{i}(-1)^i\sum_{p=0}^{\infty}\binom{i}{p}\sum_{q=0}^{\infty}\binom{j}{q}\sum_{r=0}^{\infty}\binom{k}{r}3^q\,.\,2^r\,.\,\alpha^{2p-q-r}\,.\,y^{2p-q-r}$$
$$\sum_{i=0}^{\infty}\binom{\beta-1}{i}\sum_{j=0}^{\infty}\binom{\lambda}{j}\sum_{k=0}^{\infty}\binom{\theta}{k}(-1)^{i+j+k}\sum_{p=0}^{\infty}\binom{i}{p}\sum_{q=0}^{\infty}\binom{j}{q}\sum_{r=0}^{\infty}\binom{k}{r}\frac{\lambda.\beta.\theta.3^q\,.\,2^r\,.\,\alpha^{3j-q-k+4}}{(\alpha^4+6)^{p+q+r+1}}$$
$$\int_0^{\infty}y^{s+3j-q-k+3}\,e^{-\alpha y(i+j+k+1)}dy. \qquad \text{(p)}$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i}{p}\binom{j}{q}\binom{k}{r}\frac{\alpha y^{(p+q+r)}}{(\alpha^4+6)^{p+q+r}}\sum_{q=0}^{p}\binom{p}{q}\sum_{r=0}^{q}\binom{q}{r}\int_0^{\infty}e^{-\alpha y(i+j+k+1)}\,dy. \qquad \text{(q)}$$

Then,

$$\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta i}\right]^{\theta(j+1)-1}=\sum_{p}^{\infty}\sum_{q}^{p}\sum_{r}^{q}\frac{(-1)^{p+q+r+1}(\beta i)(\theta(i+1))}{(\beta i)(\theta(i+1)-1-p)}$$
$$\binom{q-(\theta(i+1)-1)}{q}\binom{q}{p}\binom{(\theta(i+1)-1)+q}{6}\int_0^{\infty}e^{-\alpha y(i+j+k+1)}\,dy \qquad \text{(r)}$$

$$e^{-\lambda\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta}\right]^{\theta}}=\sum_{i}^{\infty}\frac{(-1)^i\lambda^i}{i!}\left[-log\left[1-\left[1+\frac{\alpha^3y^3+3\alpha^2y^2+6\alpha y}{\alpha^4+6}\right]e^{-\alpha y}\right]^{\beta i}\right]^{\theta j}$$
$$=\sum_{i}^{\infty}\sum_{p}^{\infty}\sum_{q}^{p}\sum_{r}^{q}\frac{(-1)^{i+p+q+r+1}(\lambda i)(\beta i)(\theta(i+1))}{i!\,(\beta i)(\theta(i+1)-1-p)}\binom{q-(\theta(i+1)-1)}{q}\binom{q}{p}\binom{(\theta(i+1)-1)+q}{6}$$
$$\int_0^{\infty}e^{-\alpha y(i+j+k+1)}\,dy \qquad \text{(s)}$$

Since $\int_0^{\infty}y^ne^{-\beta y}dy=\frac{\Gamma(n+1)}{\beta^{n+1}}$ and $\Gamma(\beta)=(\beta-1)!$

Substituting and simplifying (o, p, q, r, and s), we have

$$= \sum_{i=0}^{\infty}\binom{\beta-1}{i}\sum_{j=0}^{\infty}\binom{\lambda}{j}\sum_{k=0}^{\infty}\binom{\theta}{k}(-1)^{i+j+k}\sum_{p=0}^{\infty}\binom{i}{p}\sum_{q=0}^{\infty}\binom{j}{q}\sum_{r=0}^{\infty}\binom{k}{r}\frac{\lambda.\beta.\theta.3^{q}.2^{r}.\alpha^{3j-q-k+5}}{(\alpha^4+6)^{j+q+k+1}} .\alpha^{-(s+3j-q-k+1)}\frac{(s+3j-q-k)!}{[i+1]^{s+3j-q-k+1}} \quad (t)$$

$$= \sum_{i=0}^{\infty}\binom{\beta-1}{i}\sum_{j=0}^{\infty}\binom{\lambda}{j}\sum_{k=0}^{\infty}\binom{\theta}{k}(-1)^{i+j+k}\sum_{p=0}^{\infty}\binom{i}{p}\sum_{q=0}^{\infty}\binom{j}{q}\sum_{r=0}^{\infty}\binom{k}{r}\frac{\lambda.\beta.\theta.3^{q}.2^{r}.\alpha^{3j-q-k+4}}{(\alpha^4+6)^{p+q+r+1}} .\alpha^{-(s+3j-q-k+4)}\frac{(s+3j-q-k+3)!}{[i+1]^{s+3j-q-k+4}} \quad (u)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i}{p}\binom{j}{q}\binom{k}{r}\frac{\alpha y^{(p+q+r)}}{(\alpha^4+6)^{p+q+r}}\sum_{q=0}^{p}\binom{p}{q}\sum_{r=0}^{q}\binom{q}{r}\frac{\beta.\theta.2^{q}.2^{r}.\alpha^{3j-q-k+3}}{(\alpha^4+6)^{p+q+r+1}}.\frac{(s+3j-q-k+3)!}{[i+1]^{s+3j-q-k+3}} \quad (v)$$

$$= \sum_{p}^{\infty}\sum_{q}^{p}\sum_{r}^{q}\frac{(-1)^{p+q+r+1}(\beta i)(\theta(i+1))}{(\beta i)(\theta(i+1)-1-p)}\binom{q-(\theta(i+1)-1)}{q}\binom{q}{p}\binom{(\theta(i+1)-1)+q}{6}\frac{\beta.\theta.2^{q}.2^{r}.\alpha^{3j-q-k+3}}{(\alpha^4+6)^{p+q+r+1}} .\alpha^{-(s+3j-q-k+3)}\frac{(s+3j-q-k+3)!}{[i+1]^{s+3j-q-k+3}} \quad (w)$$

$$= \sum_{i}^{\infty}\sum_{p}^{\infty}\sum_{q}^{p}\sum_{r}^{q}\frac{(-1)^{i+p+q+r+1}(\lambda i)(\beta i)(\theta(i+1))}{i!\,(\beta i)(\theta(i+1)-1-p)}\binom{q-(\theta(i+1)-1)}{q}\binom{q}{p}\binom{(\theta(i+1)-1)+q}{6} \frac{\lambda.\beta.\theta.3^{q}.2^{r}.\alpha^{3j-q-k+4}}{(\alpha^4+6)^{p+q+r+1}}.\alpha^{-(s+3j-q-k+1)}\frac{(s+3j-q-k)!}{[i+1]^{s+3j-q-k+1}} \quad (x)$$

We let A, B, C, D and E represent part of the expressions in (t, u, v, w and x) as follows:

$$A_{i,j,q,k} = \sum_{i=0}^{\infty}\binom{\beta-1}{i}\sum_{j=0}^{\infty}\binom{\lambda}{j}\sum_{k=0}^{\infty}\binom{\theta}{k}(-1)^{i+j+k}\sum_{p=0}^{\infty}\binom{i}{p}\sum_{q=0}^{\infty}\binom{j}{q}\sum_{r=0}^{\infty}\binom{k}{r}\frac{\lambda.\beta.\theta.3^{q}.2^{r}.\alpha^{3j-q-k+5}}{(\alpha^4+6)^{j+q+k+1}}.\alpha^{-(s+3j-q-k+1)} \quad (i)$$

$$B_{i,j,q,k} = \sum_{i=0}^{\infty}\binom{\beta-1}{i}\sum_{j=0}^{\infty}\binom{\lambda}{j}\sum_{k=0}^{\infty}\binom{\theta}{k}(-1)^{i+j+k}\sum_{p=0}^{\infty}\binom{i}{p}\sum_{q=0}^{\infty}\binom{j}{q}\sum_{r=0}^{\infty}\binom{k}{r}\frac{\lambda.\beta.\theta.3^{q}.2^{r}.\alpha^{3j-q-k+4}}{(\alpha^4+6)^{p+q+r+1}}.\alpha^{-(s+3j-q-k+4)} \quad (ii)$$

$$C_{i,j,q,k} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{i}{p}\binom{j}{q}\binom{k}{r}\frac{\alpha y^{(p+q+r)}}{(\alpha^4+6)^{p+q+r}}\sum_{q=0}^{p}\binom{p}{q}\sum_{r=0}^{q}\binom{q}{r}\frac{\beta.\theta.2^{q}.2^{r}.\alpha^{3j-q-k+3}}{(\alpha^4+6)^{p+q+r+1}} \quad (iii)$$

$$D_{i,j,q,k} = \sum_{p}^{\infty}\sum_{q}^{p}\sum_{r}^{q}\frac{(-1)^{p+q+r+1}(\beta i)(\theta(i+1))}{(\beta i)(\theta(i+1)-1-p)}\binom{q-(\theta(i+1)-1)}{q}\binom{q}{p}\binom{(\theta(i+1)-1)+q}{6} \frac{\beta.\theta.2^{q}.2^{r}.\alpha^{3j-q-k+3}}{(\alpha^4+6)^{p+q+r+1}}.\alpha^{-(s+3j-q-k+3)} \quad (iv)$$

and

$$E_{i,j,q,k} = \sum_{i}^{\infty}\sum_{p}^{\infty}\sum_{q}^{p}\sum_{r}^{q}\frac{(-1)^{i+p+q+r+1}(\lambda i)(\beta i)(\theta(i+1))}{i!\,(\beta i)(\theta(i+1)-1-p)}\binom{q-(\theta(i+1)-1)}{q}\binom{q}{p}\binom{(\theta(i+1)-1)+q}{6} \frac{\lambda.\beta.\theta.3^{q}.2^{r}.\alpha^{3j-q-k+4}}{(\alpha^4+6)^{p+q+r+1}}.\alpha^{-(s+3j-q-k+1)} \quad (v)$$

Hence, the s-th order moment about the origin, $E(Y^s)$ of the WEP distribution is given by in (2.5.4) above

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