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A CLASS OF POWER FUNCTION DISTRIBUTIONS: ITS PROPERTIES AND APPLICATIONS

MATTHEW EKUM*, MUMINU ADAMU, AND ENO AKARAWAK

Abstract. The T-R{Y} framework is a method of generating convoluted probability distributions; which has generalized most of the existing methods. In the T-R{Y} framework, three independent distributions, T, R, and Y are combined to form a new distribution, X, where R is the baseline distribution. The new distribution X is a weighted hazard function of the baseline distribution, R. Some distributions like Normal, Weibull, Uniform, Cauchy, and Gamma have been used as baseline distributions. However, the Power function distribution, despite its flexibility and simplicity of its functional form, has not been used as a baseline distribution in the T-R{Y} framework. In this work, we developed the T-Power function{Y} family of distributions using the T-R{Y} framework. We generated twelve convoluted distributions from the family developed, and derived the properties of Gamma-Power function{Log-logistic} distribution (GPLD) as a special case. The maximum likelihood estimation (MLE) method is used to estimate the parameters of the proposed distribution. A simulation study and application to two real-life datasets were carried out. The application results shows that the new GPLD perform favourable.

1. Introduction

One of the distributions of interest that needs to be explored is the power function distribution. It is a flexible lifetime distribution, which may be obtained through a simple transformation of the Pareto, beta, Kumaraswamy, and uniform distributions ([1]). The follwoing authors, [2], [3], [4], [5] , [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], and [17] studied power function distribution and commented on its flexibility.

[18] defined convoluted distribution as the combination of two or more distributions to form a new distribution or any transformation done to an existing distribution to form a new distribution. The new distribution is a hybrid, which

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* Correspondence.

is expected to perform better than the individual distributions. Many convoluted distributions have been derived from power function distribution but the power function distribution has not been generalized using the T-R{Y} framework. So, there is a need to generalized the power function distribution using the T-R{Y} framework.

It is observed that distributions like Normal, Weibull, Uniform, and Gamma have been used as baseline distributions (R) in the T-R{Y} framework, but the Power function distribution, despite its flexibility and simplicity of its functional form, has not been used as a baseline distribution in the T-R{Y} framework. The motivation in this research is related to the flexibility of the power function distribution with its simple functional form, as well as its upper bound parameter.

The T-R{Y} framework metamorphosed from beta-X by [19] to T-X by [20], and further extended to T-X(W) by [21] . [22] unified the T-X family to a more generalized framework, T-R{Y} and many distributions have been derived from this framework (see [21]; [22]; [23]; [24]; [25]; [26]; [27]; [28]; [29] and [30].

Take T, R and Y to be random variables with known cumulative distribution functions (cdfs), $F_T(x)$, $F_R(x)$ and $F_Y(x)$ respectively. Also, let $f_T(x)$, $f_R(x)$ and $f_Y(x)$ be their corresponding probability density functions (pdfs) with known quantile functions $Q_T(x)$, $Q_R(x)$ and $Q_Y(x)$ respectively.

Then the cdf of the T-R{Y} family of distributions is given by

$$F_X(x) = \int_a^{Q_Y[F_R(x)]} f_T(t)dt = F_T\{Q_Y[F_R(x)]\}. \tag{1.1}$$

where $f_T(t)$ is the pdf of a random variable T, $Q_Y[.]$ is the quantile function of a random variable Y and $F_R(x)$ is the cdf of a random variable R. is differentiable and monotonically non-decreasing. It is necessary that and $f_T(t)$ have the same support [22]. The pdf of T-R{Y} given in (1.1) as defined by [22] is given by

$$f_X(x) = f_R(x)\frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}}. \tag{1.2}$$

The pdf in (1.2) is a weighted hazard function of the random variable R. So, the pdf in (1.2) is the generalization of random variable R.

In this article, we propose some classes of generalized power function distributions, the T-Power{Y} family, and studied some of its properties and applications. A member of the T-power{Y} family called the Gamma-Power function distribution (GPLD) was studied in detail and applied to two real data. The remaining parts of this research are unfolded as follows. In Section 2, we consider the T-power{Y} classes of distributions and define some new generalized families. We investigate some structural properties of the newly formed GPLD, which is a special case of the T-Power{Y} family. We defined the moment, moment generating function, mean and median deviations of the proposed GPLD. The maximum likelihood estimation technique was used to estimate the parameters of the proposed GPLD. In Section 3, we explored the consistency and the usefulness of the proposed GPLD using a simulation study and two real-life applications. Finally, Section 4 offers some concluding remarks.

2. Materials and Methods

This section includes the developmental procedures of generating the proposed distribution, characterisation and derivation of its properties.

2.1. Developing the T-Power Function{Y} Distribution.

Proposition 2.1. *The pdf in Equation (1.2) is proper, that is, the integral of the pdf in Equation (1.2) is equal to 1.*

Proof.

$$\int f_X(x)dx = \int f_R(x)\frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}}dx, \tag{2.1}$$

where $F_R(x)$ is cdf of R distribution, $f_R(.)$, $f_T(.)$ and $f_Y(.)$ are the pdf of R, T, and Y distributions respectively, and $Q_Y(.)$ is the quantile function of Y distribution. Note that T, R, and Y are known distributions with true pdf.

From equation (1.2), we have

$$\frac{f_X(x)}{f_R(x)} = \frac{f_T\{Q_Y[F_R(x)]\}}{f_Y\{Q_Y[F_R(x)]\}}. \tag{2.2}$$

So that

$$\int_a^\infty f_X(x)dx = \int_a^\infty f_R(x)\frac{f_X(x)}{f_R(x)}dx. \tag{2.3}$$

Therefore,

3 errors17 warnings

$$\int_a^\infty f_X(x)dx = 1. \tag{2.4}$$

□

Equation (2.4) completes the proof. This shows that if the pdf of random variables T, R, and Y are known to be true pdfs, then the pdf developed by combining them using T-R{Y} framework is also a true pdf.

The framework of some pdf and cdf for convoluted distribution that are member of the T-R{Y} family are presented in Table 1. These frameworks can be used by any study to generate univariate distributions by rightly choosing the random variables T, R and Y. Table 1 is used to generate 12 continuous probability distributions, two from each quantile function in this study.

From Table 1, $S_R(x)$, $h_R(x)$, $H_R(x)$, and $A_R(x)$ are the survival, hazard, cumulative hazard and reverse hazard functions of X given respectively as

$$S_X(x) = 1 - F_X(x), \tag{2.5}$$

$$h_X(x) = \frac{f_X(x)}{1 - F_X x}, \tag{2.6}$$

TABLE 1. Framework of Convoluted Distributions Formed using T-R{Y} Framework

Sn	Family	PDF	CDF
1	T-R{exponential}	$f_X(x) = h_R(x) f_T[H_R(x)]$	$F_X(x) = F_T[H_R(x)]$
2	T-R{log-logistic}	$f_X(x) = \frac{f_R(x)}{[S_R(x)]^2} f_T \left[\frac{f_R(x)}{S_R(x)}\right]$	$F_X(x) = F_T \left[\frac{f_R(x)}{S_R(x)}\right]$
3	T-R{frechet}	$f_X(x) = \frac{A_R(x)}{\{ln[F_R(x)]\}^2} f_T \left[-\frac{1}{ln[F_R(x)]}\right]$	$F_X(x) = F_T \left[-\frac{1}{ln[F_R(x)]}\right]$
4	T-R{logistic}	$f_X(x) = \frac{h_R(x)}{F_R(x)} f_T \left\{ln \left[\frac{F_R(x)}{S_R(x)}\right]\right\}$	$F_X(x) = F_T \left\{ln \left[\frac{F_R(x)}{S_R(x)}\right]\right\}$
5	T-R{extreme value}	$f_X(x) = h_R(x) \frac{f_T(-ln\{-ln[F_R(x)]\})}{-F_R(x) ln[F_R(x)]}$	$F_X(x) = F_T(-ln\{-ln[F_R(x)]\})$
6	T-R{uniform}	$f_X(x) = f_R(x) f_T[F_R(x)]$	$F_X(x) = F_T[F_R(x)]$

$$H_X(x) = -ln[1 - F_X(x)]. \tag{2.7}$$

and

$$A_X(x) = \frac{f_X(x)}{F_X x}, \tag{2.8}$$

All the distributions generated using this method have their quantile function. The relationship between the new random variable X and T is given by $T = Q_Y[F_R(x)]$ and thus, $X = F_R^{-1}[F_Y(T)] = Q_R[F_Y(T)]$, where $F_R^{-1}[.]$ exists. Using this relation, random variable X can be generated by generating the random variable T and then computing $X = F_R^{-1}[F_Y(T)]$. Alternatively, the cdf in (1.1) is a composite function of the form $(T.Y.R)(x)$. The inverse function is given by

$$Q_X(p) = Q_R\{F_Y[Q_T(p)]\}, p \in [0, 1]. \tag{2.9}$$

Lemma 2.2. *If h(z) and H(z) are the hazard function and cumulative hazard function of random variable Z respectively, then $H'(x) = h(x)$ and $\int h(z)dz = H(z)$.*

Proof. By definition $H(z) = -ln[1 - F(z)]$. By using chain rule, let $u = 1 - F(z)$, so that $H(z) = -lnu$.

$$\frac{du}{dz} = -f(z) \tag{2.10}$$

and

$$\frac{dH(z)}{du} = -\frac{1}{u} \tag{2.11}$$

So that

$$\frac{dH(z)}{dz} = -\frac{f(z)}{u} \tag{2.12}$$

But $u = 1 - F(z)$, so that,

$$\frac{dH(z)}{dz} = -\frac{f(z)}{1 - F(z)} \tag{2.13}$$

□

Conversely, we need to show that $\int h(z)dz = H(z)$
By definition $\frac{f(z)}{1-F(z)}$. Hence, integrating both sides with respect to z gives

$$H(z) = \int h(z)dz = \int \frac{f(z)}{1-F(z)}dz = -ln[1-F(z)] \tag{2.14}$$

Thus, Equation (2.14) ends the proof.

Theorem 2.3. *If $h(z)$ and $H(z)$ are the hazard function and cumulative hazard function of random variable Z respectively, then* $\int[H(z)h(z)]dz = \frac{1}{2}[H(z)]^2$

Using integration by part. Let $uH(z)$ and $dv = h(z)dz$, $du = h(z)dz$ and $v = H(z)$. So,

$$\int H(z)h(z)dz = H(z)H(z) - \int H(z)h(z)dz \tag{2.15}$$

Collect like terms and solving gives the required result as

$$\int H(z)h(z)dz = \frac{1}{2}[H(z)]^2 \tag{2.16}$$

Thus, Equation (2.16) completes the proof.

2.2. T**-Power Function**$\{$**Y**$\}$ **Classes of Distributions.** Let R be a random variable that follows a two-parameter power function distribution with cdf and pdf given by

$$F_R(x) = \left(\frac{x}{\lambda}\right)^2 \tag{2.17}$$

and

$$f_R(x) = \left(\frac{k}{\lambda^k}\right)x^{k-1} \tag{2.18}$$

respectively (see [28]). The cdf of T-Power function{Y} or simply T-P{Y} class of distributions is derived by inserting Equation (2.17) into (1.1) and it is given as

$$F_X(x) = \int_0^{Q_Y\left[\left(\frac{x}{\lambda}\right)^k\right]} f_T(t)dt = F_T\left\{Q_Y\left[\left(\frac{x}{\lambda}\right)^k\right]\right\}. \tag{2.19}$$

The pdf corresponding to Equation (2.19) is derived by inserting Equations (2.17) and (2.18) into equation (1.2) and it is given as

$$f_X(x) = \frac{k}{\lambda^k}x^{k-1}\frac{f_T\left\{Q_Y\left[\left(\frac{x}{\lambda}\right)^k\right]\right\}}{f_Y\left\{Q_Y\left[\left(\frac{x}{\lambda}\right)^k\right]\right\}} \tag{2.20}$$

Thus, equations (2.19) and (2.20) are the cdf and pdf of T-P{Y} family of distributions respectively. The survival, hazard, cumulative hazard and reversed hazard functions for T-P{Y} family are respectively derived as

$$S_X(x) = 1 - F_T\left\{Q_Y\left[\left(\frac{x}{\lambda}\right)^k\right]\right\}, \tag{2.21}$$

$$h_X(x) = \frac{kx^{k-1} f_T \left\{ Q_Y \left[\left(\frac{x}{\lambda}\right)^k \right] \right\}}{\lambda^k f_Y \left\{ Q_Y \left[\left(\frac{x}{\lambda}\right)^k \right] \right\} \left(1 - F_T \left\{ Q_Y \left[\left(\frac{x}{\lambda}\right)^k \right] \right\} \right)}, \tag{2.22}$$

$$H_X(x) = 1 - ln \left(1 - F_T \left\{ Q_Y \left[\left(\frac{x}{\lambda}\right)^k \right] \right\} \right) \tag{2.23}$$

and

$$A_X(x) = \frac{kx^{k-1} f_T \left\{ Q_Y \left[\left(\frac{x}{\lambda}\right)^k \right] \right\}}{\lambda^k f_Y \left\{ Q_Y \left[\left(\frac{x}{\lambda}\right)^k \right] \right\} \left(F_T \left\{ Q_Y \left[\left(\frac{x}{\lambda}\right)^k \right] \right\} \right)}. \tag{2.24}$$

Remark 2.4. If X is T-Power{Y} distributed, then it follows that

i. $X \overset{d}{=} \lambda[F_Y(T)]^{1/k}$,
ii. $Q_X(p) = \lambda\{F_Y[Q_T(p)]\}^{1/k}$,
iii. If $T \overset{d}{=} Y$, then $X =$ Power function (k, λ) in distribution and
iv. If $Y =$ Power function (k, λ), then $X \overset{d}{=} T$.

The T-P{Y} family of distributions in Equation (2.19) can be used to generate many different classes of power function distributions. Some generalized power function families using the quantile functions displayed on Table 1 are generated.

2.2.1. *T-Power{exponential}.* Let $T \in [0, \infty]$ be any random variable. By substituting $h_R(x)$ and $H_R(x)$ into item 1 of Table 1, the cdf and pdf of T-P{exponential} family are respectively given by

$$F_X(x) = F_T \left[-ln(\lambda^k - x^k) \right] \tag{2.25}$$

and

$$f_X(x) = \frac{kx^{k-1}}{(\lambda^k - x^k)} f_T \left[-ln(\lambda^k - x^k) \right];\ k, \lambda > 0; 0 \leq x \leq \lambda. \tag{2.26}$$

2.2.2. *T-Power{log-logistic}.* Let $T \in [0, \infty]$ be any random variable. By substituting $F_R(x)$ and $S_R(x)$ into item 2 of Table 1, the cdf and pdf of T-P{log-logistic} family are respectively given by

$$F_X(x) = F_T \left(\frac{x^k}{\lambda^k - x^k} \right) \tag{2.27}$$

and

$$f_X(x) = \frac{k\lambda^k x^{k-1}}{(\lambda^k - x^k)^2} f_T \left(\frac{x^k}{\lambda^k - x^k} \right);\ k, \lambda > 0; 0 \leq x \leq \lambda. \tag{2.28}$$

2.2.3. *T-Power{frechet}.* Let $T \in [0, \infty]$ be any random variable. By substituting $F_R(x)$ and $f_R(x)$ into item 3 of Table 1, the cdf and pdf of T-P{frechet} family are respectively given by

$$F_X(x) = F_T \left[-\frac{1}{ln \left(\frac{x}{\lambda}\right)^k} \right] \tag{2.29}$$

and

$$f_X(x) = \frac{k}{x\left[ln\left(\frac{x}{\lambda}\right)^k\right]^2} f_T\left[-\frac{1}{ln\left(\frac{x}{\lambda}\right)^k}\right];\ k, \lambda > 0; 0 \le x \le \lambda. \tag{2.30}$$

The T-Power{exponential}, T-Power{log-logistic} and T-Power{frechet} will take care of any T supported on the interval $[0, \infty)$, like the Rayleigh, gamma, Weibull, Lomax and Dagum. This is because exponential, log-logistic and frechet distributions are supported on the interval $[0, \infty)$.

2.2.4. *T-Power{logistic}*. Let $T \in [-\infty, \infty]$ be any random variable. By substituting $F_R(x)$ and $h_R(x)$ into item 4 of Table 1, the cdf and pdf of T-P{logistic} family are respectively given by

$$F_X(x) = F_T\left[ln\left(\frac{x^k}{\lambda^k - x^k}\right)\right] \tag{2.31}$$

and

$$f_X(x) = \frac{k}{x\left(\lambda^k - x^k\right)} f_T\left[ln\left(\frac{x^k}{\lambda^k - x^k}\right)\right];\ k, \lambda > 0; 0 \le x \le \lambda. \tag{2.32}$$

2.2.5. *T-Power{extreme value}*. Let $T \in [-\infty, \infty]$ be any random variable. By substituting $H_R(x)$ and $h_R(x)$ into item 5 of Table 1, the cdf and pdf of T-P{extreme value} family are respectively given by

$$F_X(x) = F_T\left[-ln\left(-klnx\right)\right] \tag{2.33}$$

and

$$f_X(x) = \frac{k}{x\left(\lambda^k - x^k\right) ln\left(\frac{x}{\lambda}\right)^k} f_T\left\{-ln\left[-ln\left(\frac{x}{\lambda}\right)^k\right]\right\};\ k, \lambda > 0; 0 \le x \le \lambda. \tag{2.34}$$

The T-P{logistic} and T-PP{extreme valueP} will take care of any T supported on the open interval $(-\infty, \infty)$, like the normal, Cauchy, Laplace, Gumbel etc. This is because logistic distribution is also supported on the interval $(-\infty, \infty)$.

2.2.6. *T-Power{uniform}*. Let $T \in [0, 1]$ be any random variable. By substituting $F_R(x)$ and $f_R(x)$ into item 6 of Table 1, the cdf and pdf of T-P{uniform} family are respectively given by

$$F_X(x) = F_T\left(\frac{x}{\lambda}\right)^k \tag{2.35}$$

and

$$f_X(x) = \frac{k}{\lambda^k} x^{k-1} f_T\left(\frac{x}{\lambda}\right)^k;\ k, \lambda > 0; 0 \le x \le \lambda. \tag{2.36}$$

The T-P{uniform} will take care of any T supported on the closed interval [0, 1], like the beta, Kumaraswamy etc. This is because standard uniform distribution is supported on the closed interval [0, 1].

2.3. **General Properties of T-P{Y} Family.** In this section, some properties of T-P{Y} are investigated. The following lemmas are given to establish the relationships between X and T in order to simulate variates of X from the variates of T.

Lemma 2.5. *(**Useful Transformation**). If T is a random variate from pdf $f_T(x)$, then random variate.*

(i) $X = \lambda[1 - exp(-T)]^{1/k}$ *follows the T-P{exponential}family of distributions, provided* $T \in [0, \infty)$.
(ii) $X = \lambda\left(\frac{T}{1+T}\right)^{1/k}$ *follows the T-P{log-logistic}family of distributions, provided* $T \in [0, \infty)$.
(iii) $X = exp\left\{\lambda[-(T)^{-1}]^{1/k}\right\}$ *follows the T-P{frechet}family of distributions, provided* $T \in [0, \infty)$.
(iv) $X = \lambda\left(\frac{e^{-T}}{1+e^{-T}}\right)^{1/k}$ *follows the T-P{logistic}family of distributions, provided* $T \in (-\infty, \infty)$.
(v) $X = \lambda\left(1 - e^{-e^{T}}\right)^{1/k}$ *follows the T-P{extreme value}family of distributions, provided* $T \in (-\infty, \infty)$.
(vi) $X = \lambda T^{1/k}$ *follows the T-P{uniform}family of distributions, provided* $T \in [0, 1]$.

Proof. It result is obvious from Remark 2.4 (i). □

Lemma 2.5 will be very useful in deriving some members of the generalized power function distributions. All existing univariate continuous probability distributions must satisfy at least one of the 6 situations in Lemma 2.5. We can generate random variate X if T is known.

Lemma 2.6. *(**Quantile functions**). It follows from Lemma 2.2 that the quantile functions of (i) T-P{exponential}, (ii) T-P{log-logistic}, (iii) T-P{frechet}, (iv) T-P{logistic}, (v) T-P{exreme value} and (vi) T-P{uniform} families are given respectively by:*

(i) $Q_X(p) = \lambda\{1 - exp[-Q_T(p)]\}^{1/k}$
(ii) $Q_X(p) = \lambda\left[\frac{Q_T(p)}{1+Q_T(p)}\right]^{1/k}$
(iii) $Q_X(p) = exp\left(\lambda\{-[Q_T(p)]^{-1}\}^{1/k}\right)$
(iv) $Q_X(p) = \lambda\{1 - exp[-Q_T(p)]\}^{1/k}$
(v) $Q_X(p) = \lambda\left[1 - e^{-e^{Q_T(p)}}\right]^{1/k}$
(vi) $Q_X(p) = \lambda[Q_T(p)]^{1/k}$

Proof. It is also very obvious to see the result from Remark 2.4 (ii). □

Lemma 2.6 will be very useful in deriving the quantile functions of the generalized power function distributions. With this, we can easily derive the median and other measures of partition.

We generated the following twelve convoluted distributions from the T-P{Y} family developed.

i Exponential-power function {uniform} distribution pdf is given by

$$f_X(x) = \frac{\alpha}{\beta^\alpha}\left(\frac{x}{\beta}\right)^{\alpha-1} exp\left[-\left(\frac{x}{\beta}\right)^{\alpha}\right]; \alpha, \beta > 0; x \geq 0.$$

This concides with a Weibull distribution (see Weibull, 1951).

ii Rayleigh-Power function {log-logistic} distribution pdf is given by

$$f_X(x) = \frac{1}{\sigma^2}\left(\frac{x^k}{1-x^k}\right)^2 exp\left[-\frac{1}{2\sigma^2}\left(\frac{x^k}{1-x^k}\right)^2\right]; \sigma, k > 0; 0 \leq x \leq 1.$$

iii Four parameters Gamma-Power function {log-logistic} distribution pdf is given by

$$f_X(x) = \frac{k\lambda^k\beta^\alpha x^{\alpha k-1}}{\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right];\ \alpha, \beta, k, \lambda > 0; 0 \leq x \leq \lambda.$$

iv Four parameters Weibull-Power function {log-logistic} distribution. This concides with the distribution proposed by Tahir et al. (2014) and it is given by

$$f_X(x) = \frac{\alpha\beta k\lambda^k x^{k\alpha-1}}{(\lambda^k - x^k)^{\alpha+1}} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)^{\alpha}\right];\ \alpha, \beta, k, \lambda > 0; 0 \leq x \leq \lambda.$$

v Lomax-Power function {log-logistic} distribution pdf is given by

$$f_X(x) = \frac{1}{\lambda}\left(\frac{x^k}{1-x^k}\right)^2\left[1 + \frac{1}{\lambda}\left(\frac{x^k}{1-x^k}\right)\right]^{-\left(\frac{x^k}{1-x^k}+1\right)}; \lambda, k > 0; 0 \leq x \leq 1.$$

vi Dagum-Power function {Frechet} distribution pdf is given by

$$f_X(x) = \frac{ap}{b^{ap}}\left(-klogx\right)^{1-ap}\left[1 - bklogx\right]^{-(p+1)}; a, b, p, k > 0; 0 \leq x \leq 1.$$

vii Kumaraswamy-power function {uniform} distribution pdf is given by

$$f_X(x) = abkx^{ak-1}(1 - x^{ak})^{b-1}; a, b, k > 0; 0 \leq x \leq 1.$$

viii Cauchy-Power function {logistic} distribution pdf is given by

$$f_X(x) = \frac{k}{\pi\sigma}\left[\frac{1}{x^k(1-x^k)}\right]\left\{\frac{1}{1 - \frac{1}{\sigma^2}\left[log\left(\frac{x^k}{1-x^k}\right) - \theta\right]^2}\right\}; \sigma, \theta, k > 0; 0 \leq x \leq 1.$$

ix Three parameters Normal-power function {logistic} distribution pdf is given by

$$f_X(x) = \frac{k}{x(1-x^k)\sqrt{2\pi\sigma^2}} exp\left\{-\frac{1}{2\sigma^2}\left[log\left(\frac{x^k}{1-x^k}\right) - \mu\right]^2\right\}; \sigma, \mu, k > 0; 0 \leq x \leq 1.$$

x Four parameters Normal-power function {logistic} distribution pdf is given by

$$f_X(x) = \frac{k\lambda^k}{x(\lambda^k - x^k)\sqrt{2\pi\sigma^2}} exp\left\{-\frac{1}{2\sigma^2}\left[log\left(\frac{x^k}{\lambda^k - x^k}\right) - \mu\right]^2\right\}; \sigma, \mu, \lambda, k > 0; 0 \leq x \leq \lambda.$$

xi Laplace-Power function {logistic} distribution pdf is given by

$$f_X(x) = \frac{k}{2bx(1-x^k)} exp\left\{-\frac{1}{b}\left[log\left(\frac{x^k}{1-x^k}\right) - \mu\right]\right\}; b, \mu, k > 0; 0 \leq x \leq 1.$$

xii Gumbel-Power function {extreme value} distribution pdf is given by

$$f_X(x) = \frac{ke^{\mu}}{\beta}\left\{\frac{x^{k-1}(1-x^k)^{\beta-1}}{[log(1-x^k)]^2}\right\}; \beta, \mu, k > 0; 0 \leq x \leq 1.$$

Changing T distribution and Y distribution will produce different distributions of this family, of which two of them have the same functional form with existing distributions. A good example of an existing distribution in this family is the Weibull distribution. We can also debate that all distributions with the form x^k where k can pick any form is a transformed power function distribution, which are all members of the T-Power function{Y} family of distributions. In this article, we focused on using the standard quantile function of log-logistic distribution, which is the odd-ratio of the random variable R. In this work, we carefully study some properties of Gamma-Power function{log-logistic} and test the flexibility of its MLE parameter estimates.

2.4. **Gamma-Power{log-logistic}.**

2.4.1. *Some Characterisations of Gamma-Power{log-logistic}.*

Cumulative distribution function of Gamma-Power{log-logistic}. One of the most important ways of characterising a probability distribution is through its cumulative distribution function (cdf).

Theorem 2.7. *Let X be a random variable that follows the Gamma-Power{log-logistic} distribution where parameters α and β are from gamma distribution, and k and λ are from power function distribution, then the cdf of X defined on a closed interval $[0, \lambda]$ is given by.*

$$F_X(x) = \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]$$

Proof. Let T follows the gamma distribution with shape parameter α and scale parameter β, that is

$$f_T(t) = \frac{\beta^{\alpha}}{\Gamma(\alpha)}t^{\alpha-1}exp(-\beta t),\ \alpha, \beta > 0, t \geq 0$$

The cdf of T is given by

$$F_T(t) = \frac{1}{\Gamma(\alpha)}\gamma(\beta t) \tag{2.37}$$

Substitute equation (2.37) into (2.27) to have the cdf of GPLD

$$F_X(x) = \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right];\ \alpha, \beta, k, \lambda > 0; 0 \leq x \leq \lambda \tag{2.38}$$

where $\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]$ is an incomplete gamma function. □

Equation (2.38) completes the proof. Equation (2.38) is the cdf of the developed gamma-power function log-logistic distribution (GPLD). From now hence forth the newly developed Gamma-Power{log-logistic} distribution will be regarded as GPLD.

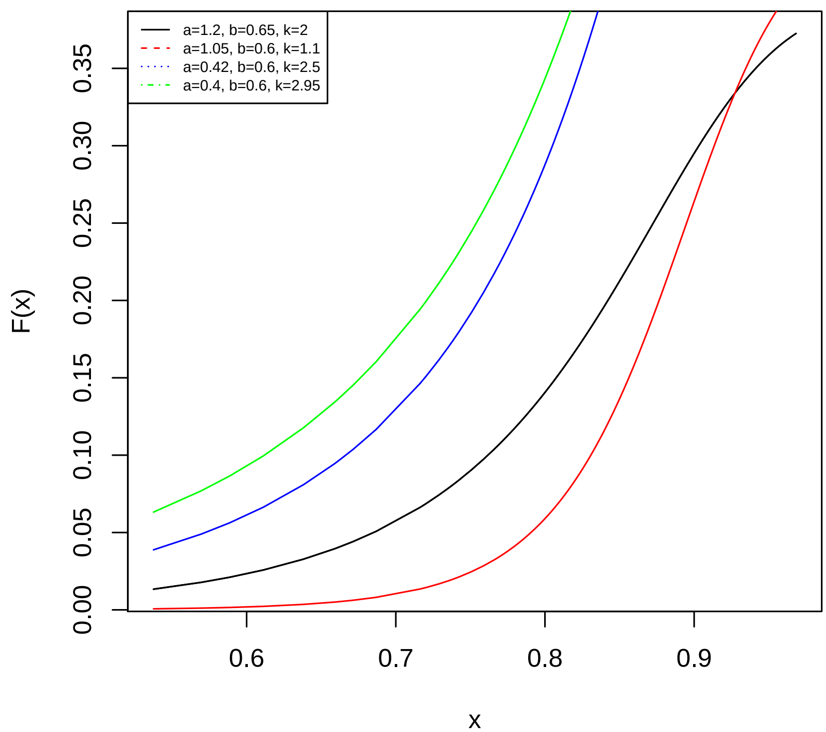

FIGURE 1. The cdf of GPLD Distribution for $\lambda = 1$.

Figure 1 shows that the cdf of is a non-decreasing function. As x increases, the cdf also increases.

Probability Density Function of GPLD. The probability density function (pdf) of GPLD is derived by differentiating the cdf in equation (2.38) with respect to x. The pdf of the proposed Gamma-Power{log-logistic} distribution is therefore given by

$$f_X(x) = \frac{k\lambda^k \beta^\alpha x^{\alpha k-1}}{\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right];\ \alpha, \beta, k, \lambda > 0; 0 \leq x \leq \lambda \tag{2.39}$$

Figures 2 illustrates some possible shapes of the density function of the GPLD, for selected parameter values. The density function can take various forms depending on the parameter values. It is obvious that the GPLD has higher flexibility than the gamma and the power function distributions, because of the additional parameters, which allow for a high degree of flexibility of the GPLD.

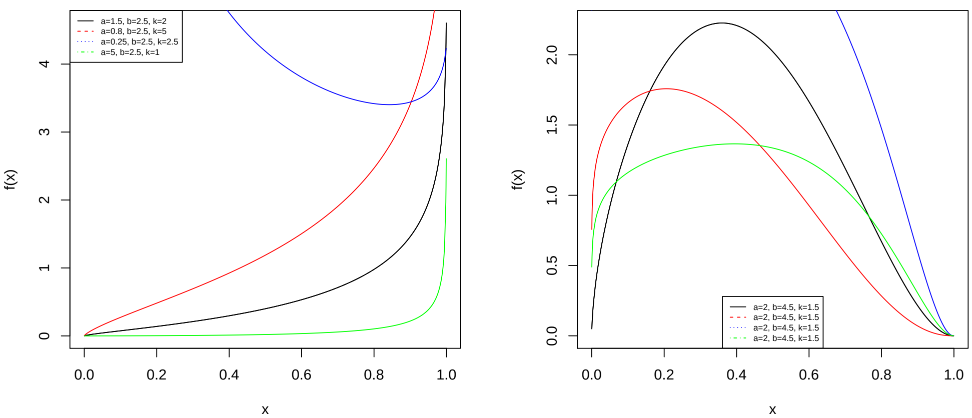

FIGURE 2. The pdf of GPLD Distribution for $\lambda = 1$.

It shows that for different parameter values α, β, k and for a constant λ, GPLD can be positively or negatively skewed and can be leptokurtic or platykurtic. The pdf has various shapes as displayed in Figure 1. So, the new distribution would be very useful in many practical situations for modelling positive real data sets.

Survival Function of GPLD. By definition, the survival function of a random variable X is given by

$$S_X(x) = 1 - F_X(x) \tag{2.40}$$

Substitute the cdf in equation (2.38) into (2.40) gives the survival function of GPLD as

$$S_X(x) = 1 - \frac{1}{\Gamma(\alpha)} \gamma \left[\beta \left(\frac{x^k}{\lambda^k - x^k} \right), \alpha \right]$$

Hazard Function of GPLD.

Theorem 2.8. *The hazard function of a random variable X that follows a GPLD with parameters α, β, k, λ exist and it is given by*

$$h_X(x) = \frac{k\lambda^k \beta^\alpha x^{\alpha k - 1} exp\left[-\beta \left(\frac{x^k}{\lambda^k - x^k} \right) \right]}{(\lambda^k - x^k)^{\alpha+1} \left\{ \Gamma(\alpha) - \gamma \left[\beta \left(\frac{x^k}{\lambda^k - x^k} \right), \alpha \right] \right\}}.$$

Proof. By definition, the hazard function of a random variable X is given by

$$h_X(x) = \frac{f_X(x)}{S_X(x)}. \tag{2.41}$$

Substitute the pdf in equation (2.39) and the survival function in equation (2.40) into equation (2.41) to derive the $h_X(x)$ of GPLD as

$$h_X(x) = \frac{k\lambda^k \beta^\alpha x^{\alpha k - 1} exp\left[-\beta \left(\frac{x^k}{\lambda^k - x^k} \right) \right]}{(\lambda^k - x^k)^{\alpha+1} \left\{ \Gamma(\alpha) - \gamma \left[\beta \left(\frac{x^k}{\lambda^k - x^k} \right), \alpha \right] \right\}}. \tag{2.42}$$

□

Equation (2.42) completes the proof.

2.5. **Cumulative Hazard Function of GPLD.** By definition, the cumulative hazard function of a random variable X is given by

$$H_X(x) = -log[S_X(x)] \tag{2.43}$$

Let X be random variable that follows a GPLD with survival function given in equation (2.40). The cumulative hazard function, $H_X(x)$ of GPLD is derived by substituting equation (2.40) into (2.43) to have

$$H_X(x) = -log\left\{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\} \tag{2.44}$$

Equation (2.44) is the cumulative hazard function of GPLD.

Reverse Hazard Function of GPLD.

Theorem 2.9. *The revered hazard function of a random variable X that follows a GPLD with parameters α, β, k, λ exist and it is given by*

$$A_X(x) = \frac{k\lambda^k\beta^\alpha x^{\alpha k-1}exp\left[-\beta\left(\frac{x^k}{\lambda^k-x^k}\right)\right]}{(\lambda^k - x^k)^{\alpha+1}\left\{\gamma\left[\beta\left(\frac{x^k}{\lambda^k-x^k}\right), \alpha\right]\right\}}.$$

Proof. By definition, the reversed hazard function of a random variable X is given by

$$A_X(x) = \frac{f_X(x)}{F_X(x)}. \tag{2.45}$$

Substitute the pdf in Equation (2.39) and the cdf in Equation (2.38) into Equation (2.45), we derive the $A_X(x)$ of GPLD as

$$\tau_X(x) = \frac{k\lambda^k\beta^\alpha x^{\alpha k-1}exp\left[-\beta\left(\frac{x^k}{\lambda^k-x^k}\right)\right]}{(\lambda^k - x^k)^{\alpha+1}\left\{\gamma\left[\beta\left(\frac{x^k}{\lambda^k-x^k}\right), \alpha\right]\right\}}. \tag{2.46}$$

□

2.6. **Quantile Function of GPLD.**

Definition 2.10. [22] The quantile function of a random variable X is the value at which the probability of the random variable is less than or equal to the given probability. It is the inverse function of the cdf and it is defined as

$$Q_X(x) = F_X^{-1}(x) \tag{2.47}$$

Recall the pdf of T-Power{log-logistic} given in Equation (41).

Lemma 2.11. *Let T be a random variable with pdf $f_T(x)$, then random variate, $X = \lambda\left(\frac{T}{1+T}\right)^{1/k}$ follows T-Power{log-logistic} family of distribution in Equation (10), provided T is supported on the interval 0 to ∞, i.e, $T \in [0, \infty)$. The log-logistic parameters, scale = shape = 1, where k and λ are the parameters from the power function distribution.*

Proof. It is easy to see the result from Remark 2.4 (i). □

Lemma 2.12. *It follows from Lemma 1 that the quantile functions of T-Power function{log-logistic} distribution is given by*

$$Q_X(p) = \lambda \left[\frac{Q_T(p)}{1 + Q_T(p)} \right]^{1/k} \tag{2.48}$$

Proof. It is easy to see the result from Remark 2.4 (i). □

Theorem 2.13. *If $T(\alpha, \beta)$ follows a gamma distribution with parameters α and β, then the quantile of GPLD with parameters $\alpha, \beta, k.\lambda$ is given by*

$$Q_X(p) = \lambda \left(\frac{Q_{T(\alpha,\beta)}(p)}{1 + Q_{T(\alpha,\beta)}(p)} \right)^{1/k}$$

where $Q_{T(\alpha,\beta)}$ is the quantile function of gamma distribution with parameters α, β; and k and λ are the parameters from the power function distribution.

Proof. Following Remarks 2.4 (i) and (ii), and Lemma (2.6) and (2.11). Substitute the quantile function of gamma distribution into Lemma (2.11) to have

$$Q_X(p) = \lambda \left(\frac{Q_{T(\alpha,\beta)}(p)}{1 + Q_{T(\alpha,\beta)}(p)} \right)^{1/k} \tag{2.49}$$

where k is a shape parameter and λ is a scale parameter from the power function distribution. □

It is easy to generate T using R codes. The rgamma generates random values of gamma distribution, T. Then, use the transformation in Theorem (2.13) with known α and β to generate random variates that follow GPLD.
The quantile function returns the value x such that

$$F(x) = P(X \leq x) = p$$

The quantile function of a particular distribution is used in Monte Carlo method to simulate random variates that follows such distribution. The quantile function can be used to partition a distribution into different non-overlapping continuous sections. We can determined the quartiles, octiles, deciles and percentiles using the quantile function.

2.7. **Moment of GPLD.** The moment of a distribution is a very important function for deriving the mean of the distribution. The series expansion of the pdf of GPLD is given by

$$f_X(x) = \frac{k\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^i (\alpha + i + j)! x^{k(\alpha+i+j)-1}}{i! j! \lambda^{k(\alpha+i+j)}} \tag{2.50}$$

If $i = j = 0$, the series expansion of the pdf of GPLD given in equation (2.50) will reduce to.

$$f_X(x) = \frac{\alpha! k\beta^\alpha}{\Gamma(\alpha)\lambda^{\alpha k}} x^{\alpha k - 1} \tag{2.51}$$

The rth moment of GPLD using the linear expansion pdf in equation (2.50) is given by

$$E(x^r) = \frac{k\lambda^r\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i(\alpha+i+j)!}{i!j![k(\alpha+i+j)+r]} \tag{2.52}$$

If $r = 1$, we have the mean of GPLD given by

$$E(x) = \frac{k\lambda\beta^\alpha}{\Gamma(\alpha)} \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^i(\alpha+i+j)!}{i!j![k(\alpha+i+j)+1]} \tag{2.53}$$

If $i = j = 0$, the mean of GPLD becomes

$$E(x) = \frac{(\alpha-1)!\alpha k\lambda\beta^\alpha}{\Gamma(\alpha)(\alpha k+1)} \tag{2.54}$$

and the variance is given by

$$Var(x) = \frac{\alpha!k\lambda^2\beta^\alpha}{\Gamma(\alpha)(\alpha k+2)} \mid 1 - \frac{\alpha!k\beta^\alpha(\alpha k+2)}{(\alpha k+1)} \mid \tag{2.55}$$

2.8. **Order Statistics of GPLD.**

2.8.1. 1^{st} *Order Statistics of GPLD.*

Lemma 2.14. *Let $X_1, X_2, ..., X_n$ be a random sample from the GPLD distribution and $X_{(1)}, X_{(2)}, , X_{(n)}$, such that, $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, are order statistics obtained from the sample. Then the pdf $f_{X_1}(x)$ of the 1^{st} order statistics, $X_{(1)}$ is given by*

$$f_{X_1}(x) = \frac{nk\lambda^k\beta^\alpha x^{\alpha k-1}}{\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}} \left\{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}^{n-1} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right]$$

Proof. By definition, 1^{st} order statistic of a random variable X is given by

$$f_{X_1}(x) = -\frac{d}{dx}\prod_{i=1}^{n}[1-F(x)]^n = n[1-F(x)]^{n-1}f(x) \tag{2.56}$$

Substitute the cdf and pdf of GPLD in Equations (2.38) and (2.39) into (2.56) to have the 1^{st} order statistic of GPLD derived as

$$f_{X_1}(x) = \frac{nk\lambda^k\beta^\alpha x^{\alpha k-1}}{\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}} \left\{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}^{n-1} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right] \tag{2.57}$$

□

Equation (2.57) completes the proof.

2.8.2. *n^{th} Order Statistics of GPLD.*

Lemma 2.15. *Let $X_1, X_2, ..., X_n$ be a random sample from the GPLD distribution and $X_{(1)}, X_{(2)}, , X_{(n)}$, such that, $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, are order statistics obtained from the sample. Then the pdf $f_{X_n}(x)$ of the n^{th} order statistics, $X_{(n)}$ is given by*

$$f_{X_n}(x) = \frac{nk\lambda^k\beta^\alpha x^{\alpha k-1}}{\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}}\left\{\frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}^{n-1} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right]$$

Proof. By definition, n^{th} order statistic of a random variable X is given by

$$f_{X_n}(x) = n\left[F(x)\right]^{n-1} f(x) \tag{2.58}$$

Substitute the cdf and pdf of GPLD in equations (2.38) and (2.39) into (2.58) to have the n^{th} order statistic of GPLD derived as

$$f_{X_n}(x) = \frac{nk\lambda^k\beta^\alpha x^{\alpha k-1}}{\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}}\left\{\frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}^{n-1} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right] \tag{2.59}$$

□Equation (2.59) completes the proof.

2.8.3. *General Order Statistics of GPLD.*

Lemma 2.16. *Let $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the order statistics of a random sample that follows GPLD distribution, $X_1, X_2, ..., X_n$, from a continuous population with cdf, $F_X(x)$ and pdf $f_X(x)$. Then the pdf $f_{X_{(j)}}(x)$ of GPLD is given by*

$$f_{X_{(j)}}(x) = \frac{n!k\lambda^k\beta^\alpha x^{\alpha k-1}}{(j-1)!(n-j)!\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right]\Lambda$$

where

$$\Lambda = \left\{\frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}^{j-1}\left\{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}^{n-j}$$

Proof. By definition, j^{th} order statistic of a random variable X is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f(x)\left\{F(x)\right\}^{j-1}\left\{1 - F(x)\right\}^{n-j} \tag{2.60}$$

Substitute the cdf and pdf of GPLD in equations (2.38) and (2.39) into (2.60) to have the j^{th} order statistic of GPLD derived as

$$f_{X_{(j)}}(x) = \frac{n!k\lambda^k\beta^\alpha x^{\alpha k-1}}{(j-1)!(n-j)!\Gamma(\alpha)(\lambda^k - x^k)^{\alpha+1}} exp\left[-\beta\left(\frac{x^k}{\lambda^k - x^k}\right)\right]\Lambda \tag{2.61}$$

where

$$\Lambda = \left\{\frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}^{j-1}\left\{1 - \frac{1}{\Gamma(\alpha)}\gamma\left[\beta\left(\frac{x^k}{\lambda^k - x^k}\right), \alpha\right]\right\}^{n-j}$$

□

Equation (2.61) completes the proof.

2.9. **Maximum Likelihood Estimation (MLE).** Taking the likelihood of the pdf in (2.39) gives

$$L(\alpha,\beta,k,\lambda)=\frac{k^n\lambda^{kn}\beta^{an}}{[\Gamma(\alpha)]^n}\prod_{i=1}^{n}\frac{x_i^{\alpha k-1}}{(\lambda^k-x_i^k)^{\alpha+1}}exp\left[-\beta\sum_{i=1}^{n}\left(\frac{x_i^k}{\lambda^k-x_i^k}\right)\right] \tag{2.62}$$

Take the log to have

$$\begin{aligned}\ell=logL=nlog\frac{k}{\Gamma(\alpha)}+nklog\lambda+\alpha nlog\beta \\ +(\alpha k-1)\sum_{i=1}^{n}logx_i-(\alpha+1)\sum_{i=1}^{n}log(\lambda^k-x_i^k)-\beta\sum_{i=1}^{n}\frac{x_i^k}{\lambda^k-x_i^k}\end{aligned} \tag{2.63}$$

Partially differentiating the log-likelihood function in equation (2.63) produced the following set of simultaneous equations.

$$0=\frac{\partial\ell}{\partial\alpha}=-n\psi(\alpha)+nlog\beta+k\sum_{i=1}^{n}log(x_i)-\sum_{i=1}^{n}log(\lambda^k-x_i^k) \tag{2.64}$$

$$0=\frac{\partial\ell}{\partial\beta}=\frac{\alpha n}{\beta}-\sum_{i=1}^{n}\frac{x_i^k}{\lambda^k-x_i^k} \tag{2.65}$$

$$0=\frac{\partial\ell}{\partial k}=\frac{n}{k}+nlog\lambda+\alpha\sum_{i=1}^{n}log(x_i)+k(\alpha+1)\sum_{i=1}^{n}\frac{x_i^{k-1}}{\lambda^k-x_i^k}-k\beta\lambda^k\sum_{i=1}^{n}\frac{x_i^{k-1}}{(\lambda^k-x_i^k)^2} \tag{2.66}$$

$$0=\frac{\partial\ell}{\partial\lambda}=\frac{nk}{\lambda}+k\lambda^{k-1}(\alpha+1)\sum_{i=1}^{n}\frac{1}{\lambda^k-x_i^k}+\beta k\lambda^{k-1}\sum_{i=1}^{n}\frac{x_i^k}{(\lambda^k-x_i^k)^2} \tag{2.67}$$

where $\psi(.)$ is called digamma function. Euler's product formula for the gamma function, combined with the functional equation and an identity for the Euler-Mascheroni constant, yields the following expression for $\psi(\alpha)$ (Abramowitz and Stegun, 1972),

$$\psi(\alpha)=-\gamma+\sum_{i=0}^{\infty}\{\frac{\alpha-1}{(i+1)(i+\alpha)}\};\alpha>0, \tag{2.68}$$

It can also be approximated to

$$\psi(\alpha)=\sum_{j=0}^{\infty}\frac{1}{(j+\alpha)^2};j=0,1,2,..., \tag{2.69}$$

where γ is the EulerMascheroni constant. The digamma function has values in closed form for rational numbers, as a result of Gauss's digamma theorem (Beal, 2003).

The maximum likelihood estimators of parameters α, β, k, and λ are the simultaneous solutions of equations (2.64) to (2.67).

However, solving for α, β, k, and λ from the first partial derivatives of Equation (2.63) is difficult. Alternatively, the best values of the parameters can be obtained by direct numerical maximisation of the log-likelihood function in Equation (2.63). This is relatively easy using a mathematical or statistical software such as MATLAB or R. The procedure was implemented in R. Note that parameter λ cannot be solved using MLE method because λ is an upper bound. This means that λ can only be estimated from order statistics or by $\lambda = max(x_i) + \sigma_{\bar{x}}$, where $\sigma_{\bar{x}}$ is the standard error of X estimated from data

We find the initial values for the remaining three parameters by first assuming that the random sample is from power function distribution. We estimate k by using the sample mean $\bar{x}$. The moment estimates for the Power function parameters k is given by $k = \frac{\bar{x}}{\lambda - \bar{x}}, \bar{x} < \lambda$ ([17]). Second, we transform the GPLD data to that of a Gamma random sample by using Theorem (2.13) to obtain $T = \frac{X^k}{\lambda^k - X^k}$, where X follows GPLD and T follows Gamma distribution. We estimate α and β by using the sample mean $\bar{t}$ and the sample standard deviation $\sigma_{\bar{t}}$. By using the moment estimates for the Gamma parameters α and β, we obtain the initial estimates for parameters α and β. These estimates are given by $\alpha = \frac{(\bar{t})^2}{(\sigma_{\bar{t}})^2}$ and $\beta = \frac{\bar{t}}{(\sigma_{\bar{t}})^2}$, where $\bar{t} = \frac{\alpha}{\beta}$ and $\sigma_{\bar{t}} = \frac{\sqrt{\alpha}}{\beta}$ are the mean and standard deviation of T respectively. The numerical procedure provided by R package (maxLik) is used to produce the results ([31]).

3. Result

3.1. **Simulation Study.**

The maximum likelihood method for estimating the performance of GPLD is evaluated using Monte Carlo simulation for a total of eighteen parameter combinations with 1000 replications. Three different sample sizes $n = 20$, 200 and 1000 were considered, for small, medium and large samples respectively. The actual values, maximum likelihood estimates, absolute bias and standard errors of the parameter estimates were presented in Table 5. From Table 5, it is noted that the maximum likelihood parameter estimates performed well for estimating the distribution parameters. As the sample size increases, the absolute bias and standard error decrease.

Consistency of the Parameter Estimates. Table 2 shows that the estimates of parameters are consistent as shown by the values of absolute biases and standard errors. The absolute biases and standard errors converge to zero as the sample size, n increases from 20 to 200 to 1000.

3.2. **Application.**

In this section, we fit the propose distribution to two real data sets to illustrate the usefulness and importance of the propose Gamma-Power function distribution (GPLD). The distribution parameters are estimated by the method of maximum likelihood and five goodness-of-fit statistics are computed to compare the flexibility of the GPLD distribution with other competing

TABLE 2. Actual values, Average Estimates and Standard errors for various parameter values

	Actual values				Estimates				Std Error.			
N	α	β	K	λ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\lambda}$	α	β	K	λ
20	0.5	0.5	1.0	5	0.39	0.42	1.04	3.70	0.1004	0.1570	0.5341	0.0650
	0.5	1.0	2.0	10	1.01	3.15	2.30	9.22	0.2823	1.1236	0.9736	0.8900
	1.0	0.5	1.0	5	1.24	0.65	1.03	5.47	0.3531	0.2272	0.5411	0.0113
	1.0	1.0	2.0	10	1.24	1.01	2.39	10.94	0.3537	0.3527	1.3898	0.0439
	1.5	0.5	1.0	5	1.41	0.65	0.90	5.31	0.4045	0.2225	0.3708	0.0049
	1.5	1.0	2.0	10	1.41	1.30	2.43	10.37	0.4045	0.4452	0.9399	0.0068
200	0.5	0.5	1.0	5	0.46	0.52	0.92	4.54	0.0381	0.0690	0.4097	0.0023
	0.5	1.0	2.0	10	0.46	0.95	0.92	11.51	0.0375	0.1256	0.8736	0.2123
	1.0	0.5	1.0	5	1.10	0.51	1.28	5.79	0.0822	0.0559	0.3411	0.0031
	1.0	1.0	2.0	10	1.06	1.01	2.04	11.33	0.0939	0.1142	0.6283	0.0089
	1.5	0.5	1.0	5	1.39	0.50	1.24	5.69	0.1255	0.0543	0.6630	0.0024
	1.5	1.0	2.0	10	1.39	0.99	2.28	11.02	0.1255	0.1085	0.9398	0.0052
1000	0.5	0.5	1.0	5	0.48	0.51	0.98	5.85	0.0178	0.0297	0.0861	0.0007
	0.5	1.0	2.0	10	0.51	1.04	1.86	11.60	0.0190	0.0603	0.4746	0.0026
	1.0	0.5	1.0	5	1.03	0.53	1.26	5.79	0.0408	0.0267	0.1965	0.0006
	1.0	1.0	2.0	10	1.00	1.04	2.02	11.53	0.0396	0.0525	0.4465	0.0023
	1.5	0.5	1.0	5	1.58	0.56	0.98	5.67	0.0646	0.0267	0.2075	0.0004
	1.5	1.0	2.0	10	1.58	1.11	2.20	11.00	0.0645	0.0535	0.9397	0.0010

distributions: Weibull-Power Cauchy distribution (WPC), Power Cauchy distribution (PC), gamma distribution and Power function distribution. The goodness-of-fit tests, Akaike information criterion (AIC), Anderson-Darling statistic (A), Cramer-von Mises statistic (W) and Kolmogorov-Smirnov statistic (K-S) are computed to compare the fitted distributions to the datasets. See [32] for detailed information of A and W. Generally, the criteria for selection of best model among competing models to the fit the data of interest, is the model with the smallest values of these statistics. The required computations are carried out in the R-language ([31]).

3.2.1. *Application 1: Breaking Strengths of 100 Yarn Data.* The first real data set represents breaking strengths of 100 yarn ([33]): 66, 117, 132, 111, 107, 85, 89, 79, 91, 97, 68, 63, 61, 86, 78, 96,93, 61, 62, 60, 95, 96, 88, 62, 65, 92, 137, 91, 84, 96, 97, 60, 65, 64, 67, 80, 64, 104, 66, 84, 92, 86, 64, 132, 94, 99, 62, 61, 64, 67, 99, 85, 95, 89, 102, 100, 98, 97, 104, 64, 61, 98, 99, 102, 91, 95, 111, 104, 97, 98, 102, 109, 88, 91, 103, 94, 75, 73, 76, 70, 71, 78, 77, 77, 71, 72, 68, 64, 60, 68, 69, 62, 62, 87, 69, 62, 92, 60, 66, 98. The data has positive skewness (0.4958) and kurtosis (2.7964). Table 3 displays the maximum likelihood estimates of the parameters with their corresponding standard errors in brackets. Table 3 shows all the parameters of the GPLD distribution and other competing distributions.

3.2.2. *Application 2: Number of Successive Failures of the Air Conditioning System of a Fleet of 213 Boeing 720 Jet Airplanes.* The second real data set consists of 213 observations on the number of successive failures of the air conditioning system of a fleet of 13 Boeing 720 jet airplanes ([34]): 50, 130, 487, 57, 102, 15,

Table 3. MLE of Parameters and Standard Errors for Breaking strengths data

Distribution	Parameter Estimates			
GPLD	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\lambda}$
	0.3424	0.0705	138.1246	1.9412
	(0.0386)	(0.0144)	(0.00698)	(0.0194)
WPC	$\hat{c}$	$\hat{\alpha}$	$\hat{\sigma}$	
	0.6964	21.2441	99.5837	
	(0.1517)	(5.6077)	(1.0215)	
PC	$\hat{\alpha}$	$\hat{\sigma}$		
	13.8617	98.6978		
	(1.2646)	(0.9599)		
GAMMA	$\hat{\alpha}$	$\hat{\beta}$		
	22.2495	0.2654		
	(3.1220)	(0.0377)		
POWER	$\hat{\lambda}$	$\hat{k}$		
	138.1246	2.9728		
	(0.0070)	(0.2996)		

Table 4. Goodness-of-fit Statistics and Criteria for Breaking strengths data

Distribution	AIC	A	W	K-S
GPLD	311.5056	0.3888	0.0750	0.0457
WPC	769.4093	0.4656	0.0792	0.0785
PC	770.0498	0.7381	0.1205	0.0870
GAMMA	860.2893	2.3244	0.3733	0.1233
POWER	950.5711	2.6500	0.4132	0.2221

14, 10, 57, 320, 261, 51, 44, 9, 254, 493, 33, 18, 209, 41, 58, 60, 48, 56, 87, 11, 102, 12, 5, 14, 14, 29, 37, 186, 29, 104, 7, 4, 72, 270, 283, 7, 61, 100, 61, 502, 220, 120, 141, 22, 603, 35, 98, 54, 100, 11, 181, 65, 49, 12, 239, 14, 18, 39, 3, 12, 5, 32, 9, 438, 43, 134, 184, 20, 386, 182, 71, 80, 188, 230, 152, 5, 36, 79, 59, 33, 246, 1, 79, 3, 27, 201, 84, 27, 156, 21, 16, 88, 130, 14, 118, 44, 15, 42, 106, 46, 230, 26, 59, 153, 104, 20, 206, 5, 66, 34, 29, 26, 35, 5, 82, 31, 118, 326, 12, 54, 36, 34, 18, 25, 120, 31, 22, 18, 216, 139, 67, 310, 3, 46, 210, 57, 76, 14, 111, 97, 62, 39, 30, 7, 44, 11, 63, 23, 22, 23, 14, 18, 13, 34, 16, 18, 130, 90, 163, 208, 1,24, 70, 16, 101, 52, 208, 95, 62, 11, 191, 14, 71.
The skewness and kurtosis of the data are 2.2332 and 8.7353 respectively. The data is positively skewed and very peaked.

TABLE 5. Maximum likelihood estimates of parameters and standard errors for Boeing Data

Distribution	Parameter Estimates			
GPLD	$\hat{\alpha}$	$\hat{\beta}$	$\hat{k}$	$\hat{\lambda}$
	0.2307	0.0147	1.0515	0.3901
	(0.0189)	(0.0026)	(0.0500)	(0.0292)
WPC	$\hat{c}$	$\hat{\alpha}$	$\hat{\sigma}$	
	3.2915	0.3467	22.6104	
	(1.5537)	(0.1648)	(12.5758)	
PC	$\hat{\alpha}$	$\hat{\sigma}$		
	1.1652	48.8228		
	(0.0776)	(4.6173)		
GAMMA	$\hat{\alpha}$	$\hat{\beta}$		
	22.2495	0.0100		
	(0.0821)	(0.0012)		
POWER	$\hat{\lambda}$	$\hat{k}$		
	1.0515	0.3901		
	(0.0503)	(0.02915)		

TABLE 6. Goodness-of-fit Statistics and Criteria for Breaking strengths data

Distribution	AIC	A	W	K-S
GPLD	826.6376	0.0474	0.0467	0.0426
WPC	1962.4300	0.4187	0.0624	0.0450
PC	1973.1370	0.1377	0.9367	0.0585
GAMMA	1968.0810	1.2851	0.2283	0.0710
POWER	2071.5000	2.2191	0.3262	0.2211

4. Discussion

In this work, we generalised the power function distribution using the T-R{Y} framework. Thus, the T-Power{Y} family was generated. The general properties of the proposed family are derived, such as the cdf, pdf, survival, hazard, cumulative hazard, reversed hazard, and quantile functions. Some useful transformation were explored to show the relationship between the new family and existing families. Six quantile functions, including exponential, log-logistic, frechet, logistic, extreme value and uniform were explored for the Y variable. Two distributions were generated from each quantile function. Thus, twelve distributions that are member of this family were developed. A special case of this family is called Gamma-Power function{Log-logistics} distribution (GPLD) and was properly discussed.

All the important characterisations and properties of GPLD were derived such characterisations as the cdf, pdf, survival, hazard, cumulative hazard, reversed hazard, and quantile functions. The moment using the series linear form of the pdf and order statistics were also derived. The MLE method was used to estimate two shape parameters and a scale parameter, while the other scale parameter, which is also an upper bound of the distribution was estimated using an approximation method, because it is an upper bound.

A simulation study was carried out to test the consistency of the MLE parameters. The simulation result shows that the parameters are consistent, in the sense that, as the sample size increases, error decreases. Two real data were used to test the flexibility of the distribution. The first appliaction data is on breaking strengths of 100 yarn, and the result clearly shows that the GPLD distribution provides the best fit to the breaking strengths data among other distributions such as WPC, PC, Gamma and Power distributions. The maximum likelihood estimates of the parameters of the fitted distributions with their corresponding standard errors in brackets are given in Table 3. All the parameters of the GPLD are significant at 5% level. The GPLD provides a better fit to the yarn data than the WPC, PC, Gamma and Power function distributions as shown in Table 4.

The second data set is on the number of successive failures of the air Conditioning System of a fleet of 213 Boeing 720 Jet Airplanes. The result also shows that the GPLD distribution provides the best fit to the second data among the competing distributions. The maximum likelihood estimates of the parameters of the fitted distributions with their corresponding standard errors in brackets are given in Table 5. All the parameters of the GPLD are significant at 5% level. The GPLD provides a better fit to the breaking strength data than the WPC, PC, Gamma and Power function distributions as shown in Table 6.

5. Conclusion

We propose a new univariate continuous probability distribution called Gamma-Power function distribution with log-logistic quantile function (GPLD) using the T-R{Y} framework. The GPLD is a member of the T-Power function{Y} family and results on its statistical properties are presented, such as the cumulative distribution function, density function, the quantile function, survival function, hazard function, cumulative hazard function, moments, and order statistics. The maximum likelihood estimation of the parameters of the model were derived. GPLD distribution was applied to two data and the results of its performance were compared favourably with WPC, PC, Gamma and Power distributions. This is a clear indication that a convoluted distribution is a better model than its sub-models or distributions combined to form the convoluted distribution.

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References

[1] A. C. Dallas. *Characterization of Pareto and power function distribution.* Annals of Mathematical Statistics 28 (1976), 491–497.

[2] M. M. Ali, J. Woo. *Inference on reliability P(Y ¡ X) in a power function distribution.* Journal of Statistics and Management Science 8 (2005), 681–686.

[3] N. L. Johnson, S. Kotz, N. Balakrishnan. *Continuous Univariate Distributions.* 2nd edn. Wiley, New York (1994).

[4] N. Balakrishnan, V. B. Nevzorov. *A Primer on Statistical Distributions.* Wiley, New York (2003).

[5] C. Kleiber, S. Kotz. *Statistical Size Distributions in Economics and Actuarial Sciences.* Wiley, New York (2003).

[6] S. K. Chang. *Characterizations of the power function distribution by the independence of record values.* Journal of Chungcheong Mathematical Society 20 (2007), 139–146.

[7] A. Mbah, M. Ahsanullah. *Some characterization of power function distribution based on lower generalized order statistics.* Pakistan Journal of Statistics 23 (2007), 139–146.

[8] S. K. Sinha, P. Singh, D. C. Singh, R. Singh. *Preliminary test estimator for scale parameter of the power function distribution.* Journal of Reliability and Statistical Studies 1 (2008), 18–27.

[9] C. Forbes, M. Evans, N. Hastings, B. Peacock. *Statistical Distributions.* Wiley, New York (2011).

[10] M. Tavangar. *Power function distribution characterized by dual generalized order statistics.* Journal of the Iranian Statistical Society 10 (2011), 13–27.

[11] A. Zaka, A.S. Akhter. *Methods for estimating the parameters of power function distribution.* Pakistan Jouranl of Statistics and Operations Research 9 (2013), 213–224.

[12] A. Zaka, N. Feroze, A.S. Akhter. *A note on modified estimators for the parameters of power function distribution.* International Journal of Advanced Science and Technology 59 (2013), 71–84.

[13] G. Azedine. *Characterization of the power function distribution based on lower records.* Applied Mathematical Sciences 7 (2013), 5259–5267.

[14] M.B. Bhatt. *Characterization of power function distribution through expectation.* Open Journal of Statistics 3 (2013), 441–443.

[15] Z. Asghar, M.N. Shahzad, F. Shehzad. *Parameter estimation of power function distribution with moments.* World Applied Science Journal 26 (2013), 205–212.

[16] M.M. Ali, J. Woo, G.E. Yoon. *The UMVUE of mean and right-tail probability in a power function distribution.* Estadistica 52 (2000), 1–10.

[17] M.I. Ekum, I.A. Adeleke, E.E.E. Akarawak. *Lambda Upper Bound Distribution, Some Properties and Application.* Benin Journal of Statistics 3 (2020a), 12–40.

[18] E.E.E. Akarawak, I.A. Adeleke, R.O. Okafor. *The Weibull-Rayleigh Distribution and Its Properties.* Journal of Engineering Research 18 (2013), 61–72.

[19] N. Eugene, C. Lee, F. Famoye. *Beta-Normal distribution and its applications.* Communications in Statistics - Theory and Methods 31 (2002), 497–512.

[20] A. Alzaatreh, C. Lee, F. Famoye. *A new method for generating families of continuous distributions.* METRON 71 (2013), 63–79.

[21] M.A. Aljarrah, C. Lee, F. Famoye. *On generating T-X family of distributions using quantile functions.* Journal of Statistical Distributions and Applications 1 (2014), : doi:10.1890/13-1452.1.

[22] A. Alzaatreh, F. Famoye, C. Lee. *The gamma-normal distribution: Properties and application.* Computational Statistics & Data Analysis 69 (2014), 67–80.
[23] A. Alzaatreh, C. Lee, F. Famoye, I. Ghosh. *The generalized Cauchy family of distributions with applications.* Journal of Statistical Distributions and Applications 3 (2016), 1–16.
[24] F. Famoye, E.E.E. Akarawak, M.I. Ekum. *Weibull-Normal Distribution and its Applications.* Journal of Statistical Theory and Applications 17 (2018), 719–727.
[25] M. Zubair, A. Alzaatreh, G,M. Cordeiro, M.H. Tahir, M. Mansoor. *On generalized classes of exponential distribution using T-X family framework.* Filomat 32:4 (2018), 1259–1272.
[26] A. A. Amalare, A.S. Ogunsanya, M.I. Ekum, T.O. Owolabi. *Lomax CauchyUniform Distribution: Properties and Application to Exceedances, of Flood Peaks of Wheaton River.* Benin Journal of Statistics 3 (2020), 66–81.
[27] A.S. Ogunsanya, O.O. Sanni, W. Yahya. *Exploring Some Properties of Odd Lomax-Exponential Distribution.* Annals of Statistical Theory and Applications (ASTA) 1 (2019), 21–30.
[28] M.I. Ekum, M.O. Adamu, E.E.E. Akarawak. *T-Dagum: A Way of Generalizing Dagum Distribution Using Lomax Quantile Function.* Estadistica (2020), ID 1641207, https://doi.org/10.1155/2020/1641207.
[29] A.S. Ogunsanya, E.E.E. Akarawak, M.I. Ekum. *On Some Properties of Rayleigh Cauchy Distribution.* Journal of Statistics and Management System DOI: 10.1080/09720510.2020.1822499.
[30] A.S. Ogunsanya, E.E.E. Akarawak, M.I. Ekum. *A New Three-Parameter Weibull Inverse Rayleigh Distribution: Theoretical Development and Applications.* Mathematics and Statistics 9 (2021), 249–272.
[31] R Development Core Team. *R Development Core Team R- A Language and Environment for Statistical Computing.* R Foundation for Statistical Computing, Austria, Vienna, 2009.
[32] G. Chen, N. Balakrishnan. *A General Purpose Approximate Goodness-of-Fit Test.* Journal of Quality Technology 27 (1995), 154–161.
[33] A.J. Duncan. *Quality Control and Industrial Statistics.* (Irwin Homewood, USA, 1974).
[34] G. Chen, N. Balakrishnan. *Theoretical explanation of observed decreasing failure rate.* Technometrics 5 (1963), 375–383.

Matthew Ekum*
Department of Mathematics and Statistics, Lagos State Polytechnic, Ikorodu, Lagos State, Nigeria.
E-mail address: matekum@yahoo.com

Muminu Adamu
Department of Mathematics, University of Lagos, Akoka, Lagos State, Nigeria.
E-mail address: madamu@unilag.edu.ng

Eno Akarawak
Department of Mathematics, University of Lagos, Akoka, Lagos State, Nigeria.
E-mail address: eakarawak@unilag.edu.ng

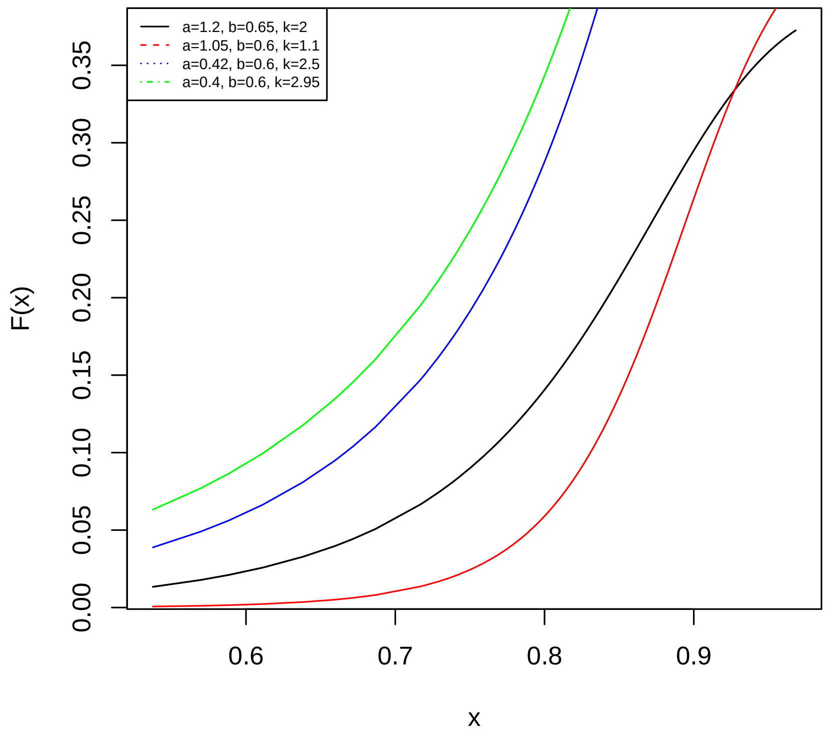

FIGURE 3. The cdf of GPLD Distribution for $\lambda = 1$.

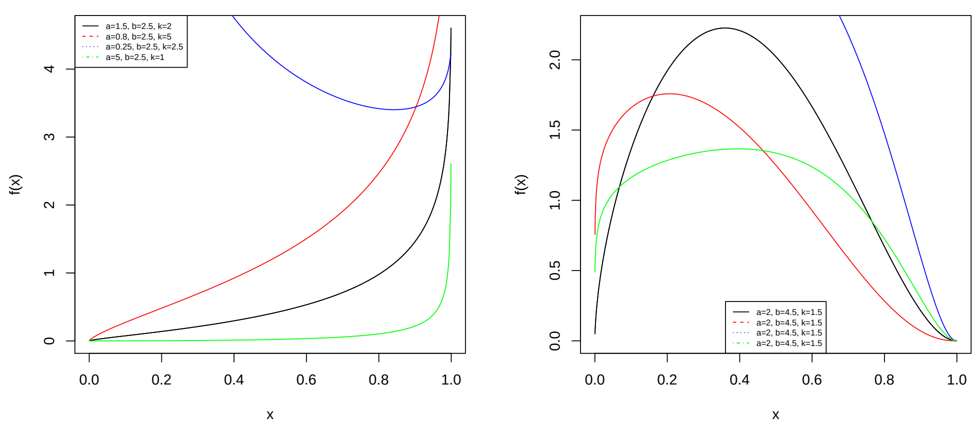

FIGURE 4. The pdf of GPLD Distribution for $\lambda = 1$.