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NUMERICAL SOLUTION OF A 3-DIMENSIONAL HEAT EQUATION BY HOMOTOPY PERTURBATION ALGORITHM

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Abstract. In this paper, a newly proposed algorithm for the numerical solutions of a 3-dimensional heat equation was presented. The formulated algorithm was developed using homotopy perturbation method (HPM). In order to test the feasibility of the proposed algorithm, four examples are considered and the numerical solutions obtain are in good agreement with available solutions in literature and exact solutions. From the computational point of view, the numerical solutions and graphs presentation show clearly that homotopy perturbation algorithm is feasible, efficiency and easy to implement.

1. Introduction

In recent years, many researches in applied mathematics and engineering are gaining more interest in understanding of natural phenomenon. Many of various processes and applications in sciences and engineering have led to understanding of physical sciences interms of partial differential equations (PDEs), system of partial differential equations (SPEs), ordinary differential equations (ODEs) and integro-differential equations (IDE). Partial differential equations involve diffusion, heat conduction, thermodynamics, gas combustions and heat exchanger have been one of the fastest growing areas in fields of mechanical and chemical engineering. Hence, the significate of the exact or approximate solutions makes us to understand the meaning of these mathematical equations in physical sciences.

Recent researches on the studies of heat equation using different exact or approximate methods of solutions are in the following works. [1] solved heat equation with strongly singular potentials,[2] obtained the control and inverse problems for the heat equation with strong singularities, [3] solved boundary value problems for the heat equation and a singular integral equation associated with it, [4] obtained the stability and stabilization of heat equation in non-cylindrical

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domain, [5] presented a DufortFrankel scheme for one-dimensional uncertain heat equation and [6] obtained systems and control letters observability on lattice points for heat equations and applications.

In the paper, we consider a 3-dimensional heat equation of the form:

$$u_t(x,y,z,t) = \lambda(u_{xx}(x,y,z,t) + u_{yy}(x,y,z,t) + u_{zz}(x,y,z,t)) \tag{1.1}$$

with initial condition:

$$u(x,y,z,0) = f(x,y,z) \tag{1.2}$$

$x, y, z \in \mathbb{R}, \lambda$ is constant, $f(x,y,z)$ is smooth function and $u(x,y,z)$ represent the temperature x, y, z are independent variables.

2. DESCRIPTION OF NUMERICAL TECHNIQUE

2.1. **Homotopy perturbation method (HPM).** The homotopy perturbation method was introduced by the Chinese researcher Prof. Ji-Huan [7] who proposed homotopy perturbation technique for solving differential and integral equations, [8] applied a coupling method of a homotopy technique and a perturbation technique for non-linear problem, [9] presented homotopy perturbation method for new nonlinear analytical technique and [10] applied homotopy perturbation method to nonlinear wave equations. In the last decade, this method becomes one of the leading, popular and acceptable numerical methods in the hands of numerical analysts because of its simplicity and high effective numerical solutions of complicated problems in many areas of applied sciences and engineering [11]. Comparing with the traditional analytic approximation tools, such as the perturbation method was presented by [12], it yields a very fast convergence of the numerical solution in series with a few iterations lead to accurate solutions. In recent time, several researchers had published some research results using HPM in applied mathematics, [13] used HPM for numerical solutions of $K(2,2)$ equation, [14] employed HPM numerical solutions of black scholes equation, [15] presented numerical solutions of the epidemic model by homotopy perturbation method, some notes on using the homotopy perturbation method for solving time-dependent differential equations was discussed by [16]. A study of thin plate vibration using homotopy perturbation algorithm was presented by [17] and [18] applied homotopy perturbation method for SIR model with vital dynamics and constant population. Thus, homotopy perturbation method is designed to obtain a quick and accurate numerical or analytical solutions of different kind of linear and nonlinear differntial equations arise from applied mathematics and engineering.

2.2. **Description of homotopy perturbation method.** In this section, we present a brief description of HPM, to illustrate the basic ideas of the homotopy perturbation method, we consider the following differential equation employed in [19, 20]

$$A(u) - f(\gamma), \quad \gamma \in \Omega \tag{2.1}$$

with boundary conditions;

$$B\left(u, \frac{\partial u}{\partial \gamma} = 0\right) \quad \gamma \in \Omega \tag{2.2}$$

where A is general differential operator, B is a boundary operator, $f(\gamma)$ is a known analytic function and ∂u is the boundary of the domain Ω. The operator A can be generally divided into two parts of L and N where L is linear part, while N is the nonlinear part in the DE. Therefore equation (2.1) can be written as

$$L(u) + N(u) - f(\gamma) = 0 \tag{2.3}$$

By using homotopy technique, One can construct a homotopy

$$Z(\gamma, m) : \Omega \times [0, 1] \mapsto \mathbb{R} \tag{2.4}$$

which satisfies

$$H(z, m) = (1 - m)[L(z) - L(u_0)] + m[L(z) + N(z) - f(\gamma)] = 0 \tag{2.5}$$

or

$$H(z, m) = L(z) - L(u_0)] + mL(u_0)) + m[N(z) - f(\gamma)] = 0 \tag{2.6}$$

where $m \in [0, 1]$, $\tau \in \Omega$ and m is called homotopy parameter and $u_0)$ is an initial approximation for the solution of equation (2.1) which satisfies the boundary conditions obviously, using equation (2.5) or (2.6), we have the following equation:

$$H(z, 0) = L(z) - L(u_0)] = 0 \tag{2.7}$$

$$H(z, 1) = L(z) + N(z) - f(\gamma) = 0 \tag{2.8}$$

Assume that the solution of (2.5) or (2.6) can be expressed as a series in m as follows:

$$Z = z_0 + mz_1 + m^2 z_2 + m^3 z_3 + \cdots = \sum_{i=0}^{\infty} m^i z_i \tag{2.9}$$

set $m \to 1$ results in the approximate solution of (2.1).

Consequently,

$$u(\gamma) = \lim_{m \to 1} Z = z_0 + z_1 + z_2 + z_3 + \cdots = \sum_{i=0}^{\infty} z_i \tag{2.10}$$

It is worth to note that the major advantage of Hes homotopy perturbation method is that the perturbation equation can be freely constructed in many ways and approximation can also be freely selected. It is well known that series (2.10) is convergent for most of the cases and also the rate of convergence is dependent on $L(z)$. The comparisons of equal powers of m give solutions of various orders. In sum, according to [21], Hes HPM considers the solution $u(x)$ of the homotopy equation in a series of m as;

$$z(x) = \sum_{i=0}^{\infty} m^i z_i = z_0 + mz_1 + m^2 z_2 + m^3 z_3 + \ldots \tag{2.11}$$

and the method considers the nonlinear $N(u)$ as

$$N(u) = \sum_{i=0}^{\infty} m^i H_i = H_0 + mH_1 + m^2 H_2 + m^3 H_3 + \dots \tag{2.12}$$

where H_n are the so-called Hes polynomials [22] which can be calculated by using the formula

$$H_n(z_0, z_1, \dots, z_n) = \frac{1}{n!}\frac{\partial^n}{\partial m^n}\left(N\left(\sum_{i=0}^{\infty} m^i z_i\right)\right)_{m=0} \tag{2.13}$$

where $n = 0, 1, 2, 3, \dots$

2.3. **Homotopy Perturbation Algorithm (HPA).** In order to reduce the evaluation of lengthy computation involve in applying homotopy perturbation method discussed in section 2.2, we formulate a four step algorithm using MAPLE18 codes software package to solve (1.1) with initial condition given in (1.2) as follows:

restart
Step 1:
Digits := $\mathbb{R}^+$
$u_0(x, y, z, 0) := f(x, y, z);$
for i from 0 to 0 do
PDE = Diff(u_0, t) + λ (Diff(u_0, x, x) + Diff(u_0, y, y) + Diff(u_0, z, z))
PDE[1] := -Int(PDE,t);
u_{i+1} := value(PDE[1]);
end do
Step 2:
for i from 1 to N do
PDE =λ (Diff(u_i, x, x) + Diff(u_i, y, y) + Diff(u_i, z, z))
PDE[1] := -Int(PDE,t);
u_{i+1} := value(PDE[1]);
end do
Step 3:
U := sum($u_j, j = 0 \dots N + 1$);
for j from 0 by 0.1 to 1 do
u[j] := evalf(eval(U,[$x = j, y = j, z = j, t = j$]));
end do
Step 4:
M := eval(U,[$x = 1, y = 1$]);
W := eval(U,[$z = 1, t = 1$]);
plot3d(M,t=-$\mathbb{R}\pi$...$\mathbb{R}\pi$, z=-$\mathbb{R}\pi$...$\mathbb{R}\pi$, grid = [100, 100], color);
plot3d(W,x=-$\mathbb{R}\pi$...$\mathbb{R}\pi$, y=-$\mathbb{R}\pi$... $\mathbb{R}\pi$, grid = [100, 100], color);
Output: See approximate solutions, tables and figures.
where λ is a constant, N is the computational length and $\mathbb{R}$ is positive integers.

3. NUMERICAL EXAMPLES AND RESULTS

Example 3.1. Consider a 3-dimensional heat equation [19]

$$u_t(x,y,z,t) = \lambda(u_{xx}(x,y,z,t) + u_{yy}(x,y,z,t) + u_{zz}(x,y,z,t)) \tag{3.1}$$

with initial condition:

$$u(x,y,z,0) = \sin x \sin y \sin z \tag{3.2}$$

Exact solution is given as;

$$u(x,y,z,0) = e^{-3\lambda} \sin x \sin y \sin z \tag{3.3}$$

Apply algorithm 2.2 for Example 3.1 when $\lambda = \frac{1}{2}$ and computational length $N = 21$, we obtained the following approximate solutions for equation (3.1) as

$$u_0 = \sin x \sin y \sin z \qquad u_1 = -1.50000000t \sin x \sin y \sin z$$
$$u_2 = 1.1250000t^2 \sin x \sin y \sin z \qquad u_3 = -0.56250000t^3 \sin x \sin y \sin z$$
$$u_4 = 0.210937500t^4 \sin x \sin y \sin z \qquad u_5 = -0.0632812500t^5 \sin x \sin y \sin z$$
$$u_6 = 0.0158203125t^6 \sin x \sin y \sin z \qquad u_7 = -0.00339006696t^7 \sin x \sin y \sin z$$
$$u_8 = 0.000635637555t^8 \sin x \sin y \sin z \qquad u_9 = -0.000105939592t^9 \sin x \sin y \sin z$$
$$u_{10} = 0.0000158909389t^{10} \sin x \sin y \sin z \qquad u_{11} = -0.00000216694t^{11} \sin x \sin y \sin z$$
$$u_{12} = 2.708682766 \times 10^{-7}t^{12} \sin x \sin y \sin z \qquad u_{13} = -3.125403192 \times 10^{-8}t^{13} \sin x \sin y \sin z$$
$$u_{14} = 3.348646277 \times 10^{-9}t^{14} \sin x \sin y \sin z \qquad u_{15} = -3.348646277 \times 10^{-10}t^{15} \sin x \sin y \sin z$$
$$u_{16} = 3.139355885 \times 10^{-11}t^{16} \sin x \sin y \sin z \qquad u_{17} = -2.770019898 \times 10^{-12}t^{17} \sin x \sin y \sin z$$
$$u_{18} = 2.308349915 \times 10^{-13}t^{18} \sin x \sin y \sin z \qquad u_{19} = -1.822381512 \times 10^{-14}t^{19} \sin x \sin y \sin z$$
$$u_{20} = 1.366786134 \times 10^{-15}t^{20} \sin x \sin y \sin z \qquad u_{21} = -9.762758100 \times 10^{-16}t^{21} \sin x \sin y \sin z$$
$$u_{22} = 6.656425977 \times 10^{-17}t^{22} \sin x \sin y \sin z$$

$$\begin{aligned}
u(x,y,z,t) \approx \sin x \sin y \sin z(&1 - 1.50000000t + 1.12500t^2 - 0.56250000t^3 \\
&+ 0.210937500t^4 - 0.0632812500t^5 + 0.0158203125t^6 - 0.0033900669t^7 \\
&+ 0.0006356376t^8 - 0.0001059396t^9 + 0.00001589093890t^{10} \\
&- 0.000002166946t^{11} + 2.708682766 \times 10^{-7}t^{12} \\
&- 3.125403192 \times 10^{-8}t^{13} + 3.348646277 \times 10^{-9}t^{14} \\
&- 3.348646277 \times 10^{-10}t^{15} + 3.139355885 \times 10^{-11}t^{16} \\
&- 2.770019898 \times 10^{-12}t^{17} + 2.308349915 \times 10^{-13}t^{18} \\
&- 1.822381512 \times 10^{-14}t^{19} + 1.366786134 \times 10^{-15}t^{20} \\
&- 9.762758100 \times 10^{-16}t^{21} + 6.656425977 \times 10^{-17}t^{22})
\end{aligned} \tag{3.4}$$

TABLE 1. Numerical solutions of $u(x, y, z, t)$ for Example 3.1.

$u(x, y, z, t)$	Exact Solution	HPA Solution	[19]
(0,0,0,0)	0.000000000000	0.000000000000	0.000000000000
(0.1,0.1,0.1,0.1)	0.000856413749	0.000856413749	0.000856413749
(0.2,0.2,0.2,0.2)	0.005809036994	0.005809036994	0.005809036994
(0.3,0.3,0.3,0.3)	0.016456179990	0.016456180000	0.016456179990
(0.4,0.4,0.4,0.4)	0.032409514260	0.032409514250	0.032409514260
(0.5,0.5,0.5,0.5)	0.052052624660	0.052052624680	0.052052624660
(0.6,0.6,0.6,0.6)	0.073190648730	0.073190648730	0.073190648730
(0.7,0.7,0.7,0.7)	0.093559679820	0.093559679820	0.093559679820
(0.8,0.8,0.8,0.8)	0.111186226800	0.111186226800	0.111186226800
(0.9,0.9,0.9,0.9)	0.124603886200	0.124603886300	0.124603886200
(1.0,1.0,1.0,1.0)	0.132946134200	0.132946134200	0.132946134200

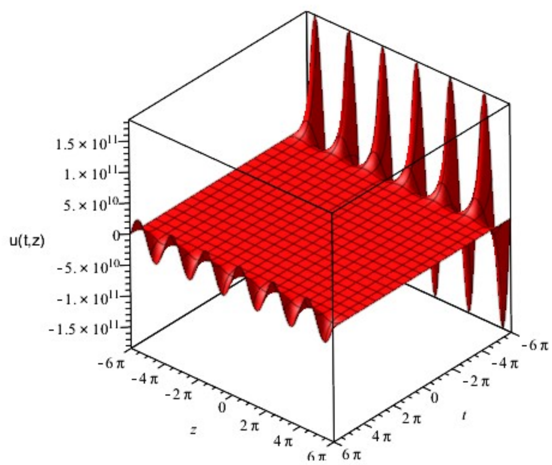

FIGURE 1. 3D-plot numerical solution $u(t, z)$ at $x = y = 1$ Example 3.1

Example 3.2. Consider a 3-dimensional heat equation [23]

$$u_t(x, y, z, t) = \lambda(u_{xx}(x, y, z, t) + u_{yy}(x, y, z, t) + u_{zz}(x, y, z, t)) \tag{3.5}$$

with initial condition:

$$u(x, y, z, 0) = (1 - y - z)e^{x+y+z} \tag{3.6}$$

Exact solution is given as

$$u(x, y, z, 0) = (1 - y - z)e^{x+t} \tag{3.7}$$

Apply algorithm 2.2 for Example 3.2 when $\lambda = -1$ and computational length $N = 15$, we obtained the following approximate solutions for equation (3.5) as

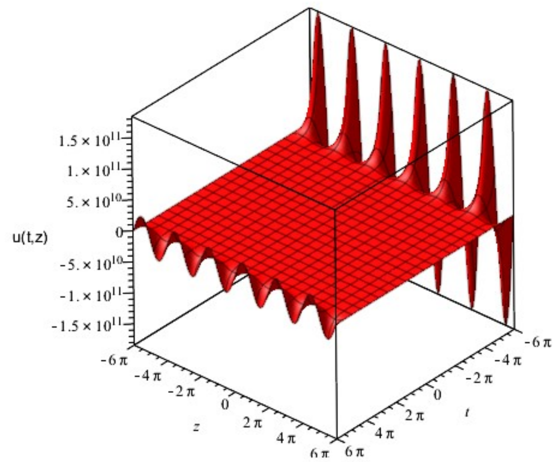

FIGURE 2. 3D-plot numerical solution $u(t,z)$ at $z=t=1$ Example 3.1

$$\begin{aligned}
u_0 &= (1-y-z)e^x & u_1 &= (1-y-z)te^x \\
u_2 &= 0.500000000(1-y-z)t^2e^x & u_3 &= 0.166666667(1-y-z)t^3e^x \\
u_4 &= 0.041666667(1-y-z)t^4e^x & u_5 &= 0.008333333(1-y-z)t^5e^x \\
u_6 &= 0.001388889(1-y-z)t^6e^x & u_7 &= 0.000198412(1-y-z)t^7e^x \\
u_8 &= 0.0000248016(1-y-z)t^8e^x & u_9 &= 0.00000275573(1-y-z)t^9e^x \\
u_{10} &= 2.755731922\times10^{-7}(1-y-z)t^{10}e^x & u_{11} &= 2.505210839\times10^{-8}(1-y-z)t^{11}e^x \\
u_{12} &= 2.087675699\times10^{-9}(1-y-z)t^{12}e^x & u_{13} &= 1.605904384\times10^{-10}(1-y-z)t^{13}e^x \\
u_{14} &= 1.147074560\times10^{-11}(1-y-z)t^{14}e^x & u_{15} &= 7.647163732\times10^{-12}(1-y-z)t^{15}e^x
\end{aligned}$$

$$\begin{aligned}
u(x,y,z,t) \approx{}& (1-y-z)e^x(1+t+0.500000000t^2+0.166666667t^3 \\
&+0.041666667t^4+0.008333333t^5+0.0013888895t^6+0.000198412t^7 \\
&+0.0000248016t^8+0.00000275573t^9+2.755731922\times10^{-7}t^{10} \\
&+2.505210839\times10^{-8}t^{11}+2.087675699\times10^{-9}t^{12}+1.605904384\times10^{-10}t^{13} \\
&+1.147074560\times10^{-11}t^{14}+7.647163732\times10^{-12}t^{15} \quad (3.8)
\end{aligned}$$

Example 3.3. Consider a 3-dimensional heat equation [23]

$$u_t(x,y,z,t) = \lambda(u_{xx}(x,y,z,t)+u_{yy}(x,y,z,t)+u_{zz}(x,y,z,t)) \quad (3.9)$$

with initial condition:

$$u(x,y,z,0) = e^{x+y+z} \quad (3.10)$$

Exact solution is given as

$$u(x,y,z,0) = e^{x+y+z+3t} \quad (3.11)$$

TABLE 2. Numerical solutions of $u(x, y, z, t)$ for Example 3.2.

$u(x, y, z, t)$	Exact Solution	HPA Solution	[19]
(0,0,0,0)	1.000000000000	1.000000000000	1.000000000000
(0.1,0.1,0.1,0.1)	0.97712220640	0.97712220640	0.97712220640
(0.2,0.2,0.2,0.2)	0.89509481880	0.89509481840	0.89509481880
(0.3,0.3,0.3,0.3)	0.72884752000	0.72884752040	0.72884752000
(0.4,0.4,0.4,0.4)	0.44510818560	0.44510818580	0.44510818560
(0.5,0.5,0.5,0.5)	0.00000000000	0.00000000000	0.00000000000
(0.6,0.6,0.6,0.6)	-0.6640233846	-0.6640233844	-0.6640233846
(0.7,0.7,0.7,0.7)	-1.6220799870	-1.6220799860	-1.6220799870
(0.8,0.8,0.8,0.8)	-2.9718194540	-2.9718194540	-2.9718194540
(0.9,0.9,0.9,0.9)	-4.8397179710	-4.8397179710	-4.8397179710
(1.0,1.0,1.0,1.0)	-7.3890560990	-7.3890560990	-7.3890560990

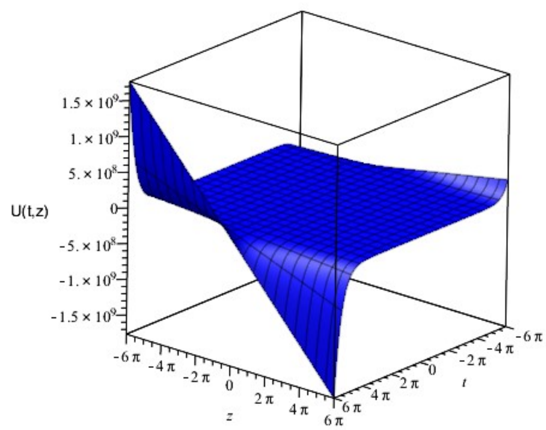

FIGURE 3. 3D-plot numerical solution $u(t, z)$ at $x = y = 1$ Example 3.2

Apply algorithm 2.2 for Example 3.3 when $\lambda = -1$ and computational length $N = 25$, we obtained the following approximate solutions for equation (3.9) as

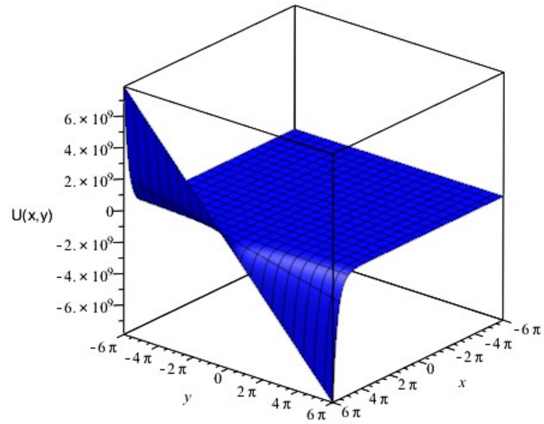

FIGURE 4. 3D-plot numerical solution $u(t, z)$ at $z = t = 1$ Example 3.2

$$
\begin{aligned}
u_0 &= e^{x+y+z} & u_1 &= 3.00000000te^{x+y+z} \\
u_2 &= 4.50000000t^2e^{x+y+z} & u_3 &= 4.50000000t^3e^{x+y+z} \\
u_4 &= 3.37500000t^4e^{x+y+z} & u_5 &= 2.02500000t^5e^{x+y+z} \\
u_6 &= 1.01250000t^6e^{x+y+z} & u_7 &= 0.43392857t^7e^{x+y+z} \\
u_8 &= 0.16272321t^8e^{x+y+z} & u_9 &= 0.05424107t^9e^{x+y+z} \\
u_{10} &= 0.01627232t^{10}e^{x+y+z} & u_{11} &= 0.004437906t^{11}e^{x+y+z} \\
u_{12} &= 0.001109476t^{12}e^{x+y+z} & u_{13} &= 0.0002560330t^{13}e^{x+y+z} \\
u_{14} &= 0.00005486422t^{14}e^{x+y+z} & u_{15} &= 0.00001097284t^{15}e^{x+y+z} \\
u_{16} &= 0.000002057408t^{16}e^{x+y+z} & u_{17} &= 3.630720481 \times 10^{-7}t^{17}e^{x+y+z} \\
u_{18} &= 6.051200802 \times 10^{-8}t^{18}e^{x+y+z} & u_{19} &= 9.55452751 \times 10^{-9}t^{19}e^{x+y+z} \\
u_{20} &= 1.43317914 \times 10^{-10}t^{20}e^{x+y+z} & u_{21} &= 2.04739877 \times 10^{-11}t^{21}e^{x+y+z} \\
u_{22} &= 2.791907410 \times 10^{-12}t^{22}e^{x+y+z} & u_{23} &= 3.641618361 \times 10^{-13}t^{23}e^{x+y+z} \\
u_{24} &= 4.552022951 \times 10^{-14}t^{24}e^{x+y+z} & u_{25} &= 5.462427542 \times 10^{-15}t^{25}e^{x+y+z}
\end{aligned}
$$

$$
\begin{gathered}
u(x,y,z,t) \approx e^{x+y+z}(1 + 3.0t + 4.5t^2 + 4.50t^3 + 3.375t^4 + 2.025t^5 + 1.0125t^6 \\
+ 0.43392857t^7 + 0.16272321t^8 + 0.05424107t^9 + 0.01627232t^{10} + 0.004437906t^{11} \\
+ 0.001109476t^{12} + 0.0002560330t^{13} + 0.00005486422t^{14} + 0.00001097284t^{15} \\
+ 0.000002057408t^{16} + 3.630720481 \times 10^{-7}t^{17} + 6.051200802 \times 10^{-8}t^{18} \\
+ 9.55452751 \times 10^{-9}t^{19} + 1.43317914 \times 10^{-10}t^{20} + 2.04739877 \times 10^{-11}t^{21} \\
+ 2.791907410 \times 10^{-12}t^{22} + 3.641618361 \times 10^{-13}t^{23} + 4.552022951 \times 10^{-14}t^{24} \\
+ 5.462427542 \times 10^{-15}t^{25} \quad (3.12)
\end{gathered}
$$

TABLE 3. Numerical solutions of $u(x, y, z, t)$ for Example 3.3.

$u(x, y, z, t)$	Exact Solution	HPA Solution	[19]
(0,0,0,0)	1.000000000000	1.000000000000	1.000000000000
(0.1,0.1,0.1,0.1)	1.8221188000	1.8221188010	1.8221188000
(0.2,0.2,0.2,0.2)	3.3201169230	3.3201169220	3.3201169230
(0.3,0.3,0.3,0.3)	6.0496474640	6.0496474640	6.0496474640
(0.4,0.4,0.4,0.4)	11.023176380	11.023176380	11.023176380
(0.5,0.5,0.5,0.5)	20.085536920	20.085536920	20.085536920
(0.6,0.6,0.6,0.6)	36.598234440	36.598234450	36.598234440
(0.7,0.7,0.7,0.7)	66.686331040	66.686331040	66.686331040
(0.8,0.8,0.8,0.8)	121.51041750	121.51041750	121.51041750
(0.9,0.9,0.9,0.9)	221.40641620	221.40641620	221.40641620
(1.0,1.0,1.0,1.0)	403.42879350	403.42879350	403.42879350

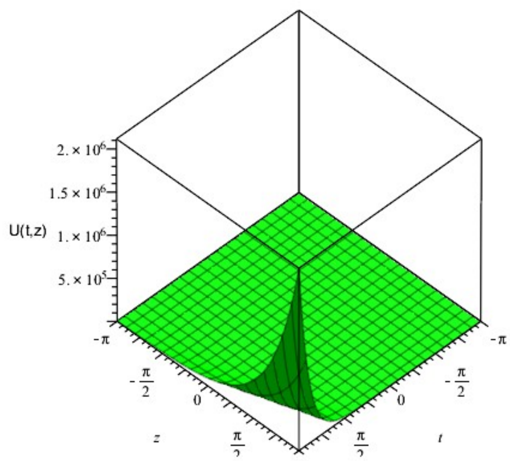

FIGURE 5. 3D-plot numerical solution $u(t, z)$ at $x = y = 1$ Example 3.3

Example 3.4. Consider a 3-dimensional heat equation [24]

$$u_t(x, y, z, t) = \lambda(u_{xx}(x, y, z, t) + u_{yy}(x, y, z, t) + u_{zz}(x, y, z, t)) \tag{3.13}$$

with initial condition:

$$u(x, y, z, 0) = (x^2 - x + y^2 - y + z^2 - z) \tag{3.14}$$

Exact solution is given as

$$u(x, y, z, 0) = (x^2 - x + y^2 - y + z^2 - z + 6t) \tag{3.15}$$

Apply algorithm 2.2 for Example 3.4 when $\lambda = -1$ and computational length $N = 2$, we obtained the following approximate solutions for equation (3.13) as

$$u_0 = x^2 - x + y^2 - y + z^2 - z \qquad u_1 = 6t$$
$$u_2 = 0 \qquad u_3 = 0$$

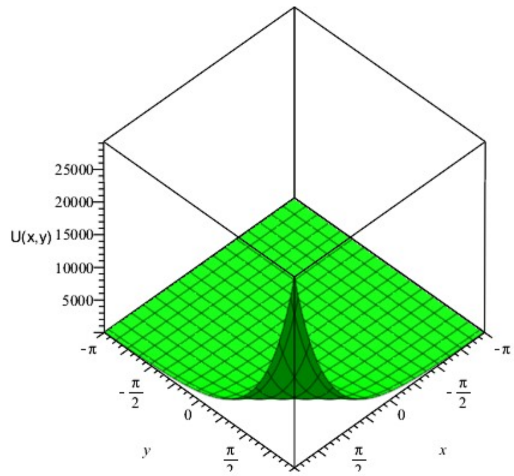

Figure 6. 3D-plot numerical solution $u(t, z)$ at $z = t = 1$ Example 3.3

$$u(x, y, z, t) \approx x^2 - x + y^2 - y + z^2 - z + 6t \tag{3.16}$$

Table 4. Numerical solutions of $u(x, y, z, t)$ for Example 3.4.

$u(x, y, z, t)$	Exact Solution	HPA Solution	[19]
(0,0,0,0)	0.000000000000	0.000000000000	0.000000000000
(0.1,0.1,0.1,0.1)	0.33000000000	0.33000000000	0.33000000000
(0.2,0.2,0.2,0.2)	0.72000000000	0.72000000000	0.72000000000
(0.3,0.3,0.3,0.3)	1.17000000000	1.17000000000	1.17000000000
(0.4,0.4,0.4,0.4)	1.68000000000	1.68000000000	1.68000000000
(0.5,0.5,0.5,0.5)	2.25000000000	2.25000000000	2.25000000000
(0.6,0.6,0.6,0.6)	2.88000000000	2.88000000000	2.88000000000
(0.7,0.7,0.7,0.7)	3.57000000000	3.57000000000	3.57000000000
(0.8,0.8,0.8,0.8)	4.32000000000	4.32000000000	4.32000000000
(0.9,0.9,0.9,0.9)	5.13000000000	5.13000000000	5.13000000000
(1.0,1.0,1.0,1.0)	6.00000000000	6.00000000000	6.00000000000

4. Conclusion

In this paper, homotopy perturbation algorithm (HPA) was formulated and employed successfully for solving a 3-dimensional heat equation. From the numerical solutions obtained, the rate of convergence was faster in Examples 2 and 4 which show the efficiency of the proposed algorithm. Moreover, the solutions obtained are in good agreement with available results in literatures and exact solutions. Therefore, this approach can be employed to solve many linear and non-linear partial differential equations arises from applied sciences and engineering. In summary we have the following highlights :

(1) Formulation of homotopy perturbation algorithm using homotopy perturbation method was developed

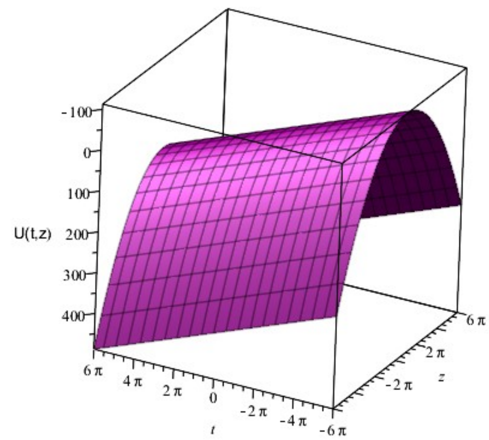

Figure 7. 3D-plot numerical solution $u(t,z)$ at $x=y=1$ Example 3.4

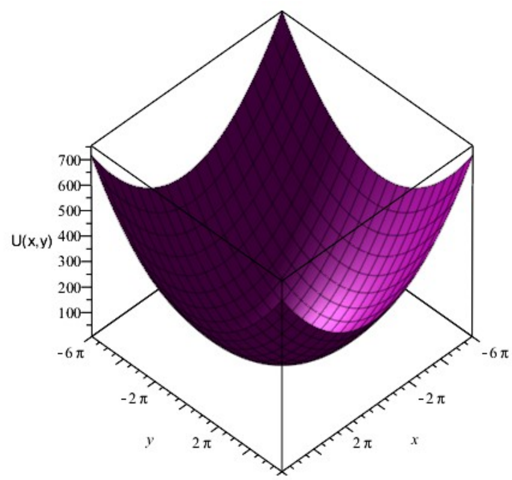

Figure 8. 3D-plot numerical solution $u(t,z)$ at $z=t=1$ Example 3.4

(2) A 3-dimensional heat equation was considered to test the feasibility of the propose algorithm.
(3) Four examples are considered from available literatures.
(4) Results compared with available literatures and exact solutions are in good agreement.
(5) MAPLE 18 software was employed to execute and display 3D plots representation.

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Authors' Conflicts of interest. The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Appendix. Maple 18 software codes for HPA Example 1

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