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FIXED POINT RESULTS FOR FUZZY SET-VALUED MAPS ON METRIC SPACE

MOHAMMED SHEHU SHAGARI

Abstract. Among various developments in fuzzy mathematics, progressive efforts have been in process to examine new fuzzy versions of the classical fixed point results and their various applications. Following this trend in this paper, some new fixed point results for fuzzy set-valued maps are established in the framework of metric space. From application viewpoint, a few corresponding fixed point theorems in ordered metric spaces as well as crisp multi-valued and single-valued mappings are pointed out and discussed. A nontrivial example is constructed to support the assertions of our obtained results. Consequently, we note that the ideas presented herein complement, unify and generalize several recently announced results in the related literature of both fuzzy and classical mathematics.

1. Introduction and preliminaries

In fixed point theory with metric structure, the contractive inequalities on underlying mappings play a significant role for solving fixed point problems. The Banach contraction mapping principle (see [4]) is one of the first well-known results in metric fixed point theory. Over the years, it has been extended in different domains (e.g., see [11, 12, 19]). Specifically, there has been more than a handful of researches concerning altering distance functions which change the distance between two points in a metric space. In 2012, Samet et al. [19] initiated the notions of $\eta-\psi$-contractive and η-admissible mappings and presented several fixed point results for such mappings.

Denote by Ψ, the class of functions $\psi : \mathbb{R}_+ = [0,\infty) \longrightarrow [0,\infty)$ such that $\sum_{n=1}^{\infty}\psi^n(t) < \infty$ for all $t > 0$, where $\psi^n(t)$ is the nth iterate of ψ. This class of functions Ψ are known in the literature as altering distance functions. The following result is well-familiar.

Lemma 1.1. *If $\psi \in \Psi$, then the following hold:*

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* Correspondence.

(i) $(\psi^n(t))_{n\in\mathbb{N}}$ *converges to* 0 *as* $n \longrightarrow \infty$ *for all* $t \in (0, \infty)$.
(ii) $\psi(t) < t$ *for all* $t > 0$.
(iii) $\psi(t) = 0$ *if and only if* $t = 0$.

Samet et al. [19] defined the concept of η-admissible mappings in the following manner.

Definition 1.2. [19] Let f be a self mapping on a nonempty set X and $\eta : X^2 \longrightarrow [0, \infty)$ be a function. We say that f is an η-admissible mapping, if

$$x, y \in X \ \eta(x, y) \geq 1 \text{ implies } \eta(fx, fy) \geq 1.$$

Let (X, d) be a metric space and $CB(X)$ denotes the set of all nonempty closed and bounded subsets of X. For $A, B \in CB(X)$, the function $H : CB(X) \times CB(X) \longrightarrow \mathbb{R}_+$ defined by

$$H(A, B) = \max\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\},$$

is called the Pompeiu-Hausdorff metric on $CB(X)$, where $d(x, A) = \inf_{x\in X}\{d(x, a) : a \in A\}$.

Remark 1.3. It is easy to see that for all $a, c \in X$ and $B, C \in CB(X)$, the following are true:

(i) $d(a, B) \leq d(a, c) + d(c, B)$.
(ii) $d(a, B) \leq d(a, c) + H(C, B)$.

Following Samet et al. [19], Asl et al. [2] introduced the notions of $\eta_* - \psi$-contractive multifunctions and η_*-admissible and proved the corresponding fixed point theorems.

Definition 1.4. Let (X, d) be a metric space and $F : X \longrightarrow CB(X)$ be a multivalued mapping. We say that F is an $\eta_* - \psi$-contractive multifunction, if there exist two functions $\eta : X^2 \longrightarrow \mathbb{R}_+$ and $\psi \in \Psi$ such that for all $x, y \in X$,

$$\eta_*(Fx, Fy)H(Fx, Fy) \leq \psi(d(x, y)),$$

where $\eta_*(A, B) = \inf\{\eta(a, b) : a \in A, b \in B\}$.

Definition 1.5. [2] Let (X, d) be a metric space, $F : X \longrightarrow CB(X)$ be a multivalued mapping and $\eta : X^2 \longrightarrow \mathbb{R}_+$ be a function. We say that F is an η_*-admissible whenever $\eta(x, y) \geq 1$ implies $\eta_*(Fx, Fy) \geq 1$.

Not long ago, Hussain et al. [10] modified the ideas of η_*-admissible and $\eta_* - \psi$-contractive mappings as follows.

Definition 1.6. [10] Let $F :\longrightarrow CB(X)$ be a multivalued mapping, $\eta, \theta : X^2 \longrightarrow \mathbb{R}_+$ be two functions, where θ is bounded. We say that F is an η_*-admissible mapping with respect to θ if for all $x, y \in X$,

$$\eta(x, y) \geq \theta(x, y) \text{ implies } \eta_*(Fx, Fy) \geq \theta_*(Fx, Fy),$$

where $\theta_*(A, B) = \sup\{\theta(a, b) : a \in A, b \in B\}$.

Note that if $\theta(x,y)=1$ for all $x,y\in X$, then Definition 1.6 reduces to Definition 1.5. For more recent results on $\eta-\psi$-contractive and η-admissible mappings with related fixed point theorems, the reader may consult [1, 11, 12, 13].

On the other hand, as a natural generalization of the concept of crisp sets, fuzzy set was introduced originally by Zadeh [23]. Since then, to use this notion, many authors have successfully extended the theory and its applications to other branches of sciences, social sciences and engineering. In 1981, Heilpern [8] used the idea of fuzzy set to initiate a class of fuzzy set-valued maps and proved a fixed point theorem for fuzzy contraction mappings which is a fuzzy analogue of the fixed point theorem of Nadler [15]. Subsequently, several authors have investigated the existence of fixed points of fuzzy set-valued maps, for example, see [3, 14, 20].

Let X be a universal set. A fuzzy set in X is a function with domain X and values in $[0,1]=I$. Denote by I^X, the collection of all fuzzy sets in X. If A is a fuzzy set in X, then the function value $A(x)$ is called the grade of membership of x in A. The α-level set of a fuzzy set A is denoted by $[A]_\alpha$ and is defined as follows:

$$[A]_\alpha=\begin{cases}\overline{\{x\in X: A(x)>0\}}, & \text{if } \alpha=0\\ \{x\in X: A(x)\geq\alpha\}, & \text{if } \alpha\in(0,1].\end{cases}$$

where by $\overline{M}$, we mean the closure of the crisp set M. We denote the family of all fuzzy sets in X by I^X.

A fuzzy set A in a metric space V is said to be an approximate quantity if and only if $[A]_\alpha$ is compact and convex in V and $\sup_{x\in V}A(x)=1$. We denote the collection of all approximate quantities in V by $W(V)$. If there exists an $\alpha\in[0,1]$ such that $[A]_\alpha,[B]_\alpha\in CB(X)$, then define

$$D_\alpha(A,B)=H([A]_\alpha,[B]_\alpha).$$

$$d_\infty(A,B)=\sup_\alpha D_\alpha(A,B).$$

Note that d_∞ is a metric on $CB(X)$ (induced by the Pompeiu-Hausdorff metric H) and the completeness of (X,d) implies the completeness of the corresponding metric space $(I_{CB(X)},d_\infty)$ (see [8]). Furthermore, $(X,d)\longmapsto(CB(X),H)\longmapsto(I_{CB(X)},d_\infty)$, are isometric embeddings via the relations $x\longrightarrow\{x\}$ (crisp set) and $M\longrightarrow\chi_M$, respectively; where

$$I_{CB(X)}=\{A\in I^X:[A]_\alpha\in CB(X),\ \text{for each } \alpha\in[0,1]\}.$$

Definition 1.7. [8] Let X be an arbitrary set and Y a metric space. A mapping $T:X\longrightarrow I^X$ is called fuzzy set-valued map. A fuzzy set-valued map T is a fuzzy subset of $X\times Y$ with membership function $T(x)(y)$. The function value $T(x)(y)$ is called the grade of membership of y in $T(x)$.

Definition 1.8. [8] Let X be a nonempty set and $T:X\longrightarrow I^X$ be a fuzzy set-valued map. A point $u\in X$ is called a fuzzy fixed point of T if there exists an $\alpha\in(0,1]$ such that $u\in[Tu]_{\alpha(u)}$.

Definition 1.9. [8] Let (X, d) be a metric space. A mapping $T : X \longrightarrow W(X)$ is called fuzzy λ-contraction, if there exists $\lambda \in (0, 1)$ such that for all $x, y \in X$,

$$d_\infty(T(x), T(y)) \leq \lambda d(x, y).$$

The following result due to Heilpern [8] is the first metric fixed point theorem for fuzzy set-valued maps.

Theorem 1.10. [8] *Every fuzzy λ-contraction on a complete metric space has a fuzzy fixed point.*

The following lemmas will be needed in the sequel.

Lemma 1.11. [15] *Let A and B be nonempty closed and bounded subsets of a metric space (X, d) and $0 < h \in \mathbb{R}$. Then, for each $b \in B$, there exists $a \in A$ such that $d(a, b) \leq H(A, B) + h$.*

Lemma 1.12. [1] *Let (X, d)be a metric space and B be a nonempty closed subset of X and $q > 1$. Then, for every $x \in X$ with $d(x, B) > 0$, there exists $b \in B$ such that $d(x, b) \leq qd(x, B)$.*

Among various developments in fuzzy mathematics, progressive efforts have been in process to investigate new fuzzy versions of the classical fixed point results and their various applications. In this direction, using novel non-crisp approaches, the aim of this paper is to introduce a new concept of fuzzy set-valued maps under the name $\eta_* - (\psi, K)$-weak fuzzy contractions and examine sufficient conditions for existence of fuzzy fixed points for such contractions in the setting of complete metric spaces. From application viewpoint, a few corresponding fixed point theorems in ordered metric space as well as crisp multi-functions and single-valued mappings are pointed out and analyzed. A nontrivial example is constructed to support the hypotheses of our obtained results. Consequently, it is noted that the ideas presented herein unify and complement several recently announced results related to $\eta - \psi$-contractive mappings and their corresponding fixed point theorems.

2. Main Results

Motivated by the ideas presented in [2, 5, 10, 9, 19], we start this section with some preliminary concepts which are useful in the establishment of our main fuzzy fixed point theorems.

Definition 2.1. Let (X, d) be a metric space, $T : X \longrightarrow I_{CB(X)}$ be a fuzzy set-valued map. Then T is called an $\eta_* - \psi$-contractive fuzzy set-valued map, if there exist two functions $\eta : X^2 \longrightarrow \mathbb{R}_+$ and $\psi \in \Psi$ such that for all $x, y \in X$ with $\alpha(x), \alpha(y) \in (0, 1]$, we have

$$\eta_*([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)})H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \psi(d(x, y)).$$

Definition 2.2. Let (X, d) be a metric space, $T : X \longrightarrow I_{CB(X)}$ be a fuzzy set-valued map and $\eta : X^2 \longrightarrow \mathbb{R}_+$ be a function. Then T is said to be η_*-admissible, if for all $x, y \in X$, there exist $\alpha(x), \alpha(y) \in (0, 1]$ such that

$$\eta(x, y) \geq 1 \text{ implies } \eta_*([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \geq 1.$$

Definition 2.3. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in a metric space (X,d) is called a trajectory of a fuzzy set-valued map $T : X \longrightarrow I^X$, starting at x_1, if $x_{n+1} \in [Tx_n]_{\alpha(x_n)}$ for each $\alpha(x_n) \in (0,1]$ and $n \in \mathbb{N}$.

We denote the family of all trajectories of a fuzzy set-valued map T at $x \in X$ by $T_{r(x)}$.

Definition 2.4. [21] Let (X,d) be a metric space and $\eta : X^2 \longrightarrow \mathbb{R}_+$ be a function. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an η-sequence if $\eta(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$

We denote the family of all η-sequence in X by $E_{X\eta}$. For a fuzzy set-valued map $T : X \longrightarrow I_{CB(X)}$, we denote the η-graph of T by $G_\eta(T)$ and is defined as:

$$G_\eta(T) = \{(x_n, x_{n+1}) \in X^2 : x_1 \in X \text{ and } \{x_n\}_{n\in\mathbb{N}} \in E_{X\eta} \cap T_{r(x_1)}\}.$$

The mapping T is called an η-closed fuzzy set-valued map if $G_\eta(T)$ is a closed subset of X^2. Note that every closed mapping is a particular kind of η-closed mapping (cf. [21]).

Definition 2.5. [21] Let (X,d) be a metric space and $\eta : X^2 \longrightarrow \mathbb{R}_+$ be a function. Then (X,d) is said to be η-complete if every Cauchy sequence in $E_{X\eta}$ is convergent in X^2.

Note that every complete metric space is η-complete, but the converse needs not be true (cf. [21]).

Throughout this paper, for all $x, y \in X$, we define $\Omega_T(x,y)$ as :

$$\Omega_T(x,y) = \max\Big\{ d(x,y), d(x,[Tx]_{\alpha(x)}), d(y,[Ty]_{\alpha(y)}), \\ d(x,[Ty]_{\alpha(y)})d(y,[Tx]_{\alpha(x)}), \frac{d(x,[Tx]_{\alpha(x)})d(y,[Ty]_{\alpha(y)})}{1+d(x,y)} \Big\}.$$

Definition 2.6. Let (X,d) be a metric space and $T : X \longrightarrow I^X$ be a fuzzy set-valued map. Then T is called (ψ, K)-weak fuzzy contraction, if there exist $\psi \in \Psi$ and $K \geq 0$ such that for all $x, y \in X$ with $\alpha(x), \alpha(y) \in (0,1]$, we have

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \psi(\Omega_T(x,y)) + Kd(y,[Tx]_{\alpha(x)}). \tag{2.1}$$

Following Berinde [5, Definition 1], we give the next definition which is an extension of Definition 2.6 and a fuzzy generalization of the main idea in [5].

Definition 2.7. Let (X,d) be a metric space, $T : X \longrightarrow I^X$ be a fuzzy set-valued map and $\eta : X^2 \longrightarrow \mathbb{R}_+$ be a function. Then T is called an $\eta_* - (\psi, K)$-weak fuzzy contraction, if there exist $\psi \in \Psi$ and $K \geq 0$ such that for all $x, y \in X$ with $\alpha(x), \alpha(y) \in (0,1]$ and $\eta_*([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \geq 1$, we have

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \psi(\Omega_T(x,y)) + Kd(y,[Tx]_{\alpha(x)}). \tag{2.2}$$

Remark 2.8. Consistent with Berinde [5, Remark 1], we note that due to the symmetry of the distance function, the contraction condition (2.1) implicitly includes the following dual one:

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \psi(\Omega_T(x,y)) + Kd(x,[Ty]_{\alpha(y)}). \tag{2.3}$$

for all $x, y \in X$. Therefore, in order to verify (ψ, K)-weak fuzzy contraction of T, it is necessary to check both the conditions (2.2) and (2.3). However, for $\eta_* - (\psi, K)$-weak fuzzy contraction, the inequality (2.2) is valid only for those $x, y \in X$ with $\alpha(x), \alpha(y) \in (0,1]$ for which $\eta_*([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \geq 1$. The advantage here is that it is not necessary to verify condition (2.2) for all $x, y \in X$.

Now, we present our main fuzzy fixed point theorem as follows.

Theorem 2.9. *Let (X, d) be an η-complete metric space and $T : X \longrightarrow I^X$ be an $\eta_* - (\psi, K)$-weak fuzzy contraction. Assume further that the following conditions are satisfied:*

- (C_1) *for each $x \in X$, there exists $\alpha(x) \in (0,1]$ such that $[Tx]_{\alpha(x)}$ is a nonempty closed and bounded subset of X;*
- (C_2) *T is η_*-admissible;*
- (C_3) *there exists $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha(x_0)}$ with $\eta(x_0, x_1) \geq 1$;*
- (C_4) *for a trajectory $\{x_n\}_{n\in\mathbb{N}}$ in $E_{X\eta}$, starting at x_1 and converging to $x \in X$, we have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, there exist $u \in X$ and $\alpha(u) \in (0,1]$ such that $u \in [Tu]_{\alpha(u)}$.

Proof. By Condition (C_3), there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha(x_0)}$ such that $\eta(x_0, x_1) \geq 1$. If $x_0 = x_1$, the proof is finished. We presume that $x_0 \neq x_1$. If there exists $x_1 \in [Tx_1]_{\alpha(x_1)}$ for some $\alpha(x_1) \in (0,1]$, then x_1 is a fuzzy fixed point of T. So suppose that $x_1 \notin [Tx_1]_{\alpha(x_1)}$. Then, given that T is η_*-admissible and $\eta(x_0, x_1) \geq 1$, we have $\eta_*([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) \geq 1$. It follows that

$$\begin{aligned} d(x_1, [Tx_1]_{\alpha(x_1)}) &\leq H([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) \\ &\leq \psi(\Omega_T(x_0, x_1)) + Kd(x_1, [Tx_0]_{\alpha(x_0)}), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \Omega_T(x_0, x_1) =& \max\Bigg\{ d(x_0, x_1), d(x_0, [Tx_0]_{\alpha(x_0)}), d(x_1, [Tx_1]_{\alpha(x_1)}), \\ & d(x_0, [Tx_1]_{\alpha(x_1)}) d(x_1, [Tx_0]_{\alpha(x_0)}), \frac{d(x_0, [Tx_0]_{\alpha(x_0)}) d(x_1, [Tx_1]_{\alpha(x_1)})}{1 + d(x_0, x_1)} \Bigg\} \\ \leq & \left\{ d(x_0, x_1), d(x_0, x_1), d(x_1, [Tx_1]_{\alpha(x_1)}), \frac{d(x_0, x_1) d(x_1, [Tx_1]_{\alpha(x_1)})}{1 + d(x_0, x_1)} \right\} \\ \leq & \max\{d(x_0, x_1), d(x_1, [Tx_1]_{\alpha(x_1)})\}. \end{aligned}$$

If $\max\{d(x_0, x_1), d(x_1, [Tx_1]_{\alpha(x_1)})\} = d(x_1, [Tx_1]_{\alpha(x_1)})$, then by Lemma 1.1, it follows from (2.4) that

$$d(x_1, [Tx_1]_{\alpha(x_1)}) \leq \psi(d(x_1, [Tx_1]_{\alpha(x_1)})) < d(x_1, [Tx_1]_{\alpha(x_1)}),$$

a contradiction. Hence, $\max\{d(x_0, x_1), d(x_1, [Tx_1]_{\alpha(x_1)})\} = d(x_0, x_1)$. Consequently,

$$d(x_1, [Tx_1]_{\alpha(x_1)}) \leq \psi(d(x_0, x_1)). \tag{2.5}$$

For any $\omega > 1$, by Lemma 1.12, there exists $x_2 \in [Tx_1]_{\alpha(x_1)}$ such that

$$d(x_1, x_2) \leq \omega d(x_1, [Tx_1]_{\alpha(x_1)}) \leq \omega\psi(d(x_0, x_1)).$$

Clearly, if $x_1 = x_2$, then T has at least one fuzzy fixed point in X and the proof is complete. Suppose that $x_1 \neq x_2$ so that $d(x_1, x_2) > 0$. Take $\omega_1 = \frac{\psi(\omega\psi(d(x_0,x_1)))}{\psi(d(x_1,x_2))}$. Then, $\omega_1 > 1$. Since $x_1 \in [Tx_0]_{\alpha(x_0)}, x_2 \in [Tx_1]_{\alpha(x_1)}$ and $\eta_*([Tx_0]_{\alpha(x_0)}, [Tx_1]_{\alpha(x_1)}) \geq 1$, we get $\eta(x_1, x_2) \geq 1$. And, η_*-admissibility of T implies that $\eta_*([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) \geq 1$. If $x_2 \in [Tx_2]_{\alpha(x_2)}$ for some $\alpha(x_2) \in (0, 1]$, then x_2 is a fuzzy fixed point of T. So, let $x_2 \notin [Tx_2]_{\alpha(x_2)}$, then

$$\begin{aligned} d(x_2, [Tx_2]_{\alpha(x_2)}) &\leq H([Tx_1]_{\alpha(x_1)}, [Tx_2]_{\alpha(x_2)}) \\ &\leq \psi(\Omega_T(x_1, x_2)) + Kd(x_2, [Tx_1]_{\alpha(x_1)}), \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} \Omega_T(x_1, x_2) =\ & \max\Bigg\{ d(x_1, x_2), d(x_1, [Tx_1]_{\alpha(x_1)}), d(x_2, [Tx_2]_{\alpha(x_2)}), \\ & d(x_1, [Tx_2]_{\alpha(x_2)})d(x_2, [Tx_1]_{\alpha(x_1)}), \frac{d(x_1, [Tx_1]_{\alpha(x_1)})d(x_2, [Tx_2]_{\alpha(x_2)})}{1 + d(x_1, x_2)} \Bigg\} \\ \leq\ & \left\{ d(x_1, x_2), d(x_1, x_2), d(x_2, [Tx_2]_{\alpha(x_2)}), \frac{d(x_1, x_2)d(x_2, [Tx_2]_{\alpha(x_2)})}{1 + d(x_1, x_2)} \right\} \\ \leq\ & \max\{d(x_1, x_2), d(x_2, [Tx_2]_{\alpha(x_2)})\}. \end{aligned}$$

Assume that $\max\{d(x_1, x_2), d(x_2, [Tx_2]_{\alpha(x_2)})\} = d(x_2, [Tx_2]_{\alpha(x_2)})$, then by Lemma 1.1, (2.6) yields

$$d(x_2, [Tx_2]_{\alpha(x_2)}) \leq \psi(d(x_2, [Tx_2]_{\alpha(x_2)})) < d(x_2, [Tx_2]_{\alpha(x_2)}),$$

a contradiction. Therefore, $\max\{d(x_1, x_2), d(x_2, [Tx_2]_{\alpha(x_2)})\} = d(x_1, x_2)$. It follows that

$$d(x_2, [Tx_2]_{\alpha(x_2)}) \leq \psi(d(x_1, x_2)). \tag{2.7}$$

Since $\omega_1 > 1$, then by Lemma 1.12, there exists $x_3 \in [Tx_2]_{\alpha(x_2)}$ such that

$$\begin{aligned} d(x_2, x_3) \leq \omega_1 d(x_2, [Tx_2]_{\alpha(x_2)}) &\leq \omega_1\psi(d(x_1, x_2)) \\ &= \psi(\omega\psi(d(x_0, x_1))). \end{aligned}$$

By continuing in this fashion, we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \in [Tx_{n-1}]_{\alpha(x_{n-1})}$, $x_n \neq x_{n-1}$, $\eta(x_n, x_{n-1}) \geq 1$ for all $n \in \mathbb{N}$ such that

$$d(x_n, x_{n-1}) \leq \psi^{n-1}(\omega\psi(d(x_0, x_1))). \tag{2.8}$$

Next, we shall show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. For this, let $m, n \in \mathbb{N}$ with $m > n$, then

$$d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq \sum_{i=n}^{m-1} \psi^{i-1}(\omega\psi(d(x_0, x_1))). \tag{2.9}$$

Since $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, it follows from (2.9) that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. Therefore, $\{x_n\}_{n\in\mathbb{N}} \in E_{X\eta} \cap T_{r(x_0)}$ is a Cauchy sequence. The η-completeness of this space implies that there exists $u \in X$ such that $x_n \longrightarrow u$ as $n \longrightarrow \infty$.

To show that u is a fuzzy fixed point of T, assume contrary that $u \notin [Tu]_{\alpha(u)}$ with $d(u, [Tu]_{\alpha(u)}) > 0$. Now, since $\{x_n\}_{n\in\mathbb{N}} \in E_{X\eta} \cap T_{r(x_0)}$ and $x_n \longrightarrow u$ as $n \longrightarrow \infty$, it follows from (C_4) that $\eta(x_n, u) \geq 1$ for each $n \in \mathbb{N}$. Moreover,

$$\begin{aligned} d(u, [Tu]_{\alpha(u)}) &\leq d(u, x_{n+1}) + d(x_{n+1}, [Tu]_{\alpha(u)}) \\ &\leq d(u, x_{n+1}) + H([Tx_n]_{\alpha(x_n)}, [Tu]_{\alpha(u)}) \\ &\leq \psi(\Omega_T(x_n, u)) + Kd(u, [Tx_n]_{\alpha(x_n)}) + d(x_{n+1}, u) \\ &\leq \psi(\Omega_T(x_n, u)) + Kd(u, x_{n+1}) + d(x_{n+1}, u), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} \Omega_T(x_n, u) = & \max\Big\{ d(x_n, u), d(x_n, [Tx_n]_{\alpha(x_n)}), d(u, [Tu]_{\alpha(u)}), \\ & d(x_n, [Tu]_{\alpha(u)})d(u, [Tx_n]_{\alpha(x_n)}), \frac{d(x_n, [Tx_n]_{\alpha(x_n)})d(u, [Tu]_{\alpha(u)})}{1 + d(x_n, u)} \Big\} \\ \leq & \max\Big\{ d(x_n, u), d(x_n, x_{n+1}), d(u, [Tu]_{\alpha(u)}), \\ & d(x_n, [Tu]_{\alpha(u)})d(u, x_{n+1}), \frac{d(x_n, x_{n+1})d(u, [Tu]_{\alpha(u)})}{1 + d(x_n, u)} \Big\}. \end{aligned}$$

Taking limit in (2.10) as $n \longrightarrow \infty$, there exists $n_0 \in \mathbb{N}$ such that $\Omega_T(x_n, u) = d(u, [Tu]_{\alpha(u)})$ as $n \longrightarrow \infty$ for all $n \geq n_0$. Hence, Lemma 1.1 implies that

$$d(u, [Tu]_{\alpha(u)}) \leq \psi(d(u, [Tu]_{\alpha(u)})) < d(u, [Tu]_{\alpha(u)}),$$

is a contradiction. Consequently, there exist $u \in X$ and $\alpha(u) \in (0, 1]$ such that $u \in [Tu]_{\alpha(u)}$; that is, u is a fuzzy fixed point of T. □

Corollary 2.10. *Let (X, d) be an η-complete metric space and $T : X \longrightarrow I^X$ be a fuzzy set-valued map. Assume further that the following conditions are satisfied:*

- (C_1) *for each $x \in X$, there exists $\alpha(x) \in (0, 1]$ such that $[Tx]_{\alpha(x)} \in CB(X)$;*
- (C_2) *there exist $\gamma \in (0, 1)$ and $K \geq 0$ such that for all $x, y \in X$,*

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \gamma d(x, y) + Kd(y, [Tx]_{\alpha(x)});$$

- (C_3) *T is η_*-admissible;*
- (C_4) *there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha(x_0)}$ with $\eta(x_0, x_1) \geq 1$;*
- (C_5) *for a trajectory $\{x_n\}_{n\in\mathbb{N}}$ in $E_{X\eta}$, starting at x_1 and converging to $x \in X$, we have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, there exist $u \in X$ and $\alpha(u) \in (0, 1]$ such that $u \in [Tu]_{\alpha(u)}$.

Proof. Take $\psi(t) = \gamma t$ for all $t \geq 0$ in Theorem 2.9. □

The following result in connection with d_∞-metric for fuzzy sets improves the results presented in [3, 8, 17]. It is noteworthy that fuzzy fixed point results in the setting of d_∞-metric is very useful in computing Hausdorff dimensions. These dimensions help us to understand the concepts of ε^∞ -space which is of tremendous importance in higher energy physics (see, e.g. [6, 7]).

For all $x, y \in X$, we define $\Omega_T(x, y, p)$ as:

$$\Omega_T(x,y,p) = \max\Bigg\{ d(x,y), p(x,T(x)), p(y,T(y)), \\ p(x,T(y))p(y,T(x)), \frac{p(x,T(x))p(y,T(y))}{1+d(x,y)} \Bigg\}.$$

Theorem 2.11. *Let (X, d) be an η-complete metric space and $T : X \longrightarrow W(X)$ be a fuzzy set-valued map. Assume that the following conditions are satisfied:*

(C_1) *there exists $K \geq 0$ such that for all $x, y \in X$,*

$$d_\infty(Tx, Ty) \leq \psi(\Omega_T(x,y,p)) + Kp(y, T(x)); \tag{2.11}$$

(C_2) *T is η_*-admissible;*

(C_3) *there exist $x_0 \in X$ and $x_1 \in T(x_0)$ with $\eta(x_0, x_1) \geq 1$;*

(C_4) *for a trajectory $\{x_n\}_{n\in\mathbb{N}}$ in $E_{X\eta}$, starting at x_1 and converging to $x \in X$, we have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then, there exist $u \in X$ such that $\{u\} \subset T(u)$.

Proof. Let $x \in X$, by assumption, $[Tx]_1$ is a nonempty closed and bounded subset of X. Now, for all $x, y \in X$,

$$\begin{aligned} D_1(T(x), T(y)) &\leq d_\infty(T(x), T(y)) \\ &\leq \psi(\Omega_T(x,y,p)) + Kp(y, T(x)). \end{aligned}$$

Since $[Tx]_1 \subseteq [Tx]_{\alpha(x)} \in CB(X)$ for each $\alpha(x) \in (0, 1]$, therefore, $d(x, [Tx]_{\alpha(x)}) \leq d(x, [Tx]_1)$ for each $\alpha(x) \in (0, 1]$. This implies that $p(x, T(x)) \leq d(x, [Tx]_1)$. On similar arguments, $p(y, T(x)) \leq d(y, [Tx]_1)$. Hence,

$$\begin{aligned} H([Tx]_1, [Ty]_1) &\leq D_1(T(x), T(y)) \\ &\leq \psi(\Omega_T(x,y,p)) + Kp(y, T(x)) \\ &\leq \psi(\Omega_T(x,y)) + Kd(y, [Tx]_1). \end{aligned}$$

Consequently, Theorem 2.9 can be applied to find $u \in X$ and $\alpha(u) = 1$ such that $u \in [Tu]_1$. □

In the next result, we replace Condition (C_4) of Theorem 2.9 with η-closedness of the mapping T.

Theorem 2.12. *Let (X, d) be an η-complete metric space and $T : X \longrightarrow I^X$ be an $\eta_* - (\psi, K)$-weak fuzzy contraction. Assume further that the following conditions are satisfied:*

(C_1) *for each $x \in X$, there exists $\alpha(x) \in (0, 1]$ such that $[Tx]_{\alpha(x)} \in CB(X)$;*

(C_2) *T is η_*-admissible;*

(C_3) *there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha(x_0)}$ with $\eta(x_0, x_1) \geq 1$;*

(C_4) *T is an η-closed fuzzy set-valued map.*

Then, there exist $u \in X$ and $\alpha(u) \in (0, 1]$ such that $u \in [Tu]_{\alpha(u)}$.

Proof. In line with the proof of Theorem 2.9, we have a sequence $\{x_n\}_{n\in\mathbb{N}} \in E_{X\eta} \cap T_{r(x_0)}$ such that $x_n \longrightarrow u \in X$ as $n \longrightarrow \infty$. To see that u is a fuzzy fixed point of T, first notice that since $\{x_n\}_{n\in\mathbb{N}} \in E_{X\eta} \cap T_{r(x_0)}$, then by definition of $G_\eta(T)$, we get $(x_{n-1}, x_n) \in G_\eta(T)$ for all $n \in \mathbb{N}$. Given that T is an η-closed fuzzy set-valued map, the η-graph $G_\eta(T)$ is a closed subset of X^2. Thus, letting $n \longrightarrow \infty$ and noting that $x_n \longrightarrow u$ as $n \longrightarrow \infty$, we have

$$\lim_{n\longrightarrow\infty} (x_{n-1}, x_n) \in G_\eta(T) \text{ implies } (u, u) \in G_\eta(T).$$

By definition of $G_\eta(T)$, we obtain some $u \in X$ and $\alpha(u) \in (0, 1]$ such that $u \in [Tu]_{\alpha(u)}$. □

In what follows, we provide an example to support the hypotheses of Theorem 2.9.

Example 2.13. Let $X = [0, \infty)$ and $d : X^2 \longrightarrow \mathbb{R}_+$ be defined by

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ \max\{x, y\}, & \text{if } x \neq y. \end{cases}$$

Then, (X, d) is an η-complete metric space. Let $\lambda \in (0, 1]$ and $T(x) : X \longrightarrow [0, 1]$ be a fuzzy set-valued map defined as follows:
if $x = 0$, then

$$T(x)(t) = \begin{cases} \lambda, & \text{if } t = 0 \\ 0, & \text{if } t \neq 0, \end{cases}$$

if $x \in (0, 3]$, then

$$T(x)(t) = \begin{cases} \lambda, & \text{if } 0 \leq t < \frac{x}{40} \\ \frac{\lambda}{5}, & \text{if } \frac{x}{40} \leq t < \frac{x}{16} \\ \frac{\lambda}{8}, & \text{if } \frac{x}{16} \leq t < \frac{x}{2} \\ 0, & \text{if } \frac{x}{2} \leq t < \infty, \end{cases}$$

if $x > 3$, then

$$T(x)(t) = \begin{cases} \lambda, & \text{if } t = \frac{1}{5} \\ 0, & \text{if } t \neq \frac{1}{5}. \end{cases}$$

Suppose that $\alpha(x) = \frac{\lambda}{5}$ for each $x \in X$, then

$$[Tx]_{\alpha(x)} = \begin{cases} \{0\}, & \text{if } x = 0 \\ \left[0, \frac{x}{16}\right], & \text{if } x \in (0, 3] \\ \left\{\frac{1}{5}\right\}, & \text{if } x > 3. \end{cases}$$

Clearly, $[Tx]_{\alpha(x)} \in CB(X)$ for each $x \in X$. Define the functions $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and $\eta : X^2 \longrightarrow \mathbb{R}_+$ respectively as follows:

$$\psi(t) = \begin{cases} \frac{t}{4}, & \text{if } t \in [0, 3] \\ \frac{1}{4}, & \text{if } t > 3. \end{cases}$$

$$\eta(x, y) = \begin{cases} 2, & \text{if } x, y \in [0, 3] \\ \max\{x, y\}, & \text{if } x, y \in (3, \infty). \end{cases}$$

Now, we verify Conditions $(C_2), (C_3)$ and (C_4) of Theorem 2.9. Let $x, y \in (0, 3]$ and $\eta(x, y) \geq 1$. Then,

$$\eta_*([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) = \inf\{\eta(p, q) : p \in [Tx]_{\alpha(x)}, q \in [Ty]_{\alpha(y)}\} = 2;$$

that is, T is η_*-admissible. Hence, (C_2) holds good. Take $x_0 \in (0, 3]$, then $[Tx_0]_{\alpha(x_0)} = \left[0, \frac{x}{16}\right]$. If $x_1 \in [Tx_0]_{\alpha(x_0)}$, then $x_1 \in \left[0, \frac{x_0}{16}\right] \subseteq [0, 3]$ and $\eta(x_0, x_1) = 2$. Thus, (C_3) is satisfied. Now, consider a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $\eta(x_n, x_{n+1}) \geq 1$ such that $\lim_{n\longrightarrow\infty} x_n = x \in X$. Then, we are sure that $x_n \in [0, 3]$. It follows that $x \in [0, 3]$. Therefore, $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$, proving (C_4). To see that T is an $\eta_* - (\psi, K)$-weak fuzzy contraction, first observe that for $x = y = 0$ or $x, y \in (3, \infty)$, there is nothing to show, since

$$\begin{aligned} H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) &= H(\{0\}, \{0\}) = 0 \\ &= H\left(\left\{\frac{1}{5}\right\}, \left\{\frac{1}{5}\right\}\right). \end{aligned}$$

So, if $\eta(x, y) \geq 1$ and $x, y \in [0, 3]$ with $x \neq y$, we consider the following possibilities:

Case I. If $x < y$, then $H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) = H(\left[0, \frac{x}{16}\right], \left[0, \frac{y}{16}\right]) = \frac{y}{16}$, $d(x, y) = y, d(x, [Tx]_{\alpha(x)}) = x, d(y, [Ty]_{\alpha(y)}) = y, d(y, [Tx]_{\alpha(x)}) = y$, and

$$d(x, [Ty]_{\alpha(y)}) = \begin{cases} 0, & \text{if } x \leq \frac{y}{16} \\ x, & \text{if } x > \frac{y}{16}. \end{cases}$$

So, for each $K \geq \frac{1}{4}$, it is easy to verify that the contraction Condition (2.2) is satisfied.

Case II. If $x, y \in [0, 3]$ and $x > y$, then $H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) = H(\left[0, \frac{x}{16}\right], \left[0, \frac{y}{16}\right]) = \frac{x}{16}$, $d(x, y) = x, d(x, [Tx]_{\alpha(x)}) = x, d(y, [Ty]_{\alpha(y)}) = y, d(x, [Ty]_{\alpha(x)}) = x$, and

$$d(y, [Tx]_{\alpha(y)}) = \begin{cases} 0, & \text{if } x \leq \frac{y}{16} \\ y, & \text{if } x > \frac{y}{16}. \end{cases}$$

Thus, the inequality (2.2) holds for all $K \geq \frac{1}{4}$. Consequently, T is an $\eta_* - (\psi, K)$-weak fuzzy contraction. Therefore, all the hypotheses of Theorem 2.9 are satisfied. Hence, there exist $u = 0 \in X$ and $\alpha(0) = \frac{\lambda}{5}$ such that $0 \in [T0]_{\alpha(0)}$.

3. Applications in ordered metric spaces

The study of existence of fixed points on metric spaces endowed with a partial order is one of the very interesting improvements in the area of fixed point theory. This trend was introduced by Turinici [22] in 1986, but it became one of the core research subjects after the publications of the results of Ran and Reurings [18] and Nieto and Rodriguez [16].

In this section, we study the analogue of our main result in the setting of ordered metric spaces. Some preliminary concepts, introduced as follows, are necessary. Accordingly, $(X, d, \preceq)$ is called an ordered metric space, if:

(i) (X, d) is a metric space, and
(ii) $(X, \preceq)$ is a partially ordered set (poset).

Definition 3.1. Let $(X, \preceq)$ be a poset, $A, B \subseteq X$ and $T : X \longrightarrow I^X$ be a fuzzy set-valued map. Then, we write $A \sqsubseteq B$ if an only if $a \preceq b$ for all $a \in A$ and $b \in B$. The mapping T is called order preserving if for each $x, y \in X$, there exist $\alpha(x), \alpha(y) \in (0, 1]$ such that $x \preceq y$ implies $[Tx]_{\alpha(x)} \sqsubseteq [Ty]_{\alpha(y)}$.

Definition 3.2. Let $(X, d, \preceq)$ be an ordered metric space. A fuzzy set-valued map $T : X \longrightarrow I^X$ is called an ordered (ψ, K)-weak fuzzy contraction, if for all $x, y \in X$, there exist $\alpha(x), \alpha(y) \in (0, 1]$ with $[Tx]_{\alpha(x)} \sqsubseteq [Ty]_{\alpha(y)}$, $\psi \in \Psi$ and $K \geq 0$ such that

$$H([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) \leq \psi(\Omega_T(x, y)) + Kd(x, [Tx]_{\alpha(x)}). \tag{3.1}$$

Consistent with Shukla et al. [21], we say that a sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an ordered sequence if $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$. The class of all ordered sequences in X is denoted by $E_{X\preceq}$. For a fuzzy set-valued map $T : X \longrightarrow I^X$, we denote by $G^{\preceq}(T)$, the $\preceq$-graph of T, where

$$G^{\preceq}(T) = \{(x_n, x_{n+1}) \in X^2 : x_1 \in X \text{ and } \{x_n\}_{n\in\mathbb{N}} \in E_{X\preceq} \cap T_{r(x_1)}\}.$$

The fuzzy set-valued map T is said to be $\preceq$-closed, if $G^{\preceq}(T)$ is a closed subset of X^2. The metric space $(X, d, \preceq)$ is called $\preceq$-complete, if every Cauchy sequence in $E_{X\preceq}$ is convergent in X^2.

The following result is an extension of the main results of Ran and Reurings [18] and Nieto and Rodriguez [16] into fuzzy domain.

Corollary 3.3. *Let $(X, d, \preceq)$ be an $\preceq$-complete metric space and $T : X \longrightarrow I^X$ be an ordered (ψ, K)-weak fuzzy contraction. Assume that the following conditions are satisfied:*

- (C_1) *for each $x \in X$, there exists $\alpha(x) \in (0, 1]$ such that $[Tx]_{\alpha(x)} \in CB(X)$;*
- (C_2) *T is order preserving;*
- (C_3) *there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha(x_0)}$ with $x_0 \preceq x_1$;*
- (C_4) *at least one of the following hypotheses holds:*
  - (i) *for a trajectory $\{x_n\}_{n\in\mathbb{N}} \in E_{X\preceq}$, starting at x_1 and converging to $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$;*
  - (ii) *T is an $\preceq$-closed fuzzy set-valued map.*

Then, there exist $u \in X$ and $\alpha(u) \in (0, 1]$ such that $u \in [Tu]_{\alpha(u)}$.

Proof. Consider a mapping $\eta : X^2 \longrightarrow \mathbb{R}_+$ defined by

$$\eta(x, y) = \begin{cases} 1, & \text{if } x \preceq y \\ 0, & \text{otherwise.} \end{cases}$$

If $\eta(x, y) \geq 1$, then, by order preserveness of T, $[Tx]_{\alpha(x)} \sqsubseteq [Ty]_{\alpha(y)}$ for some $\alpha(x), \alpha(y) \in (0, 1]$. Hence,

$$\eta_*([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}) = \inf\{\eta(p, q) : p \in [Tx]_{\alpha(x)}, q \in [Ty]_{\alpha(y)}\} = 1.$$

This implies that T is an η_*-admissible fuzzy set-valued map. Moreover, from Condition (C_3), there exist $x_0 \in X$ and $x_1 \in [Tx_0]_{\alpha(x_0)}$ such that $\eta(x_0, x_1) \geq 1$. And, by (i) of Condition (C_4), for any trajectory $\{x_n\}_{n\in\mathbb{N}} \in E_{X\preceq}$ converging to $x \in X$, we must have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$. In addition, it is clear that

$\preceq$-closedness of T implies the η-closednes of T and $\preceq$-completeness of $(X, d, \preceq)$ yields η-completeness. Since T is an ordered (ψ, K)-weak fuzzy contraction, by definition of η, the mapping T is an $\eta_* - (\psi, K)$-weak fuzzy contraction. Consequently, Theorem 2.9 or 2.12 can be applied to obtain $u \in X$ and some $\alpha(u) \in (0, 1]$ such that $u \in [Tu]_{\alpha(u)}$. □

4. Applications in multivalued and single-valued mappings

In this section, we show that some existing results in the framework of crisp set-valued and single-valued mappings can be obtained from our fuzzy fixed point theorems.

Corollary 4.1. [9, Theorem 14] *Let (X, d) be a complete metric space and $F : X \longrightarrow CB(X)$ be an η_*-admissible multivalued mapping. Assume that for $\psi \in \Psi$, we have*

$$\eta_*(Fx, Fy)H(Fx, Fy) \leq \psi\left(\max\left\{d(x, y), d(x, Fx), d(y, Fy), \frac{d(x, Fx)d(y, Fy)}{1 + d(x, y)}\right\}\right),$$

for all $x, y \in X$. Suppose further that the following conditions are satisfied:

- (C_1) *there exist $x_0 \in X$ and $x_1 \in Fx_0$ with $\eta(x_0, x_1) \geq 1$;*
- (C_2) *for a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x \in X$ and $\eta(x_n, x_{n+1}) \geq 1$, we have $\eta(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.*

Then F has at least one fixed point in X.

Proof. Let $\alpha : X \longrightarrow (0, 1]$ be a mapping and consider a fuzzy set-valued map $T : X \longrightarrow I^X$ defined as

$$T(x)(t) = \begin{cases} \alpha(x), & \text{if } t \in Fx \\ 0, & \text{if } t \notin Fx. \end{cases}$$

Then, for each $x \in X$, there exists $\alpha(x) \in (0, 1]$ such that

$$[Tx]_{\alpha(x)} = \{t \in X : (Tx)(t) \geq \alpha(x)\} = Fx.$$

Consequently, Theorem 2.9 can be applied to find $u \in X$ and some $\alpha(u) \in (0, 1]$ such that $u \in [Tu]_{\alpha(u)} = Fu$. □

Corollary 4.2. [9, Theorem 18] *Let (X, d) be a complete metric space and $f : X \longrightarrow X$ be an η-admissible single-valued mapping. Assume that for $\psi \in \Psi$, we have*

$$\eta(fx, fy)d(fx, fy) \leq \psi\left(\max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fx)d(y, fy)}{1 + d(x, y)}\right\}\right),$$

for all $x, y \in X$. Also, suppose that the following conditions are satisfied:

(C_1) *there exists* $x_0 \in X$ *with* $\eta(x_0, fx_0) \geq 1$*;*
(C_2) *for a sequence* $\{x_n\}_{n\in\mathbb{N}}$ *in* X *converging to* $x \in X$ *and* $\eta(x_n, x_{n+1}) \geq 1$, *we have* $\eta(x_n, x) \geq 1$ *for all* $n \in \mathbb{N}$.

Then f *has a fixed point in* X.

Proof. For each $x \in X$, define a fuzzy set-valued map $T(x) : X \longrightarrow [0, 1]$ by

$$T(x)(t) = \begin{cases} 1, & \text{if } t = fx \\ 0, & \text{if } t \neq fx. \end{cases}$$

Then, for each $x \in X$, there exists $\alpha(x) = 1 \in (0, 1]$ such that $[Tx]_1 = \{fx\}$. Clearly, $\{fx\} \in CB(X)$. Note that in this case, $H([Tx]_1, [Ty]_1) = d(fx, fy)$ for all $x, y \in X$. Hence, all the asumptions of Theorem 2.9 and Corollary 4.2 coincide. Consequently, Theorem 2.9 can be applied to find $u \in X$ and some $\alpha(u) = 1$ such that $u \in [Tu]_1 = \{fu\}$, which further implies that $fu = u$. □

Remark 4.3. In line with the proofs of Corollaries 4.1 and 4.2, we can also deduce other results due to Hussain et al. [9], Samet et al. [19], Asl et al. [2] and some references therein.

Conclusion

In this note, a generalized new concept of fuzzy set-valued maps under the name $\eta_* - (\psi, K)$-weak fuzzy contractions is introduced and sufficient criteria for existence of fuzzy fixed points for such contractions in the setting of complete metric spaces are investigated. From application viewpoint, a few corresponding fixed point theorems in ordered metric spaces as well as crisp multi-valued and single-valued mappings are pointed out and discussed. An example is provided to support the assumptions of our obtained results. It is noted consequently that the notions presented herein unify, extend and complement several recently announced results related to $\eta - \psi$-contractive mappings and their corresponding fixed point theorems.

Competing Interests

The authors declare that they have no competing interests.

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Mohammed Shehu Shagari
Department of Mathematics, Faculty of Physical Sciences, Ahmadu Bello University, Nigeria.
E-mail address: shagaris@ymail.com